Community in Algebraic and Enumerative Combinatorics (24w5302)

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1 Context and format of the workshop

The objective of this five-day workshop was to provide a supportive and vitalizing environment for gender minorities in algebraic combinatorics from a wide variety of backgrounds and experiences, to share their expertise and to strengthen their presence in the mathematical community. Algebraic combinatorics is a branch of mathematics that engages algebra in various combinatorial settings and, conversely, uses combinatorial methods to solve algebraic problems. The field has significant connections to representation theory, mathematical physics, algebraic geometry, number theory, knots and links, mathematical biology, statistical mechanics, symmetric functions, invariant theory, computer science, and other areas.

Ultimately, the goal of this workshop was to foster collaboration and advance the state of knowledge in several important and interconnected topics of combinatorics. Moreover, this workshop will improve the visibility and success of women and people from underrepresented gender identities in combinatorics and mathematics more generally.

We chose a very active format, which began with group leaders proposing research problems. Participants indicated their preferences, and were matched to a group. There was some preliminary work completed in advance of the meeting. The majority of the time on Tuesday through Friday was dedicated to research on the projects, and each group presented a summary of progress on Thursday afternoon. The preliminary progress of the groups is reported in this document in the sections that follow. The group members are indicated, and the group leaders are identified in bold.

2 Exploring combinatorial models for quantum Schubert polynomials

Laura Colmenarejo, Olya Mandelshtam, *Angela Hicks, Nancy Wallace*

The (classical) *Schubert polynomials* $\mathfrak{S}_w(x)$ were defined by Lascoux and Schützenberger [2] as polynomials in the ring $\mathbb{Z}[x_1, \ldots, x_n]$ that generalize the Schur functions $\{s_\lambda\}_{\lambda \vdash n}$, and are polynomial representatives for the Schubert classes $\mathfrak{S}_w$ in the full flag variety $\mathbb{F}\ell_n$. They are indexed by permutations $w \in S_n$, and can be computed recursively through divided difference operators ∂_j, defined as

$$\partial_j(f(\mathbf{x})) = \frac{f(\mathbf{x}) - s_i(f(\mathbf{x}))}{x_i - x_{i+1}}.$$

For $w \in S_n$, with $w = s_{i_k} \cdots s_{i_1} w_0$ and $w_0 = (n, n-1, \ldots, 1)$, define

$$\mathfrak{S}_w(\mathbf{x}) = \partial_{i_k} \cdots \partial_{i_1}(x_1^{n-1} x_2^{n-2} \cdots x_{n-1}).$$

Example 2.1. $\mathfrak{S}_{23154}(\mathbf{x}) = \partial_1\partial_3\partial_2\partial_1\partial_4\partial_3\partial_2(x_1^4 x_2^3 x_3^2 x_4^1) = x_4x_2x_1 + x_3x_2x_1 + x_2^2x_1 + x_2x_1^2.$

The **quantum Schubert polynomials** $\mathfrak{S}_w^q(x) := \mathfrak{S}_w^q(x_1, \ldots, x_n; q_1, \ldots, q_{n-1})$ are a generalization in the indeterminates $q_1, \ldots, q_{n-1}$ and are the representatives of Schubert classes in the quotient ring $\mathbb{Z}[x_1, \ldots, x_n; q_1, \ldots, q_{n-1}]/I_n^q$, where I_n^q is a quantum generalization of the ideal I_n. They were introduced by Fomin, Gelfand, and Postnikov [1] and setting $q_i = 0$ for all i recovers the classical polynomials.

Example 2.2. $\mathfrak{S}_{23154}^q(\mathbf{x}) = x_4x_2x_1 + x_3x_2x_1 + x_2^2x_1 + x_2x_1^2 + q_1x_4 + q_1x_3 + q_1x_2 + q_1x_1.$

In the quantum framework, there is no analog of the divided difference operator construction or any other *easy* way to generate the quantum Schubert polynomials. Moreover, there are no good combinatorial models to generate the quantum Schubert polynomials. **Our goal, during this workshop, is to expand the known classical combinatorial models to the quantum framework.**

Schubert polynomials can be expressed in terms of **standard elementary polynomials** (SEP), $e_{i_1}^1 e_{i_2}^2 \cdots e_{i_n}^n$, where $e_i^j := e_i(x_1, \ldots, x_j)$ is the i^{th} elementary symmetric polynomial j variables. The SEP expansion is unique, and can be obtained algebraically through the **straightening algorithm** of Fomin-Gelfand-Postnikov [1].

While at Banff, we found a combinatorial model that follows the algorithm. Using this model, we have begun studying how to simplify the algorithm in certain cases to obtain the SEP expansion of $\mathfrak{S}_w$ more directly.

Example 2.3. $\mathfrak{S}_{1432}(x) = x_2^2x_3 + x_1x_2x_3 + x_1^2x_3 + x_1x_2^2 + x_1^2x_2 = e_1^2e_2^3e_3^4 - e_3^3e_3^4 - e_1^2e_1^3e_4^4$

The quantum Schubert polynomials are computed from the SEP expansion by sending $e_i^j \mapsto E_i^j$, where $E_i^j := E_i^j(x_1, \ldots, x_j; q_1, \ldots, q_{n-1})$ is the **quantum elementary polynomial**. These polynomials are defined by a certain matrix, which is equivalent to adding terms that replace x_ix_{i+1}, in a given e_j^k, with q_i. We can model this last step combinatorially by adding certain vertical dominoes in our classical combinatorial model.

Example 2.4. $E_4^4 = e_4^4 + q_1x_3x_4 + q_2x_1x_4 + q_3x_1x_2 + q_1q_3$

$$\mathfrak{S}_{4123}^q(x) = \mathfrak{S}_{4123} - q_1(2x_1 + x_2) = E_1^1E_1^2E_1^3 - E_1^1E_2^3 - E_2^2E_1^3 + E_3^3$$

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3 From counting spanning trees to quantum field theory and back

Karen Yeats, *Melanie Fraser, Elizabeth Kelley, Greta Panova, Mei Yin*

We focused on a lemma from Panzer and Yeats (2304.05299, Lemma 6.16):

Lemma 3.1. *Let G be a graph on $n \geq 4$ vertices with $m = 2(n-1)$ edges, and let $p = r+1$ be a prime. Then*

$$[a_1^r \dots a_m^r]\Psi_G^{2r} \equiv -3p^2 \cdot c_2^{(p)}(G) \text{ mod } p^3$$

Here $\Psi_G = \sum_{T \text{ spanning tree}} \prod_{e \notin T} a_e$ is the dual Kirchhoff polynomial and $p^2 c_2^{(p)}(G)$ mod p is the number of points $\Psi_G = 0$ in $\mathbb{F}_p$. The situation with the Kirchhoff polynomial (defined likewise but with $e \in T$) is similar and we moved between them as convenient.

Note that for $p = 2$, we can think of $[a_1 \dots a_m]\Psi_G^2$ as the number of partitions of G into two edge disjoint spanning trees. On the other hand for $c_2^{(2)}(G)$, we can consider the number of spanning subgraphs with an even number of spanning trees. We worked on two different problems related to the above Lemma:

1. Can we construct a map between the two spanning tree interpretations above in order to prove the lemma combinatorially for $p = 2$?
   - One strategy for a combinatorial proof would be to show (1) that every spanning tree whose complement is a spanning tree appears in the set counted by the righthand side, and (2) that the number of "extra" spanning trees in this collection is divisible by p^3.
2. Can we give a combinatorial reason why $[a_1 \dots a_m]\Psi_G^2$ is divisible by 4?
   - Because the order of the disjoint spanning trees in $[a_1 \dots a_m]\Psi_G^2$ matters, that gives an immediate power of two. To get the other power of two, we created an involution between partitions into disjoint spanning trees. If the degree of all vertices $\deg(v) \geq 3$, then because of the number of edges, we are guaranteed at least four vertices of degree 3. Label the vertices on G and take the smallest degree 3 vertex v. One edge will be in one of the trees, T, and the other two will be in the other tree T'. The involution merely switches the single edge in T with the unique edge in T' that results in two spanning trees when swapped.

We also worked on the situation in which G has $m = k(n-1)$ edges. In this case, continuing the focus on $p = 2$, the second question becomes:

Question 3.2. How many powers of 2 divide the number of ways to partition G into k disjoint spanning trees?

Using Sage, we ran a few examples for $k = 3$, with the lowest power of 2 dividing the number of partitions being 2^6. Doing some rough estimates by counting symmetries and involutions, it appears that the number of k-tuples of edge-disjoint trees of G should be divisible by $k!2^{\binom{k}{2}}$.

Returning to general p, when $G = K_{2t}$ then the point count, i.e. $p^2 c_2$, is divisible by $p^{t(t-1)}$ (math/9806055, Theorem 4.1). What can we say about the number of t-tuples of edge-disjoint trees partitioning the complete graph?

4 Type B_n crystal graphs, keys, virtualization and cacti

Olga Azenhas, *Daoji Huang, Nicolle Gonzalez, Jacinta Torres*

In our project, we want to study various combinatorial aspects of crystals of type B. We aim to mostly generalize the results in [AFT22, San21b, AS, San21a], which were carried out for the symplectic group. These include:

- Evacuation algorithm on orthogonal Kashiwara–Nakashima tableaux [KN94] yielding the Schützenberger–Lusztig involution.
- Algorithms for partial Schützenberger–Lusztig involutions and therefore an explicit description of the cactus group in type B (which coincides with the one in type C)
- Computation of orthogonal left and right keys and applications

During the workshop we have discussed the tasks presented above, and made progress which is outlined below:

- We have developed a signature rule defining the crystal structure in orthogonal tableaux. The next task is to write a proof for it using the definition of the tensor product of two crystals.
- We have discussed the case of the spin crystal and incorporated this into the general framework.
- It remains to use the virtualization map in [PPSS23], which "transforms" type B_n crystals into type C_n crystals, and apply the algorithms developed in [AFT22, San21b, AS, San21a]. De-virtualize using the inverse of Baker's type B_n virtualization map [Bak00].

We plan to continue our collaboration and meet once per week online until we have finished our article.

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5 The ubiquity of crystal bases

Anne Schilling, *Sarah Brauner, Sylvie Corteel, Zajj Daugherty*

We read the recent preprint of Mass-Gariépy [2] and proved one of its conjectures [2, Conjecture 5.3].

Theorem 5.1. *Given a partition λ, the dual equivalence graph of λ is a subgraph of the crystal skeleton of the crystal graph $B(\lambda)$.*

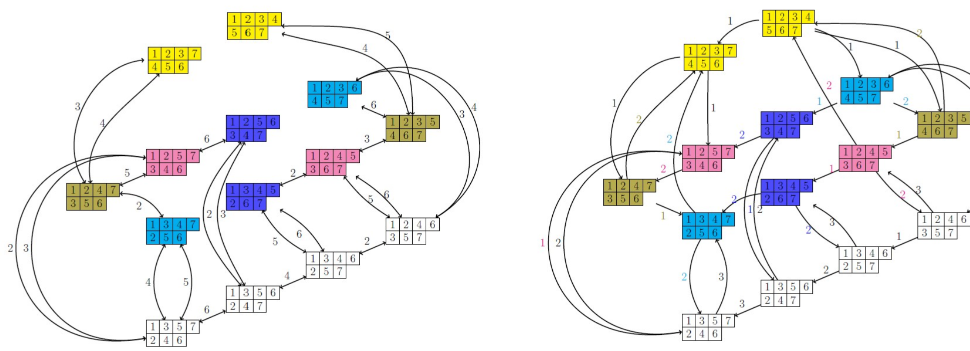

Figure 1: On the left is the dual equivalence graph for $\lambda = (4, 3)$ and on the right the skeleton graph for the same λ

The vertices of these two graphs are the standard Young tableaux of shape λ. In the crystal skeleton graph, we have an edge between two standard tableaux if there is a crystal operator between two semi-standard Young tableaux in the crystal $B(\lambda)$ whose standardizations are our two standard tableaux. In the dual equivalence graph, there is an edge between two semi-standard Young tableaux if there is a dual equivalence move between them. Assaf [1] showed that the dual equivalence moves can be written as crystal operators and this was our main tool to prove the conjecture. An example for $\lambda = (4, 3)$ is given on Figure 1 taken from [2].

We are now investigating the structure of the crystal skeleton. We were able to characterize all the edges that are in bijection with dual equivalence moves and are currently studying all the other edges.

We wrote several programs in SageMath and we have several interesting conjectures that we plan to investigate in the coming months. Understanding this graph may help to prove famous problems in algebraic combinatorics: given a symmetric polynomial, if we know its expansion in the fundamental quasisymmetric polynomial basis, could we know its expansion in the Schur polynomial basis?

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6 Differential transcendence in combinatorics

***Charlotte Hardouin, Marni Mishna**, Eva-Maria Hainzl, Sarah Selkirk*

In the study of discrete structures, enumerative data can be encoded in the coefficients of a formal power series, its *generating function.* These generating functions are often implicitly described by a functional equation that encode a recurrence relation. In particular, we consider *iterative equations* for the generating function $f(t)$ that are of the form

$$f(t) = P\left(t, f\big(R(t)\big), f\big(R(R(t))\big), \ldots, f\left(R^k(t)\right)\right)$$

where $R(t)$ is a rational function, P is a polynomial and n is the *order* of the equation.

For example, we might be interested in the number of complete 2-3 trees with n leaves (see page 3 in [2]). Since we can generate this trees by taking a complete 2-3 tree and replace each leaf with a vertex of degree 2 or a vertex of degree three, their generating function $f(t)$, where the variable t counts the number of leaves, satisfies the equation

$$f(t) = t + f\left(t^2 + t^3\right).$$

In general, solving such an equation is quite the challenge. So it became a very active area of research to extract information on the properties of the solution $f(t)$ from the functional equation itself. One property that impacts in particular the computational complexity of computing the coefficients is whether a solution is differentially transcendental or not.

A generating function f is called *differentially transcendental* over the field of rational functions $K = \mathbb{C}(t)$ if, for any integer n and any polynomial P over K with $n{+}1$ arguments, $P(t, f, f', \ldots, f^{(n)}) \neq 0$. Recently, solutions to linear iterative functional equation, where $R(t) = t^p$ for $p \in \mathbb{Z}_{>1}$, $R(t) = t{+}q$ or $R(t) = qt$ for $q \in \mathbb{C}^\times$ were proven to satisfy a dichotomy: either they are rational or differentially transcendental [1]. Further, if the equation is of the form

$$f(R(t)) = a(t)f(t) + b(t)$$

where $a(t), b(t)$ are rational and $R(t)$ satisfies certain conditions, the solution either satisfies a *linear* differential equation or it is differentially transcendental [2].

We identified several interesting directions to extend these results:

1. Second (and higher) order linear equations motivated by examples in tree enumeration
2. First order non-linear equations motivated by results of Nishioka and Nishioka [3] and Borwein [4]
3. Linear systems.

Indeed we found several compelling examples of equations in these categories describing for example the generating function of non-plane binary trees (OEIS A001190) or the Molien series of upper triangular matrices over $GF(k)$ (OEIS A008645, A008648, A008650, A008652).

Our next goal is to prove that the Borwein condition in [4] is generally satisfied for linear higher order equations which do not fall into the regime in [1] and prove that solutions are either rational or differentially algebraic.

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7 Coxeter combinatorics and posets

Susanna Fishel, *Jennifer Elder, Pamela Harris, Shaheen Nazir, Bridget Tenner*

7.1 Background

A *bond lattice* for a graph G on n vertices is a sublattice of the set partition lattice of $[n]$. We form the bond lattice as follows: consider a set partition $B_1|B_2|\dots|B_k$. If the induced subgraphs of G formed by the vertex sets $B_1, B_2, \dots, B_k$ are all connected, the set partition remains in the lattice. Otherwise, we do not include it in the bond lattice.

The bond lattices which are sublattices of the noncrossing partition lattice are particularly interesting, for several reasons. One of those reasons is Stanley's bijection from the maximal chains in the noncrossing partition lattice to parking functions [2]. The graphs whose bond lattices are sublattices of the noncrossing partition lattices are characterized in [1]; the family of *triangulation graphs* all produce sublattices of the noncrossing partition lattice and, in fact, they all produce the same bond lattice.

We define a *spider graph on n vertices g_n* to be the graph formed from an n-cycle and all edges of the form $1-i$ for $3 \leq i \leq n-1$. For example, the spider graph on five vertices is seen in Figure 2.

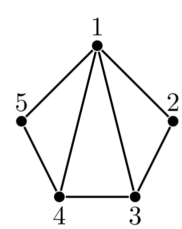

Figure 2: Spider Graph g_5.

Common tools to study posets include the f- and h-polynomials. The h-polynomial of a poset is defined as

$$h(q) = (1-q)^{\deg f} f\left(\frac{q}{1-q}\right),$$

where f is the chain-polynomial of the poset, with the coefficient of q^k in f enumerating the $(k-1)$-element chains in the poset.

7.2 Project

In this project, we are investigating the h-polynomial of the bond lattice of the spider graph. We use the spider graph, which is a triangulation graph, since all triangulation graphs produce the same bond lattice.

Our collaboration is focused on the following:

1. Give a characterization of the parking functions produced from Stanley's bijection and the maximal chains in the bond lattice.

2. Prove that the coefficients of q^k in the h-polynomial are the bond lattice parking functions with k weak descents.
3. Show that the f- and h-polynomials are real rooted.
4. Investigate what it means to have that the bond lattice of our spider graph is supersolvable.
5. Give a recursion for the h-polynomial of the bond lattice of the spider graph.

During our week in Banff, we developed (2) and (3). We have preliminary results related to all of our goals, and will continue to work together on the problems throughout the year. We have set up a weekly zoom meeting.

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8 Critical Groups of Complexes

Caroline Klivans*, Ayah Almousa, Mercedes Rosas, Martha Yip*

Let G be a finite graph on n vertices. The graph Laplacian $L(G)$ associated to G is an $n \times n$ matrix whose diagonal is the degree sequence of G and the off diagonal entries form the negative adjacency matrix. The critical group of G can be defined as the cokernel of the reduced Laplacian formed by striking out one row and one column from $L(G)$. It is a finite abelian group with connections to the combinatorics of the graph. For example, the size of the group equals the number of spanning trees of G. A well explored, but difficult, problem is to determine the invariant factors of the group for special classes of graphs. Equivalently, one seeks to determine the Smith normal form of the Laplacian.

In previous work, the critical group was extended to cell complexes of higher dimension. One considers the cokernel of an appropriately reduced combinatorial Laplacian associated to the complex. The critical group is again strongly associated to the combinatorics of the complex. For example, the size of the group can be expressed in terms of the number of higher dimensional trees of the complex. The eigenvalues of the Laplacian are also strongly related to these ideas. In this project, we will look to determine the invariant factors (smith normal form) of well motivated classes of complexes, especially those with known integral spectra.

9 Dynamical algebraic combinatorics in multi-dimensional settings

Jessica Striker*, Esther Banaian, Emily Barnard, Sunita Chepuri*

Dynamical algebraic combinatorics studies actions on objects with beautiful counting formulas and/or algebraic significance. Often, the study of the *orbits* of an action provides insight into the structure of the objects, revealing hidden symmetries and connections. One typically first seeks to understand the order n of the action and then finds interesting properties the action exhibits.

The paper [SW12] brought to light the *toggle group* of Cameron and Fon-der-Flaass [CF95] as a powerful tool for studying the dynamics of combinatorial actions on order ideals. (A *toggle* for a given poset element acts on an order ideal as the symmetric difference of the element and the order ideal if the result is an order ideal, and as the identity otherwise.) More specifically, it showed that a convex closure action we called *rowmotion* (equivalent to toggling top to bottom) on order ideals of posets is conjugate to another toggle group action, toggling left to right, which is often easier to analyze.

This was the first in the series of papers [SW12, DPS17, DSV19, BSV21, BSV23] finding ever increasing domains in which promotion on tableaux or tableaux-like objects corresponds to rowmotion on order ideals or generalizations thereof.

The following table summarizes these bijections.

Promotion	Rowmotion
$2 \times n$ standard Young tableaux	Order ideals of the Type A_{n-1} root poset
$a \times b$ increasing tableaux with entries at most $a+b+c-1$	Order ideals of $[a] \times [b] \times [c]$
Increasing labelings of P with restriction R	Order ideals of $\Gamma(P, R)$
P-strict labelings of a convex subposet of $P \times [\ell]$ with restriction R	B-bounded $\Gamma(P, R)$-partitions

The paper [BSV21] gave a vast simultaneous generalization of [DSV19] and the standard bijection between *semistandard Young tableaux* and *Gelfand-Tsetlin patterns* to give an equivariant bijection on generalized objects and actions: generalized promotion on *P-strict labelings* and piecewise-linear rowmotion on *B-bounded Q-partitions.*

Problem 9.1. *Specialize the equivariant bijection between promotion on P-strict labelings and rowmotion on B-bounded Q-partitions to more multidimensional cases of interest.*

Aside from this bijection, both P-strict labelings and B-bounded P-partitions are intriguing new objects worthy of further study. P-strict labelings are new generalizations of semistandard Young tableaux, thus there are many analogous avenues for research.

Problem 9.2. *Enumerate interesting families of P-strict labelings. Investigate the orbit structure of promotion.*

Our progress on these problems centers around the following conjectures.

We define the zigzag poset $\mathcal{Z}_n$ as $(\{x_1, \ldots, x_n\}, \leq)$ such that $x_1 < x_2 > x_3 < x_4 > \cdots x_n$. The order of promotion on P-strict labelings of $\mathcal{Z}_3 \times [\ell]$ with labels in $[q]$ is $2q$ for $q \geq 3$ and $\ell \geq 1$ [A23]. Our first two conjectures extend this idea to larger zigzag posets.

Conjecture 9.3. *The order of promotion on P-strict labelings of $\mathcal{Z}_4 \times [\ell]$ with labels in* $[3]$ *is* 15 *for* $\ell \geq 1$.

Conjecture 9.4. *The order of promotion on P-strict labelings of $\mathcal{Z}_4$ with labels in* $[q]$ *is* $15q$ *for* $q \geq 4$.

Conjecture 9.5. *The order of promotion on P-strict labelings of $\mathcal{Z}_5$ with labels in* $[q]$ *is* $120q$ *for* $q \geq 5$.

From initial computations, the arithmetic progression seen in zigzag posets seems to be a more general phenomenon.

Conjecture 9.6. *Given any finite ranked poset P, the order of promotion on P-strict labelings of P with labels in* $[q]$ *grows arithmetically with q.*

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10 Branching rules for Schur-like functions

Sheila Sundaram, Stephanie van Willigenburg, *Nadia Lafrenière, Rosa Orellana, Ying Anna Pun, Tamsen Whitehead McGinley*

Schur functions are ubiquitous, arising in combinatorics, representation theory and algebraic geometry, in addition to being a lauded basis in the algebra of symmetric functions. Within the last decade a flourishing area has been that of Schur-like functions, namely generalisations of Schur functions that exhibit various properties of the classical Schur functions. The first such Schur-like functions were quasisymmetric Schur functions, forming a basis in the algebra of quasisymmetric functions, whose combinatorics, representation theory and rules, such as Pieri and branching rules, were established. Other Schur-like functions were subsequently discovered, for example the (row-strict) dual immaculate functions and their skew generalisations. It is these latter functions that we researched, making the following discoveries, which will form the basis of a substantial paper.

1. Discovered the skew Hecke posets $P^{R\mathfrak{S}^*}_{\alpha/\beta}, P^{\mathfrak{S}^*}_{\alpha/\beta}$, indexed by skew compositions α/β.
2. Proved that $\left(P^{R\mathfrak{S}^*}_{\alpha/\beta}\right)^* = P^{\mathfrak{S}^*}_{\alpha/\beta}$.
3. Determined the unique maximum and minimum of each of these posets.
4. Enumerated the rank of the posets.
5. Proved the posets satisfied the 0-Hecke relations.
6. Verified the posets are subposets of the weak Bruhat order.
7. Created modules from the poset, $V_{\alpha/\beta}, W_{\alpha/\beta}$, whose quasisymmetric characteristic are the row-strict dual immaculate functions and dual immaculate functions, respectively.
8. Proved that $V_{\alpha/\beta}, W_{\alpha/\beta}$ are cyclic.
9. Extended all these results to (row-strict) extended Schur functions.
10. Discovered branching rules in all these instances.

11 Mentorship and community building activities

11.1 Icebreaker

We aimed to foster connections and understanding among participants through an engaging icebreaker session inspired by Eric de Groot's Snowball fight. Our adaptation involved a unique twist to encourage both introduction and the sharing of personal/professional challenges anonymously.

Preparation Before the workshop, we prepared two types of papers: one lined and the other plain. Participants were instructed to write their introduction—name, affiliation, position, area of research, or any relevant information—on plain paper. On the lined paper, participants were asked to share personal or professional challenges they faced in their careers, without revealing their identity.

Execution During the icebreaker session, participants crumpled both papers into balls, holding them in their hands. The atmosphere was light-hearted and enthusiastic as participants were ready to engage in the activity.

Snowball Fight Participants engaged in a playful snowball fight, tossing the crumpled papers around the room. The activity created an enjoyable environment, fostering camaraderie and laughter among participants.

Introductions and Challenges Following the snowball fight, each participant were asked to grab two balls—one plain and one lined. Participants opened the plain paper to introduce the individual whose information was written on the page. Additionally, participants read aloud the challenges from the other paper.

Outcomes The icebreaker injected energy and enthusiasm into the workshop, setting a positive tone for collaboration and discussion. Participants gained insight into each other's backgrounds and areas of expertise through the introduction phase. The activity facilitated open dialogue about personal and professional challenges faced by team members. We collected valuable data on the challenges encountered by participants, which served as a basis for deeper discussions in subsequent sessions.

Conclusion The activity's emphasis on anonymous sharing challenges played a vital role in creating a supportive and inclusive environment where participants felt empowered to speak up about their challenges and offer support to one another. It exemplified the power of empathy, understanding, and collective problem-solving in overcoming obstacles and fostering resilience within a team or community.

11.2 Early Career Researcher Talks

On the first day, early career researchers had the opportunity to give a 20 minute talk on their work, to share their research area with the workshop participants.

Daoji Huang Promotion and standard monomials of positroid varieties.

Esther Banaian c-singleton Birkhoff polytopes and order polytopes of heaps

Sarah Brauner Card shuffling, q-analogues and derangements.

Nicolle Gonzalez Calibrated Representations of the double Dyck path algebra (TCPL 201)

Jennifer Elder Cyclic sieving on permutations - an analysis of maps and statistics in the FindStat database

Elizabeth Kelley Skein Relations for Punctured Surfaces

Emily Barnard The pop-stack sorting method on c-Cambrian lattices

Jacinta Torres Atoms and charge beyond type A

11.3 Panel 1: Navigating the path to career success

The panel was organized by Olya Mandelshtam and Rosa Orellana, with the aim of offering guidance to junior participants for navigating the academic landscape. The panel contained five seasoned mathematicians (Anne Schilling, Stephanie van Willigenburg, Jessica Striker, Pamela Harris, Zajj Daugherty), each responding to a curated list of questions. The panelists provided expert advice and diverse opinions from varied perspectives. The informal style of the panel encouraged additional thoughtful questions and responses from audience members. The answers to the questions were recorded in a shared document, and several topics sparked further discussion over Discord.

The curated list of questions included the following subtopics:

- Picking Journals
- Handling Rejections
- Building Networks
- Strategies for Remote Research
- Writing Competitive Grants and Proposals
- Productivity During Sabbaticals

11.4 Panel 2: Preparing and presenting your work for maximum impact

Panelists: Sylvie Corteel, Charlotte Hardouin, Greta Panova, and Karen Yeats
Moderator: Marni Mishna. Our panellists shared the editor's perspective on the publication process. They offered advice and strategies on how successfully publish in high impact venues.