Mathieu Moonshine and $T^4/\mathbb{Z}_3$ sigma-models

Ida Zadeh

Mainz Institute for Theoretical Physics

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Our group

Anne Taormina

Katrin Wendland

Kasia Budzik

Mara Ungureanu

Ida Zadeh

Main goal

Find all symmetries of $T^4/\mathbb{Z}_3$ sigma-models:

Construct the Kummer-like lattice. Give the full description of the integral homology lattice.

Map this lattice into a Niemeier lattice.

Main tool: Lattice theory

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The connection is realised via the *gluing technique*: how to construct an even self dual lattice through correlating two even lattices. [Nikulin]

With Kasia we will discuss:

- Motivation
- $T^4/\mathbb{Z}_3$ subspace
- Future

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The integral homology $H_*(X,\mathbb{Z})$ of a K3 surface X, together with the intersection form, is an even self dual lattice of signature $(4,20)$:

$$H_*(X,\mathbb{Z}) \simeq U^4 \oplus E_8^2(-1) \ ,$$

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Symplectic automorphisms: leave the complex and and Kähler structures invariant.

Elliptic genus of K3

The elliptic genus of K3 is a weak Jacobi form of weight 0 and index 1: [Eguchi, Ooguri, Taormina, Yang]

$$\mathcal{E}_{\mathrm{K3}} = 8\left[\left(\frac{\vartheta_2(\tau,z)}{\vartheta_2(\tau,0)}\right)^2 + \left(\frac{\vartheta_3(\tau,z)}{\vartheta_3(\tau,0)}\right)^2 + \left(\frac{\vartheta_4(\tau,z)}{\vartheta_4(\tau,0)}\right)^2\right]$$

$q := e^{2\pi i\tau}$, $y := e^{2\pi i z}$, $\tau \in \mathbb{H}$, $z \in \mathbb{C}$

ϑ_i are Jacobi theta functions, *e.g.* $\vartheta_3 = \sum_{n=-\infty}^{\infty} q^{n^2}$, etc.

The elliptic genus is a topological quantity.

$$\mathcal{E}_{\mathrm{K3}} = 2y + 20 + \frac{2}{y} + q\left(20y^2 - 128y + 216 - \frac{128}{y} + \frac{20}{y^2}\right) + O(q^2)$$

Mathieu moonshine

Dimensions of representations of Mathieu group $\mathbb{M}_{24}$ appear in the decomposition of $\mathcal{E}_{\mathrm{K3}}$ into small $\mathcal{N}=4$ superconformal characters:

[Eguchi, Ooguri, Tachikawa]

$$\mathcal{E}_{\mathrm{K3}} = 20\mathrm{ch}_{\frac{1}{4},0} - 2\mathrm{ch}_{\frac{1}{4},\frac{1}{2}} + \mathbf{90}\mathrm{ch}_{\frac{1}{4}+1,0} + \mathbf{462}\mathrm{ch}_{\frac{1}{4}+2,0} + \mathbf{1540}\mathrm{ch}_{\frac{1}{4}+3,0} + \cdots ,$$

$$\mathrm{ch}^{\mathcal{N}=4}_{h,\ell} := q^{h-\frac{3}{8}} \frac{\vartheta_1(\tau,z)^2}{\eta(\tau)^3} , \qquad \eta = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n) .$$

[Eguchi, Taormina]

$$\begin{aligned} \mathbf{90} &= \mathbf{45} + \overline{\mathbf{45}} \\ \mathbf{462} &= \mathbf{231} + \overline{\mathbf{231}} \\ \mathbf{1540} &= \mathbf{770} + \overline{\mathbf{770}} \end{aligned}$$

Mathieu moonshine: cont'd

Decomposition of the elliptic genus in terms of $\mathbb{M}_{24}$ representations is understood. [Gannon]

The vertex algebra underlying the $\mathbb{M}_{24}$ group is still unknown!

Elliptic genus arises in K3 sigma-models: What can we learn about Mathieu moonshine from K3 sigma-models?

K3 Conformal field theory

K3 sigma-models: 2d CFTs defined on a Riemann surface and with target space a K3 surface.

Sigma-model on K3 has an 80-d moduli space

$$\mathcal{M}_{K3} = \mathrm{O}(4,20;\mathbb{Z})\backslash\mathrm{O}(4,20;\mathbb{R})/\mathrm{O}(4;\mathbb{R})\times\mathrm{O}(20;\mathbb{R})$$

[Aspinwall, Morrison; Wendland; Nahm, Wendland]

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[Aspinwall, Morrison; Wendland; Nahm, Wendland]

There is however no K3 sigma-model with $\mathbb{M}_{24}$ as its automorphism group. Many of them have symmetries which are not even included in $\mathbb{M}_{24}$!

[Gaberdiel,Hohenegger,Volpato]

Symmetry surfing

Combine geometric symmetries from different points of $\mathcal{M}_{\mathsf{K3}}$ moduli space in the hope of pinning down the action of $\mathbb{M}_{24}$.

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Extensive work done on Kummer surfaces, namely the $T^4/\mathbb{Z}_2$ locus.

Our goal: understand geometric symmetries of K3 sigma models on $T^4/\mathbb{Z}_3$ subspace of K3 moduli space.

Thank You!