Summer School
Inclusive Paths in Explicit Number Theory

July 2-7, 2023

Zero-free regions close to the real axis

Habiba Kadiri
University of Lethbridge
habiba.kadiri@uleth.ca https://www.cs.uleth.ca/~kadiri/

Contents

Abstract: *The zero-free regions for the Riemann zeta function uses bounds for zeta at large enough heights. This is not feasible when we look at the case of other Dirichlet L-functions or Dedekind zeta functions. Instead, we must consider the existence of low-lying zeros, real ones, and even of an exceptional one close to 1. In this lecture, we will assume the audience to be familiar with classical proof for zero-free regions for zeta, and will focus on the techniques to establish regions close to the real line with at most one (or even a finite number of) zero(s).*

For the sake of improving these notes, please do not hesitate to ask for clarification, or to point out any typo or factual error.

1 Introduction

Theorem 1. *Let $L(s,\chi)$ be a Dirichlet character modulo q. Then there exists $R > 0$ such that $L(s,\chi)$ is non-vanishing in the region $s = \sigma + it$ where*

$$\sigma \geq 1 - \frac{1}{R\log q} \quad \text{and} \quad |t| \leq 1$$

with the exception of at most one simple zero in the case χ is a quadratic character.

Here we will prove Theorem 1 for $R = 35$ generalizing de la Vallée Poussin's method for the Riemann zeta function.

Riemann zeta function:

- Numerical verification of RH/GRH:
  Zeta for $|t| \leq 3 \cdot 10^{12}$: Platt, Trudgian [PT21] (2021),
  Dirichlet L-functions for $q \leq 400\,000$ and $|t| \leq \frac{10^8}{q}$: Platt [Pla16] (2016).
- Classic zero-free region for Riemann zeta function:
  There exists a constant $R > 0$ s.t. $\zeta(\sigma + it)$ does not vanish in

  $$\sigma \geq 1 - \frac{1}{R\log|t|} \quad \text{and } |t| \geq 2.$$

  by de la Vallée Poussin (1899).
  - Stechkin [Ste70] (1970) and Rosser and Schoenfeld [RS75] (1975): $R = 9.65$ (used "Stechkin's trick").
  - Kondratev [Ke77] (1977) $R = 9.55$ (used a degree 8 trigonometric polynomial).
  - Ford [For02b] (2002): $R = 8.43$ (consequence of Korobov-Vinogradov)
  - Kadiri [Kad05] (2005): $R = 5.71$ (smoothed ζ-function, Stechkin's trick)
  - Jang and Kwon [JK14] (2014): $R = 5.69$ (partial numerical verification of RH)
  - Mossinghoff and Trudgian [MT15] (2015): $R = 5.58$ (degree 16 trigonometric polynomial).
  - Mossinghoff, Trudgian and Yang [MTY23] (2022): $R = 5.56$ (consequence of Korobov-Vinogradov).
    Best region for $3 \cdot 10^{12} \leq |t| \leq e^{208.4}$.
- Littlewood's zero-free region for Riemann zeta function:
  - Littlewood (1922): : existence of constant $c > 0$ s.t. $\zeta(\sigma + it)$ does not vanish in $|t| \geq 3$ and

    $$\Re s \geq 1 - \frac{(\log\log|\Im s|)}{c(\log|\Im s|)}$$

(needs sub-convexity bounds: $\zeta(\sigma+it) \ll t^{\frac{1}{2^k-2}}(\log t)$ with $k \geq 4$).

- Yang [Yan23] (Arxiv January 2023): $c = 21.44$.
  Best region for $e^{208.4} \leq |t| \leq e^{511\,174}$.

- Korobov-Vinogradov zero-free region for Riemann zeta function:
  - Korobov-Vinogradov (1958): : existence of constant $r > 0$ s.t. $\zeta(\sigma+it)$ does not vanish in $|t| \geq 3$ and
  $$\sigma \geq 1 - \frac{1}{r(\log|t|)^{2/3}(\log\log|t|)^{1/3}}$$
  (needs sub-convexity bounds: $\zeta(\sigma+it) \leq A|t|^{B(1-\sigma)^{2/3}}$ for $1/2 \leq \sigma \leq 1$)
  - Ford [For02a] (2002, cor. 2022); $r = 57.54$
  - Mossinghoff, Trudgian and Yang [MTY23] (2022): $r = 55.241$.
  - Bellotti [Bel23] (Arxiv June 2023): $r = 54.004$.
    Best region for $|t| \geq e^{511\,174}$.

- A main difference between L-functions and $\zeta(s)$: its first zero occurs at $\Im\varrho \approx 14.1347$. On the other hand, Dirichlet L-functions can vanish as low as the real line. From the Generalized Riemann Hypothesis, it is expected that Dirichlet L-functions $L(s,\chi)$ do not vanish on $\frac{1}{2} < \Re s \leq 1$. It is actually also expected that $L(s,\chi)$ does not vanish at $s = \frac{1}{2}$ (Chowla's conjecture for quadratic characters) For more on the topic, see Conrey and Soundararajan's *Real zeros of quadratic Dirichlet L-functions* [**?**], Conrey, Iwaniec, Sound [CIS13], etc.

Dirichlet L-functions

- History of explicit versions of Theorem 1:
  - McCurley [McC84] proved $R = 9.65$ for Dirichlet L-functions, generalizing work of Stechkin (1970) [Ste70] and Rosser and Schoenfeld (1975)[RS75] about $\zeta(s)$.
  - I proved $R = 5.70$ in (Ph.D. thesis [Kad02], 2002).

- Assuming q is large enough, admissible values for R down to 2.88 (Heath-Brown, 1992) [HB92]), 2.75 (Liu and Wang, 1998) and then 2.28 (Xylouris, 2011).

- Korobov-Vinogradov zero-free region for Dirichlet L-functions and others:
  For all $q \geq 3$ and χ mod q, the Dirichlet L-function $L(\sigma+it,\chi)$ does not vanish in the region
  $$\sigma \geq 1 - \frac{1}{10.5\log q + 61.5(\log|t|)^{2/3}(\log\log|t|)^{1/3}}, \quad |t| \geq 10,$$
  (Khale [Kha22], Arxiv October 2022)
  See Coleman *A zero-free region for the Hecke L-function* (1990).

Application to primes in arithmetic progression

- *Explicit bounds for primes in arithmetic progressions* by Bennett, Martin, O'Bryant, Rechnitzer [BMOR18] (2018)
  Let $q \geq 10^5$. Then for all $x \geq \exp(4R(\log q)^2)$
  $$\left|\frac{\psi(x;q,a) - x/\varphi(q)}{x/\varphi(q)}\right| \leq \frac{1.02}{\phi(q)} x^{\beta_0} + 1.457x\sqrt{\frac{\log x}{R}}\exp\left(-\sqrt{\frac{\log x}{R}}\right),$$
  where β_0 term is present only if some Dirichlet L-function (mod$q$) has an exceptional zero β_0, R constant from zero-free region for Dirichlet L-functions: the smaller R, the sharper the bound for $\psi(x;q,a)$.

- Linnik's theorem (1944):
  There exists an absolute constant $A > 0$, s.t. for any arithmetic progression a mod q, there exists a prime $p \equiv a$ mod q with
  $$P(a,q) \ll q^A.$$
  For q sufficiently large: $A = 5.5$ by Heath-Brown [HB92] *Zero-free regions for Dirichlet L-functions and the least prime in an arithmetic progression* (1992)
  $A = 5.2$ (Xylouris [Xyl11], 2009)
  For all moduli q: $P(a,q) \leq eq^{7(\log q)}$ (Kadiri, 2008).

- Languasco and Zaccagnini [LZ07] (2007) *A note on Mertens' formula for arithmetic progressions*

Dedekind ζ-functions

- $\zeta_L(s)$ vanishes at most at the "exceptional zero" ϱ_0 in the region
  $$\Re s > 1 - \frac{1}{c\log d_L} \text{ and } |\Im s| < \frac{1}{c\log d_L},$$
  For d_L is sufficiently large: Stark [Sta74] (1974) $c = 4$.
  For all $L \neq \mathbb{Q}$: Ahn and Kwon [AK19] (2019) $c = 2$, Kadiri and Wong [KW22] (2021) $c = 1.7$.

- $\zeta_L(s)$ vanishes at most at the "exceptional zero" ϱ_0 in the region
  $$\Re s > 1 - \frac{1}{r\log d_L} \text{ and } |\Im s| \leq 1.$$
  For d_L is sufficiently large: Kadiri [Kad12] (2012) $r = 12.8$, Lee [Lee21] (2021) $r = 12.5$.
  For all $L \neq \mathbb{Q}$: Ahn and Kwon [AK19] (2019) $r = 29.6$.

- $\zeta_L(s)$ does not vanish in
  $$\Re s > 1 - \frac{[}{c_1\log d_L + c_2 n_L \log|\Im s| + c_3 n_L + c_4}.$$
  For d_L is sufficiently large: Kadiri [Kad12] (2012) $c_1 = 12.6, c_2 = 9.7, c_3 = 3.1, c_4 = 58.7$, and Lee [Lee21] (2021) $c_1 = 12.3, c_2 = 9.6, c_3 = 0.1, c_4 = 2.3$.

The case $|t| \leq 1$ is important inexplicit bounds for $\pi(x;q,a)$ and also in the proof of Linnik's theorem.

2 Explicit Inequalities

A so-called explicit inequality essentially relates an L-function (more specifically the primes associated to it) with the pole(s) and zeros of this L-function.

2.1 Notation

Let $q \geq 3$, χ is a character modulo q, and χ' the primitive character modulo q' inducing χ. We recall $q' \mid q$. We assume $\sigma > 1$ and propose to express $-\frac{\zeta'}{\zeta}(s)$ and $-\frac{L'}{L}(s,\chi)$ in terms of its singularities (i.e. poles and zeros of $\zeta(s)$ and $L(s,\chi)$). We denote χ_0 the principal character modulo q: $\chi^0(n) = \chi_0(n) = 1$ if $(n,q) = 1$ and $\chi^0(n) = \chi_0(n) = 0$ if $(n,q) > 1$.

2.2 A "global" formula

From Davenport [Dav00, Chapter 12, Equations (8) and (17)], we have the explicit identities for the logarithmic derivative of the zeta and the Dirichlet L-function $L(s,\chi)$, where χ is a primitive character modulo q:

$$-\frac{\zeta'}{\zeta}(s) = \frac{1}{s-1} - \frac{1}{2}\log\pi + \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s+2}{2}\right) - B - \sum_{\zeta(\varrho)=0}\left(\frac{1}{s-\varrho} + \frac{1}{\varrho}\right), \tag{1}$$

$$-\frac{L'}{L}(s,\chi) = \frac{1}{2}\log\left(\frac{q}{\pi}\right) + \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s+\mathfrak{a}}{2}\right) - B(\chi) - \sum_{L(\varrho,\chi)=0}\left(\frac{1}{s-\varrho} + \frac{1}{\varrho}\right), \tag{2}$$

where the sums are over the non-trivial zeros of respectively $\zeta(s)$ and $L(s,\chi)$, and where

$$\mathfrak{a} = \begin{cases} 0 & \text{if } \chi(-1) = 1, (\chi \text{ is even}), \\ 1 & \text{if } \chi(-1) = -1, (\chi \text{ is odd}), \\ 2 & \text{if } \chi \text{ is principal}). \end{cases} \tag{3}$$

Here, recall that

$$\Re B = -\Re \sum_{\zeta(\varrho)=0} \frac{1}{\varrho}, \text{ and } \Re B(\chi) = -\Re \sum_{L(\varrho,\chi)=0} \frac{1}{\varrho}. \tag{4}$$

So by taking the real parts and using (4), we have

$$-\Re\frac{\zeta'}{\zeta}(s) = \Re\left(\frac{1}{s-1}\right) - \frac{1}{2}\log\pi + \frac{1}{2}\Re\frac{\Gamma'}{\Gamma}\left(\frac{s+2}{2}\right) - \sum_{\zeta(\varrho)=0}\Re\left(\frac{1}{s-\varrho}\right), \tag{5}$$

$$\text{and } -\Re\frac{L'}{L}(s,\chi) = \frac{1}{2}\log\left(\frac{q}{\pi}\right) + \frac{1}{2}\Re\frac{\Gamma'}{\Gamma}\left(\frac{s+\mathfrak{a}}{2}\right) - \sum_{L(\varrho,\chi)=0}\Re\left(\frac{1}{s-\varrho}\right). \tag{6}$$

Remark 1. *Both sums over the zeros are non-negative for $\sigma > 1 > \Re\varrho$ (for any zero ϱ).*

2.3 Stechkin explicit inequalities

Adopting McCurley [McC84]'s notation: Consider the difference

$$f(t,\chi) = f(t,\chi,\sigma) = -\Re\left(\frac{L'}{L}(s+it,\chi) - \kappa\frac{L'}{L}(s_1+it,\chi)\right), \tag{7}$$

where $s = \sigma + it$, with $1 < \sigma < 1.15, t \geq 0$, and $s_1 = \sigma_1 + it$ and κ are given by

$$\sigma_1 = \frac{1}{2} + \frac{1}{2}\sqrt{1+4\sigma^2}, \tag{8}$$

$$\kappa = \frac{1}{\sqrt{5}} \approx 0.4472. \tag{9}$$

In addition we define

$$\kappa' = \frac{1-\kappa}{2} = \frac{1-\frac{1}{\sqrt{5}}}{2} \approx 0.2764.$$

Now the sum over the zeros is of the shape

$$\Re\left(\frac{1}{s-\varrho} - \frac{\kappa}{s_1-\varrho}\right),$$

Stechkin's Lemma [Ste70] insures that for s_1 and κ chosen above, this remains non-negative:

Lemma 2. *Let* $\sigma > 1, \sigma_1 = \frac{1}{2}+\frac{1}{2}\sqrt{1+4\sigma^2}, s = \sigma+it, s_1 = \sigma_1+it$. *Then, for all* $0 \leq \Re z < 1$,

$$\Re\left(\frac{1}{s-z} + \frac{1}{s-1+\bar{z}}\right) - \kappa\Re\left(\frac{1}{s_1-z} + \frac{1}{s_1-1+\bar{z}}\right) \geq 0.$$

2.4 Bounding the Γ-terms:

We recall Stirling formulas (see [Dav00, Chapter 10]): for any $\epsilon > 0$,

$$\log\Gamma(z) = (z - \frac{1}{2})\log z - z + \log\sqrt{2\pi} + \mathcal{O}(|z|^{-1})$$

for all $|z| \geq 1$ and $|\arg z| \leq \pi - \epsilon$.
In addition, (by using a Cauchy's integral formula for $\frac{\Gamma'}{\Gamma}(z) = \frac{\partial}{\partial z}\log\Gamma(z)$), we have

$$\frac{\Gamma'}{\Gamma}(z) = \log z - \frac{1}{2z} + \mathcal{O}(|z|^{-2}). \tag{10}$$

The following bound (see [**?**, Equation (4) page 113]) gives an explicit version of (10).

Lemma 3. *Let $z = x + it$ with $x \geq 0$ and $t \neq 0$. Then*

$$\frac{\Gamma'}{\Gamma}(z) = \log z + \frac{1}{2z} + \mathcal{O}^\star\left(\frac{1}{4t^2}\right).$$

Note that, for $\sigma \geq 1, |t| \leq 4$,

$$\frac{\Gamma'}{\Gamma}(\sigma + it) = \mathcal{O}(1),$$

which the following makes explicit:

Problem 1. *Using*

$$\left|\frac{\Gamma'}{\Gamma}(s) - \log(s) + \frac{1}{2s}\right| \leq \frac{1}{12|s^2|} + \frac{1}{6}\int_0^\infty \frac{dx}{((\sigma + x)^2 + t^2)^{\frac{3}{2}}} \tag{11}$$

prove that, for $1 < \sigma < 1.15$ and $|t| \leq 4$, $\mathfrak{a} = 0, 1, 2$, then

$$\left|\Re\frac{\Gamma'}{\Gamma}\left(\frac{s+\mathfrak{a}}{2}\right)\right| \leq 2.$$

Solution *The following bound (see [**?**, Equation (4) page 113]) give an explicit version of* (10).

$$\left|\frac{\Gamma'}{\Gamma}(s) - \log(s) + \frac{1}{2s}\right| \leq \frac{1}{12|s^2|} + \frac{1}{6}\int_0^\infty \frac{dx}{((\sigma + x)^2 + t^2)^{\frac{3}{2}}} \tag{12}$$

Let $s = \sigma + it$ with $|t| \leq 1$. Then

$$\left|\frac{\Gamma'}{\Gamma}(s) - \log(s) + \frac{1}{2s}\right| \leq \frac{1}{12\sigma^2} + \frac{1}{6}\int_0^\infty \frac{dx}{(\sigma + x)^3} = \frac{1}{12\sigma^2} + \frac{1}{6}(-1)\frac{1}{2}(\sigma + x)^{-2}\Big|_0^\infty = \frac{1}{6\sigma^2}. \tag{13}$$

Replacing s by $\frac{s+\mathfrak{a}}{2}$ this becomes

$$\left|\frac{\Gamma'}{\Gamma}\left(\frac{s+\mathfrak{a}}{2}\right) - \log\left(\frac{s+\mathfrak{a}}{2}\right) + \frac{1}{2(\frac{s+\mathfrak{a}}{2})}\right| \leq \frac{2}{3(\sigma + \mathfrak{a})^2}. \tag{14}$$

It follows that

$$\begin{aligned}
\Re \frac{\Gamma'}{\Gamma}\Big(\frac{s+\mathfrak{a}}{2}\Big) &= \Re \log\Big(\frac{s+\mathfrak{a}}{2}\Big) + \mathcal{O}^*\Big(\frac{1}{\sigma+\mathfrak{a}} + \frac{2}{3(\sigma+\mathfrak{a})^2}\Big) \\
&= \ln\Big|\frac{\sigma+\mathfrak{a}}{2} + i\frac{t}{2}\Big| + \mathcal{O}^*\Big(\frac{1}{\sigma+\mathfrak{a}} + \frac{2}{3(\sigma+\mathfrak{a})^2}\Big) \\
&= \frac{1}{2}\ln\Big|\Big(\frac{\sigma+\mathfrak{a}}{2}\Big)^2 + \Big(\frac{t}{2}\Big)^2\Big| + \mathcal{O}^*\Big(\frac{1}{\sigma+\mathfrak{a}} + \frac{2}{3(\sigma+\mathfrak{a})^2}\Big) \\
&\leq \frac{1}{2}\ln\Big|\Big(\frac{\sigma+\mathfrak{a}}{2}\Big)^2 + 4\Big| + \mathcal{O}^*\Big(\frac{1}{\sigma+\mathfrak{a}} + \frac{2}{3(\sigma+\mathfrak{a})^2}\Big).
\end{aligned}$$

Let

$$g_a(\sigma) := \frac{1}{2}\ln\Big|\Big(\frac{\sigma+\mathfrak{a}}{2}\Big)^2 + 4\Big| + \frac{1}{\sigma+\mathfrak{a}} + \frac{2}{3(\sigma+\mathfrak{a})^2}$$

We conclude by verifying this is ≤ 3 *under our conditions.*

We have that [McC84, Lemma 1] and [McC84, Lemma 2] give bounds for

$$\frac{1}{2}\Re\left(\frac{\Gamma'}{\Gamma}(\frac{s+\mathfrak{a}}{2}) - \kappa\frac{\Gamma'}{\Gamma}(\frac{s_1+\mathfrak{a}}{2})\right)$$

for $|t| < 1$ and $|t| \geq 1$ respectively.

Lemma 4. *Let* $1 < \sigma < 1.15$, $|t| < 1$, $1 \leq k \leq 4$, $\mathfrak{a} = 0, 1,$ *or* 2, $s = \sigma + ikt$ *and* $s_1 = \sigma_1 + ikt$. *We recall*

$$\kappa = \frac{1}{\sqrt{5}} \approx 0.4472, \ \kappa' = \frac{1-\frac{1}{\sqrt{5}}}{2} \approx 0.2764.$$

Then

$$\frac{1}{2}\Re\left(\frac{\Gamma'}{\Gamma}(\frac{s+\mathfrak{a}}{2}) - \kappa\frac{\Gamma'}{\Gamma}(\frac{s_1+\mathfrak{a}}{2})\right) \leq \begin{cases} \kappa'\log|t| + c(\mathfrak{a},k) & \text{if } |t| \geq 1, \\ d(\mathfrak{a},k) & \text{if } |t| < 1. \end{cases}$$

Admissible values for $c(\mathfrak{a}, k)$ and $d(\mathfrak{a}, k)$ are given in the table:

Table 1: Values for $c(\mathfrak{a}, k)$

m	$\mathfrak{a} = 0$ or 1	$\mathfrak{a} = 2$
1	0.3918	0.3316
2	0.3915	0.3530
3	0.4062	0.3780
4	0.4266	0.4080

Table 2: Values for $d(\mathfrak{a}, k)$

m	$\mathfrak{a} = 0$ or 1	$\mathfrak{a} = 2$
1	−0.0390	0.0615
2	0.2469	0.1565
3	0.4452	0.2638
4	0.5842	0.3636

Problem 2. *Let* $1 < \sigma < 1.15, 0 \leq t < 1$, *and* χ *a primitive character modulo* q. *Prove the explicit formulas:*

$$-\Re\frac{L'}{L}(s+it,\chi_0) + \kappa\Re\frac{L'}{L}(s_1+it,\chi_0) < \frac{\sigma-1}{(\sigma-1)^2+t^2} + d(2,1) - \kappa'\log\pi + s_0(q), \tag{15}$$

$$-\Re\frac{L'}{L}(s+it,\chi) + \kappa\Re\frac{L'}{L}(s_1+it,\chi) < \kappa'\log q + d(1,1) - \kappa'\log\pi, \tag{16}$$

with $\kappa' \approx 0.2764$.

2.5 A "local" formula

In [HB92, Lemma 3.1.], Heath-Brown proves a Jensen type formula relating L-function to its singularities inside a small disc around a point to close to the vertical 1-line. The sub-convexity bound used for $L(s,\chi)$ is proven from some Burgess bounds for character sums (see [HB92, Lemma 2.5.]):
For any integer $k \geq 3$ and any $\epsilon > 0$

$$L(\sigma + it) \ll_{\epsilon,k} q^{\phi(1-\sigma)(1+\frac{1}{k})+\epsilon}(1 + |t|)$$

uniformly for $1 - \frac{1}{k} \leq \sigma \leq 1 + \frac{\log\log q}{\log q}$.
Here ϕ is a constant defined as

$$\phi = \begin{cases} \frac{1}{4} & \text{if } q \text{ is cube-free or the order of } \chi \text{ is at most } \log q, \\ \frac{1}{3} & \text{otherwise.} \end{cases} \tag{17}$$

Lemma 5. *Let χ be a non-principal character modulo q and let ϕ be defined as in (17). Then for every $\epsilon > 0$, there exists a $\delta = \delta(\epsilon)$ such that*

$$-\Re\frac{L'}{L}(s,\chi) \leq -\sum_{|1+it-\varrho|\leq\delta} \Re\frac{1}{s-\varrho} + (\frac{\phi}{2} + \epsilon)(\log q)$$

uniformly for $1 + \frac{1}{(\log q)(\log\log q)} \leq \sigma \leq 1 + \frac{\log\log q}{\log q}$ and $|t| \leq \log q$, providing that q sufficiently large.

Here $\phi = \frac{1}{8} = 0.125$ or $\frac{1}{6} \approx 0.167$.

Remark 2. *The the factor of $\log q$ directly determines the size of the zero-free region (smaller factor gives larger region). To date there is no version of Heath-Brown's explicit formula valid for all $q \geq 3$ (and thus zero-free region).*

2.6 Handling the principal and non primitive characters

2.6.1 From $L(s,\chi_0)$ to $\zeta(s)$

If χ_0 is the principal character modulo q, then

Lemma 6. *Let* $\sigma \geq 1$.

$$\left|\Re \frac{L'}{L}(s+it,\chi_0) - \Re \frac{\zeta'}{\zeta}(\sigma+it)\right| \leq s_0(q) \tag{18}$$

where $s_0(q)$ *is defined (as in [McC84, page 10]):*

$$s_0(q) = \sum_{p|q} \frac{\log p}{p^\sigma - 1}. \tag{19}$$

Problem 3. *Prove*

$$s_0(q) \leq \begin{cases} 2(\log\log q + 1) & \text{if } \sigma \geq 1, \\ \frac{6\cdot 2^\sigma}{2^\sigma - 1}(\log q)^{1-\sigma} & \text{if } 3/4 < \sigma < 1. \end{cases} \tag{20}$$

Solution *Consider the sum*

$$T(\sigma) = \sum_{p|q} \frac{\log p}{p^\sigma}$$

where $\sigma > 0$. *Let* $2 \leq x \leq q$ *be a parameter. Note that*

$$T(\sigma) = \sum_{\substack{p|q \\ p \leq x}} \frac{\log p}{p^\sigma} + \sum_{\substack{p|q \\ p > x}} \frac{\log p}{p^\sigma}.$$

Observe that

$$\sum_{\substack{p|q \\ p > x}} \frac{\log p}{p^\sigma} \leq \frac{1}{x^\sigma} \sum_{\substack{p|q \\ p > x}} \log p \leq \frac{\log q}{x^\sigma}$$

and thus

$$T(\sigma) \leq \sum_{p \leq x} \frac{\log p}{p^\sigma} + \frac{\log q}{x^\sigma}. \tag{21}$$

Let us split in two cases.
Case 1. $\sigma = 1$. *In this case we recall a result of Rosser and Schoenfeld* [**?**]

$$\sum_{p \leq x} \frac{\log p}{p} \leq \log x \text{ if } x > 1.$$

It follows that

$$T(1) \leq \log x + \frac{\log q}{x}$$

and choosing $x = \log q$

$$T(1) \leq \log\log q + 1.$$

Case 2. $0 < \sigma < 1$. *In this case we recall a result of Broadbent et al. [BKL+21]*

$$\vartheta(x) \le c_0 x \text{ if } x > 1$$

where

$$c_0 = 1 + 1.93378 \cdot 10^{-8}.$$

Here we apply partial summation to obtain

$$\begin{aligned}
\sum_{p \le x} \frac{\log p}{p^\sigma} &= \frac{\vartheta(x)}{x^\sigma} + \sigma \int_{\frac{3}{2}}^{x} \frac{\vartheta(t)}{t^{\sigma+1}}\, dt \\
&\le \frac{c_0 x}{x^\sigma} + c_0\sigma \int_{\frac{3}{2}}^{x} \frac{t}{t^{\sigma+1}}\, dt \\
&= c_0 x^{1-\sigma} + c_0\sigma \int_{\frac{3}{2}}^{x} t^{-\sigma}\, dt \\
&= c_0 x^{1-\sigma} + \frac{c_0\sigma}{1-\sigma} t^{1-\sigma}\Big|_{\frac{3}{2}}^{x} \\
&\le c_0 x^{1-\sigma} + \frac{c_0\sigma}{1-\sigma} x^{1-\sigma} \\
&= c_0 x^{1-\sigma}\Big(1 + \frac{\sigma}{1-\sigma}\Big) \\
&= \frac{c_0}{1-\sigma} x^{1-\sigma}.
\end{aligned}$$

It follows from (21) *that*

$$T(\sigma) \le \frac{c_0}{1-\sigma} x^{1-\sigma} + \frac{\log q}{x^\sigma}.$$

Setting $x = \log q$ *yields*

$$T(\sigma) \le \Big(\frac{c_0}{1-\sigma} + 1\Big)(\log q)^{1-\sigma}.$$

With $\sigma = \frac{3}{4}$, *it follows that*

$$T(\tfrac{3}{4}) \le (4c_0 + 1)(\log q)^{1-\sigma} \le 6(\log q)^{1-\sigma}.$$

Thus we obtain

$$\sum_{p|q} \frac{\log p}{p^\sigma} \le \begin{cases} \log\log q + 1 \text{ if } \sigma = 1, \\ 6(\log q)^{1-\sigma} \text{ if } 0 < \sigma < 1. \end{cases} \tag{22}$$

2.6.2 From imprimitive to primitive character

Lemma 7. *Let χ' the primitive character inducing χ modulo q'. Let $\frac{1}{2} \le \sigma \le 2, |t| \le 1$. Then*

$$\left|\frac{L'}{L}(s,\chi) - \frac{L'}{L}(s,\chi')\right| \le s_0(q/q'). \tag{23}$$

For the proof, note that

$$L(s,\chi) = L(s,\chi') \prod_{p|q}(1 - \chi'(p)p^{-s})$$

valid for all s which implies

$$\frac{L'}{L}(s,\chi) = \frac{L'}{L}(s,\chi') + \sum_{p|q} \frac{\chi'(p)\log p}{p^s - \chi'(p)}.$$

2.7 Explicit bounds for $-\Re\frac{L'}{L}(\sigma+it,\chi_0)$ and $-\Re\frac{L'}{L}(\sigma+it,\chi)$

Note that taking $\sigma > 1$ makes the sum over the zeros $\varrho = \beta + i\gamma$ non-negative:

$$\Re\Big(\frac{1}{s-\varrho}\Big) = \frac{\sigma-\beta}{(\sigma-\beta)^2+(t-\gamma)^2} \geq 0 \ \text{ since } \ \sigma > 1 \geq \beta. \tag{24}$$

Explicit formulas as produced in the previous chapter establish that for $\sigma \geq 1$ and $|t| \leq 1$ give

$$-\Re\frac{\zeta'}{\zeta}(\sigma+it) = \Re\frac{1}{\sigma+it-1} + O(1), \ -\Re\frac{\zeta'}{\zeta}(\sigma) = \frac{1}{\sigma-1} + O(1), \tag{25}$$

Let $\sigma > 1$, $0 \leq t < \log q$. Let $q \geq 3$ and χ_0 the principal character modulo q. Then

$$-\Re\frac{L'}{L}(\sigma+it,\chi_0) \leq \Re\Big(\frac{1}{\sigma+it-1}\Big) + o(\log q). \tag{26}$$

In addition, if χ is a non-principal character modulo q, then

$$-\Re\frac{L'}{L}(\sigma+it,\chi) \leq \frac{A}{2}\log q - \sum_{\varrho\in Z_A(\chi)} \frac{\sigma-\beta}{(\sigma-\beta)^2+(t-\gamma)^2} + o(\log q), \tag{27}$$

where $Z_A(\chi)$ is the set of all the non-trivial zeros $\varrho = \beta + i\gamma$ of $L(s,\chi)$ satisfying

$$\begin{cases} 0 \leq \beta \leq 1 & \text{if } A/2 = 0.5 \text{ (for all } q\text{)},\\ 0 \leq \beta \leq 1 & \text{if } A/2 = \kappa'/2 \approx 0.27 \text{ (Stechkin, for all } q\text{)},\\ |1+it-\varrho| \leq \delta_\epsilon & \text{if } A/2 = (\phi/2+\epsilon) \leq 0.25 \text{ (Heath-Brown, for } q \text{ sufficiently large).} \end{cases} \tag{28}$$

2.8 Introducing non-negative trigonometric polynomials

Zero-free regions proofs all rely on the use of a non-negative cosine polynomial:

$$P(\theta) = \sum_{k=0}^{m} a_k \cos(k\theta) \geq 0 \text{ with all coefficients } \; a_k > 0. \tag{29}$$

For instance the trigonometric polynomial

$$P(\theta) = 3 + 4\cos\theta + \cos(2\theta) = 2(1+\cos\theta)^2 \tag{30}$$

origins in the work of de la Valée Poussin [dlVP99] for zeta and can be used to more general L-functions.

Lemma 8. *Let χ be a Dirichlet characters modulo q and let $s = \sigma + it$ with $\sigma > 1$. Then*

$$a_0 \Re\left(-\frac{L'}{L}(\sigma + ik\gamma, \chi_0)\right) + \sum_{k=1}^{m} a_k \Re\left(-\frac{L'}{L}(\sigma + ik\gamma, \chi^k)\right) \geq 0. \tag{31}$$

In particular, for the polynomial as defined in (30), we have

$$3\Re\left(-\frac{L'}{L}(\sigma, \chi_0)\right) + 4\Re\left(-\frac{L'}{L}(\sigma + it, \chi)\right) + \Re\left(-\frac{L'}{L}(\sigma + 2it, \chi^2)\right) \geq 0. \tag{32}$$

Problem 4. *Prove Lemma 8.*
Prove McCurley's version:

$$a_0 f(0, \chi_0) + \sum_{k=1}^{m} a_k f(k\gamma, \chi^k) \geq 0, \tag{33}$$

where we denote

$$f(t, \chi) = -\Re\left(\frac{L'}{L}(s + it, \chi) - \kappa \frac{L'}{L}(s_1 + it, \chi)\right)$$

Hints: Replace θ with $\arg\left(\frac{\chi(n))}{n^{it}}\right)$. Note $1 - \frac{\kappa}{n^{\sigma_1 - \sigma}} \geq 0$.

2.9 Examples

Here are some of the polynomials used

Author	Non-negative trigonometric polynomial $P(\theta)$	Coefficients
Riemann zeta $\zeta(s)$ de la Vallée Poussin	$2(1 + \cos\theta)^2$	$a_0 = 3$ $a_1 = 4$ $a_2 = 1$
Dirichlet L-functions McCurley (1984)	$8(0.9126 + \cos\theta)^2(0.2766 + \cos\theta)^2$	$a_0 = 11.1859355312082048$ $a_1 = 19.073344004352$ $a_2 = 11.67618784$ $a_3 = 4.7568$ $a_4 = 1$
Riemann zeta $\zeta(s)$ Mossinghof-Trudgian	$c_0 = 1$ $c_1 = -2.09100370089199$ $c_2 = 0.414661861733616$ $c_3 = 4.94973437766435$ $c_4 = 2.26052224951171$ $c_5 = 8.58599241204357$ $c_6 = 6.87053689828658$ $c_7 = 22.6412990090005$ $c_8 = 6.76222005424994$ $c_9 = 50.2233943767588$ $c_{10} = 8.07550113395201$ $c_{11} = 223.771572768515$ $c_{12} = 487.278135806977$ $c_{13} = 597.268928658734$ $c_{14} = 473.937203439807$ $c_{15} = 237.271715181426$ $c_{16} = 59.6961898512813$	$a_0 = 1$ $a_1 = 1.74126664022806$ $a_2 = 1.128282822804652$ $a_3 = 0.5065272432186642$ $a_4 = 0.1253566902628852$ $a_5 = 9.35696526707405 \cdot 10^{-13}$ $a_6 = 4.546614790384321 \cdot 10^{-13}$ $a_7 = 0.01201214561729989$ $a_8 = 0.006875849760911001$ $a_9 = 7.77030543093611 \cdot 10^{-12}$ $a_{10} = 2.846662294985367 \cdot 10^{-7}$ $a_{11} = 0.001608306592372963$ $a_{12} = 0.001017994683287104$ $a_{13} = 2.838909054508971 \cdot 10^{-7}$ $a_{14} = 5.482482041999887 \cdot 10^{-6}$ $a_{15} = 2.412958794855076 \cdot 10^{-4}$ $a_{16} = 1.281001290654868 \cdot 10^{-4}$

In the last example, $P(\theta) = \left(\sum_{k=0}^{16} c_k e^{ik\theta}\right)^2 = \sum_{k=0}^{16} a_k \cos(k\theta)$.

2.10 A smoothing argument

Consider a "smoothed version" of $-\Re \frac{L'}{L}(s,\chi)$:

$$\Re \sum_{n\geq 1} \frac{\Lambda(n)\chi(n)f(\log n)\left(1-\frac{\kappa}{n^{\delta}}\right)}{n^s}, \tag{34}$$

We establish a version of explicit formula of the form

$$\Re \sum_{n\geq 1} \frac{\Lambda(n)\chi(n)f(\log n)\left(1-\frac{\kappa}{n^{\delta}}\right)}{n^s} = \frac{(1-\kappa)}{2} f(0)\log(q|\Im s|) - \sum_{\varrho\in Z(\chi)} \Re\left(F(s-\varrho)-F(s+\delta-\varrho)\right) + E_q(s), \tag{35}$$

where F is the Laplace transform of f, χ is non-principal, $Z(\chi)$ is the set of non-trivial zeros of $L(s,\chi)$, and $E_q(s)$ is an error term.
In addition, when χ is principal, the term $a_0 \Re F(s-1)$ arises for $k=0$ from the pole of $\zeta(s)$ at $s=1$.
To compare with the classical proof, κ and δ would each be 0, f would be identically 1, $\Re F(s-1)$ would be $\frac{1}{\Re s-1}$, and $-\sum_{\varrho\in Z(\chi)} \Re \frac{1}{s-\varrho}$ would be the sum over the zeros. We compare (34) for various values of s on a vertical line passing near ϱ_0 by means of a trigonometric inequality of the form

$$P(t) = \sum_{k=0}^{n_0} a_k \cos(kt) \geq 0 \;\text{ with }\; a_k \geq 0 \;\text{ for all }\; k = 0,\ldots,n_0.$$

We deduce

$$\sum_{n\geq 1} \frac{\Lambda(n)f(\log n)\left(1-\frac{\kappa}{n^{\delta}}\right)}{n^{\sigma}} \sum_{k=0}^{n_0} a_k \cos\left(k \arg\left(\frac{\chi(n)}{n^{i_0}}\right)\right) \geq 0.$$

It remains to give accurate upper bounds to the right hand side of (35) for $s = \sigma + ik_0$, $k = 0,\ldots,n_0$.

$$\frac{1-\kappa}{2} f(0)\log(q) \sum_{k=1}^{n_0} a_k - a_1 F(\sigma-\beta_0) + a_0 F(\sigma-1) + \epsilon \geq 0, \tag{36}$$

where ϵ is an error term. We choose f to depend on β_0 by setting $f(0) = h(0)(1-\beta_0)$, where $h(0)$ is independent of ϱ_0 and h is a smooth function chosen appropriately. We also choose the polynomial coefficients a_i, and the parameter σ. Then the inequality

$$(1-\beta_0)\log(q) \geq \frac{a_1 F(\sigma-\beta_0) - a_0 F(\sigma-1) - \epsilon}{\frac{1-\kappa}{2} h(0) \sum_{k=1}^{n_0} a_k} \tag{37}$$

provides a formula where the zero-free constant R^{-1} is given by the right term.
This replaces the classical proof's conclusion

$$(1-\beta_0)\log(q) \geq \frac{\frac{a_1}{\sigma-\beta_0} - \frac{a_0}{\sigma-1}}{\frac{1}{2}(1+o(1))(a_1+\ldots+a_m)}.$$

Advantage of smoothing method: one can take $\sigma = 1$ (even $\sigma < 1$).

3 Zero-free region for $L(s,\chi)$ when $0 \leq \gamma < 1$

Let $q \geq 3$ and χ a primitive character modulo q. Then there is at most one zero of $L(s,\chi)$ in the region

$$\Re s \geq 1 - \frac{1}{R\log q} \text{ and } |\Im s| \leq 1,$$

This zero if it exists is real and arises from a real character.
The goal of this section is to prove this theorem with $R = 35$.

Let $q \geq 3$. Assume χ is a primitive character modulo q and consider ϱ_0 be a non-trivial zero of $L(s,\chi)$:

$$\varrho_0 = \beta_0 + i\gamma_0 \ \text{ with } \ 1/2 \leq \beta_0 < 1 \ \text{ and } \ 0 \leq \gamma_0 < 1.$$

Note that we can assume $\gamma_0 \geq 0$, as the zeros of $L(s,\chi)$ with $\gamma_0 \leq 0$ are the complex conjugates of the zeros of $L(s,\overline{\chi})$ with $\gamma_0 > 0$, since $L(\overline{s},\chi) = \overline{L(s,\overline{\chi})}$. We also introduce the parameter σ and consider the following points just on the right of the vertical 1-line:

$$\sigma + ik\gamma_0 \ \text{ with } \ \sigma > 1 \ \text{ and } \ k = 0, 1, \ldots, m.$$

Remark 3. *Note that for $k = 1$, $\sigma + i\gamma_0$ is close to the zero $\varrho_0 = \beta_0 + i\gamma_0$.*
We present here the classical proof using the explicit formulas (26) *and* (27) *with $A/2 = 1/2$:*

$$- \Re\frac{L'}{L}(\sigma + it, \chi_0) \leq \Re\Big(\frac{1}{\sigma + it - 1}\Big) + o(\log q).$$
$$- \Re\frac{L'}{L}(\sigma + it, \chi) \leq \frac{1}{2}\log q - \sum_{L(\varrho,\chi)=0} \frac{\sigma - \beta}{(\sigma-\beta)^2 + (t-\gamma)^2} + o(\log q),$$

To estimate the sum over the zeros $-\sum_{\varrho\in Z_A(\chi)} \frac{\sigma-\beta}{(\sigma-\beta)^2+(t-\gamma)^2}$, we isolate the largest terms, ie when $\sigma+it$ is the closest to a zero of $L(s,\chi)$ and bound the others by 0.

3.1 When $0 \leq \gamma < 1$ and χ complex

We fix $s = \sigma + i\gamma_0$ with $\sigma > 1$. We recall bounds (26) and(27). In addition, for $k = 1$, we isolate the zero $\varrho = \varrho_0$ in the sum over the zeros, and use that the rest of the sum is non-negative:

$$-\sum_{\varrho\in Z(\chi)} \frac{\sigma-\beta}{(\sigma-\beta)^2+(t-\gamma)^2} \leq -\frac{1}{\sigma-\beta_0} \text{ and } -\sum_{\varrho\in Z(\chi_2)} \frac{\sigma-\beta}{(\sigma-\beta)^2+(t-\gamma)^2} \leq 0.$$

We recall that here χ_2 is the primitive character associated to χ^2 modulo $q_2 \mid q$ (so $\log q_2 \leq \log q$). Thus

$$-\Re\frac{L'}{L}(\sigma,\chi_0) \leq \frac{1}{\sigma-1} + o(\log q). \tag{38}$$

$$-\Re\frac{L'}{L}(\sigma+i\gamma_0,\chi) \leq \frac{1}{2}\log q - \frac{1}{\sigma-\beta_0} + o(\log q), \tag{39}$$

$$-\Re\frac{L'}{L}(\sigma+2i\gamma_0,\chi^2) \leq \frac{1}{2}\log q + o(\log q), \tag{40}$$

Together with Lemma 8;

$$3\left(-\frac{L'}{L}(\sigma,\chi_0)\right) + 4\left(-\frac{L'}{L}(\sigma+it,\chi)\right) + \left(-\frac{L'}{L}(\sigma+2it,\chi^2)\right) \geq 0,$$

we obtain

$$\frac{3}{\sigma-1} - \frac{4}{\sigma-\beta_0} + \frac{(4+1)}{2}\log q + o(\log q) \geq 0 \tag{41}$$

Taking $(\sigma-1) = x(1-\beta_0)$, we have

$$\frac{3}{x} - \frac{4}{(x+1)} + \frac{5}{2}(1-\beta_0)\log q + o((1-\beta_0)\log q) \geq 0,$$

i.e.

$$\frac{3}{x} - \frac{4}{(x+1)} + \left(\frac{5}{2} + o(1)\right)(1-\beta_0)(\log q) \geq 0$$

i.e.

$$(1-\beta_0)(\log q) \geq \frac{\frac{4}{(x+1)} - \frac{3}{x}}{\frac{5}{2} + o(1)}$$

Problem 5. *Prove that an optimal choice for x is $x = 3 + 2\sqrt{3}$ and deduce the zero free region for $L(s,\chi)$:*

$$(1-\beta_0)\log q \geq \frac{1}{(7+4\sqrt{3})(\frac{5}{2}+o(1))} \quad \textit{with} \quad \frac{1}{(7+4\sqrt{3})\frac{5}{2}} \approx \frac{1}{34.82}$$

Remark 4. *Note the role of the trigonometric polynomial: R is obtained by optimizing* $\frac{\frac{5}{2}}{\frac{4}{x+1}-\frac{3}{x}}$, *or more generally*

$$\frac{\frac{(a_1+a_2+...+a_m)}{2}}{\frac{a_1}{x+1}-\frac{a_0}{x}}.$$

So in addition to the conditions of positivity on the coefficients a_k *and on the trigonometric polynomial, we add that* $a_1 + a_2 + \ldots + a_m$ *is as small as possible while* $a_1 > a_0$.

Remark 5. *This achieves the proof that for any* $q \geq 3$, *and any* **complex** *primitive Dirichlet character* χ *modulo* q, $L(s,\chi)$ *has no zero in the region*

$$\beta \geq 1 - \frac{1}{35 \log q} \text{ and } 0 \leq \gamma < 1.$$

3.2 When $0 \leq \gamma < 1$ and χ real

In this case χ^2 is the principal character modulo q, so the associated primitive character is the trivial character. In this case, an extra pole contribution appears from χ^2, so we have

$$-\Re\frac{L'}{L}(\sigma,\chi_0) \leq \frac{1}{\sigma-1} + o(\log q), \tag{42}$$

$$-\Re\frac{L'}{L}(\sigma+i\gamma_0,\chi) \leq \frac{1}{2}\log q - \frac{1}{\sigma-\beta_0} + o(\log q), \tag{43}$$

$$-\Re\frac{L'}{L}(\sigma+2i\gamma_0,\chi^2) \leq \frac{\sigma-1}{(\sigma-1)^2+4\gamma_0^2} + o(\log q) \tag{44}$$

Together with Lemma 8, we obtain

$$\frac{3}{\sigma-1} + \frac{\sigma-1}{(\sigma-1)^2+4\gamma_0^2} - \frac{4}{\sigma-\beta_0} + \frac{4}{2}\log q + o(\log q) \geq 0. \tag{45}$$

In this case, the bound for $\gamma_0 \geq 0$ gives that (45) becomes

$$\frac{3}{\sigma-1} + \frac{(\sigma-1)}{(\sigma-1)^2+0^2} - \frac{4}{\sigma-\beta_0} + 2\log q + o(\log q) \geq 0, \tag{46}$$

i.e

$$\frac{4}{\sigma-1} - \frac{4}{(\sigma-1)+(1-\beta_0)} + 2\log q + o(\log q) \geq 0, \tag{47}$$

Note that this becomes a trivial inequality as for $\sigma - 1 = x(1-\beta_0)$, it leads to

$$\frac{4}{x} - \frac{4}{(x+1)} + 2(1-\beta_0)(\log q) + o(\log q)(1-\beta_0) \geq 0, \tag{48}$$

$$(1-\beta_0)(\log q) \geq \frac{\frac{2}{x+1} - \frac{2}{x}}{(1++o(1))}, \tag{49}$$

where

$$\frac{2}{x+1} - \frac{2}{x} < 0.$$

This makes the previous proof fail.
Note that as γ_0 gets closer to zero, the pole contribution $\frac{\sigma-1}{(\sigma-1)^2+4\gamma_0^2}$ gets close to $\frac{1}{\sigma-1}$, which is problematic. So, we first split the cases depending on the size of γ_0 in comparison to $\frac{1}{\log q}$.

3.2.1 When $c(1-\beta_0) < \gamma \leq 1$ and χ real

We assume $c(1-\beta_0) < \gamma \leq 1$ for some positive constant c. Here c is an extra parameter which we will choose later. [1]

In this case we replace (45) with

$$\frac{3}{\sigma-1} + \frac{\sigma-1}{(\sigma-1)^2 + 4c^2(1-\beta_0)^2} - \frac{4}{\sigma-\beta_0} + \frac{4}{2}\log q + o(\log q) \geq 0, \tag{50}$$

We choose $\sigma - 1 = x(1-\beta_0)$, so that

$$\frac{3}{x(1-\beta_0)} + \frac{x(1-\beta_0)}{x^2(1-\beta_0)^2 + 4c^2(1-\beta_0)^2} - \frac{4}{(x+1)(1-\beta_0)} + 2(1+o(1))\log q \geq 0, \tag{51}$$

so

$$\frac{3}{x} + \frac{x}{x^2+4c^2} - \frac{4}{(x+1)} + 2(1+o(1))(1-\beta_0)(\log q) \geq 0, \tag{52}$$

i.e

$$(1+o(1))(1-\beta_0)(\log q) \geq \frac{2}{(x+1)} - \frac{3}{2x} - \frac{x}{2(x^2+4c^2)}, \tag{53}$$

i.e

$$(1-\beta_0)(\log q) \geq \frac{\frac{2}{(x+1)} - \frac{3}{2x} - \frac{x}{2(x^2+4c^2)}}{(1+o(1))}. \tag{54}$$

For $c = 17$, we find an optimal value at $x \approx 6.2271$, for which $\frac{2}{(x+1)} - \frac{3}{2x} - \frac{x}{2(x^2+4c^2)} \approx 30.07$, so

$$(1-\beta_0)(\log q) \geq \frac{1}{30.07 + o(1)}.$$

[1]This assumption is to simplify the exposition. The same argument can be made with comparing γ with $\frac{1}{\log q}$

3.2.2 When $0 < \gamma \leq c(1 - \beta_0)$ and χ real

One can counter this by noticing that both ϱ and $1-\overline{\varrho}$ are zeros of $L(s,\chi)$ since χ is real. Thus we have both terms $\frac{\sigma-\beta_0}{(\sigma-\beta_0)^2+\gamma_0^2}$ and $\frac{\sigma-\beta_0}{(\sigma-\beta_0)^2+(-\gamma_0)^2}$ appearing. Both are bounded above with $\frac{2}{\sigma-\beta_0}$. We use the parameter σ in place of $\sigma+i\gamma_0$. We have the bounds

$$-\Re\frac{L'}{L}(\sigma,\chi_0)\leq\frac{1}{\sigma-1}+o(\log q), \tag{55}$$

$$-\Re\frac{L'}{L}(\sigma,\chi)\leq\frac{1}{2}\log q-2\frac{\sigma-\beta_0}{(\sigma-\beta_0)^2+\gamma_0^2}+o(\log q)\leq\frac{1}{2}\log q-\frac{2(\sigma-\beta_0)}{(\sigma-\beta_0)^2+c^2(1-\beta_0)^2}+o(\log q), \tag{56}$$

$$-\Re\frac{L'}{L}(\sigma,\chi^2)\leq\frac{1}{\sigma-1}+o(\log q) \tag{57}$$

Thus, in place of (50), we have

$$\frac{3+1}{\sigma-1}-\frac{4\times 2(\sigma-\beta_0)}{(\sigma-\beta_0)^2+c^2(1-\beta_0)^2}+\frac{4}{2}(\log q)+o(\log q)\geq 0, \tag{58}$$

i.e.

$$\frac{4}{x}-\frac{8(x+1)}{(x+1)^2+c^2}+(2+o(1))(1-\beta_0)(\log q)\geq 0,$$

i.e.

$$(1+o(1))(1-\beta_0)(\log q)\geq\frac{4(x+1)}{(x+1)^2+c^2}-\frac{2}{x}$$

where the optimum value for $c=17$ is given by $x\approx 35.4888$, giving $\frac{2}{x}-\frac{4(x+1)}{(x+1)^2+c^2}\approx 29.6595$.

Remark 6. *Note that one can just take* $(1+\cos\theta)\geq 0$*, i.e.* $-\Re\frac{L'}{L}(\sigma,\chi_0)-\Re\frac{L'}{L}(\sigma,\chi)\geq 0$ *as it gives*

$$\frac{1}{x}-\frac{2(x+1)}{(x+1)^2+c^2}+(\frac{1}{2}+o(1))(1-\beta_0)(\log q)\geq 0,$$

(the same equation as above).

Problem 6. *Prove that a value for c of* 17 *(or close to it) gives a final value for R is as small as possible for both cases* $\gamma_0>c(1-\beta_0)$ *and* $\gamma_0\leq c(1-\beta_0)$*.*
Solution *Note that the final R decreases in the first case, and decreases in the second, with respect to c.*

Remark 7. *At this point we have proven regions free of zeros for complex characters, and free of non-real zeros for real characters, with a constant* ≤ 35*. Note that in this argument fails if there is only one real zero, instead of 2 conjugate ones. This brings back the same issue as in Section 3.2.2.*

3.2.3 When $\gamma_0=0$ and χ real

In this case, we consider two real zeros β_1 of $L(s, \chi)$, where χ is a real character. This time, we isolate both β_1 and β_2 and proceed in a similar argument to Section 3.2.2:

$$-\Re\frac{L'}{L}(\sigma, \chi_0) \leq \frac{1}{\sigma - 1} + o(\log q), \tag{59}$$

$$-\Re\frac{L'}{L}(\sigma, \chi) \leq \frac{1}{2}\log q - \frac{1}{\sigma - \beta_1} - \frac{1}{\sigma - \beta_2} + o(\log q) \leq \frac{1}{2}\log q - \frac{2}{\sigma - \min(\beta_1, \beta_2)} + o(\log q), \tag{60}$$

$$-\Re\frac{L'}{L}(\sigma, \chi^2) \leq \frac{1}{\sigma - 2} + o(\log q) \tag{61}$$

Thus, in place of (58), we have

$$\frac{4}{\sigma - 1} - \frac{8}{\sigma - \min(\beta_1, \beta_2)} + (2 + o(1))(\log q) \geq 0. \tag{62}$$

giving for $\sigma - 1 = x(1 - \min(\beta_1, \beta_2))$

$$\frac{4}{x} - \frac{8}{(x+1)} + (2 + o(1))(1 - \min(\beta_1, \beta_2))(\log q) \geq 0,$$

i.e. the largest zero falls in the same region described in Section 3.2.2:

$$1 - \min(\beta_1, \beta_2) \geq \frac{1}{2.91(1 + o(1))(\log q)}.$$

Remark 8. *This achieves the proof that for any* **real** *primitive Dirichlet character χ modulo q, $L(s, \chi)$ has* **at most one real zero** *in the region*

$$\beta \geq 1 - \frac{1}{2.92\log q} \text{ and } \gamma = 0.$$

We call the zeros outside this region ”exceptional". The next section describes how rare those are:

- *there is at most one exceptional zero per modulus,*
- *exceptional moduli are rare.*

Problem 7. 1. *Prove a zero free region using Stechkin's device.*

2. *Prove a zero free region using this time the trigonometric polynomial*

$$8(0.9126 + \cos\theta)^2(0.2766 + \cos\theta)^2.$$

   Hint: we recall that

   - *when χ is of order 2, then χ is real,*
   - *when χ is of order 3, then $\chi^2 = \overline{\chi}$, and*
   - *when χ is of order 4, then χ^2 is real.*

   So, all these cases need a separate study.

3. *Prove a zero free region using both.*

3.3 Sparcity of exceptional zeros

Theorem 9. *Let $q_1, q_2 \geq 3$. Let χ_1, χ_2 be two distinct real primitive characters modulo q_1 and q_2 respectively. Assume that $L(s,\chi_1)$ has an exceptional zero β_1, and that $L(s,\chi_2)$ has an exceptional zero β_2. Then*

$$\min(\beta_1,\beta_2) < 1 - \frac{1}{r\log(q_1q_2)}.$$

Corollary 10. *There is at most one real non-principal character χ modulo q for which $L(s,\chi)$ has an exceptional real zero.*

Corollary 11. *If $q_1 < q_2$ are exceptional moduli, then $q_2 > q_1^2$.*

Proof. Since the product of the primitive characters $\chi_1\chi_2$ is non-principal, we denote χ' the primitive character modulo $q' \mid (q_1q_2)$ inducing $\chi_1\chi_2$. So both β_1 and β_2 are zeros of $L(s,\chi')$. We use the bounds

$$-\frac{\zeta'}{\zeta}(\sigma) \leq \frac{1}{\sigma-1} + o(\log q), \tag{63}$$

$$-\Re\frac{L'}{L}(\sigma,\chi_1) \leq \frac{1}{2}\log q_1 - \frac{1}{\sigma-\beta_1} + o(\log q) \tag{64}$$

$$-\Re\frac{L'}{L}(\sigma,\chi_2) \leq \frac{1}{2}\log q_2 - \frac{1}{\sigma-\beta_2} + o(\log q) \tag{65}$$

$$-\Re\frac{L'}{L}(\sigma,\chi') \leq \frac{1}{2}\log q' - \frac{1}{\sigma-\beta_1} - \frac{1}{\sigma-\beta_2} + o(\log q) \tag{66}$$

and combine them thanks to the inequality

$$-\Re\frac{\zeta'}{\zeta}(\sigma) - \Re\frac{L'}{L}(\sigma,\chi_1) - \Re\frac{L'}{L}(\sigma,\chi_2) - \Re\frac{L'}{L}(\sigma,\chi_1\chi_2) = \Re\sum_{n\geq 1}\Lambda(n)(1+\chi_1(n))(1+\chi_2(n)) \geq 0. \tag{67}$$

Noting $q' \leq q_1q_2$ and $-\frac{1}{\sigma-\beta_i} \leq -\frac{1}{\sigma-\min(\beta_1,\beta_2)}$, we deduce

$$\frac{1}{\sigma-1} - \frac{4}{\sigma-\min(\beta_1,\beta_2)} + (1+o(1))\log(q_1q_2) \geq 0. \tag{68}$$

Taking $\sigma - 1 = x(1-\min(\beta_1,\beta_2))$, we obtain as in in Section 3.2.2

$$\frac{1}{x} - \frac{4}{(x+1)} + (1-\min(\beta_1,\beta_2))(1+o(1))\left(\log(q_1q_2)\right) \geq 0,$$

leading to

$$(1-\min(\beta_1,\beta_2))(\log(q_1q_2)) \geq \frac{1}{2.92\,(1+o(1))}.$$

□

References

[AK19] J-H. Ahn and S-H Kwon. An explicit upper bound for the least prime ideal in the Chebotarev density theorem. *Ann. Inst. Fourier (Grenoble)*, 69(3):1411–1458, 2019.

[Bel23] C. Bellotti. Explicit bounds for the riemann zeta function and a new zero-free region. https://arxiv.org/abs/2306.10680, 2023.

[BKL+21] Samuel Broadbent, Habiba Kadiri, Allysa Lumley, Nathan Ng, and Kirsten Wilk. Sharper bounds for the Chebyshev function $\theta(x)$. *Math. Comp.*, 90(331):2281–2315, 2021.

[BMOR18] M. Bennett, G. Martin, K. O'Bryant, and A. Rechnitzer. Explicit bounds for primes in arithmetic progressions. *Illinois J. Math.*, 62(1-4):427–532, 2018.

[CIS13] B Conrey, H Iwaniec, and K Soundararajan. Critical zeros of Dirichlet L-functions. *J. reine angew. Math.*, 681:175–198, 2013.

[Dav00] H Davenport. *Multiplicative Number Theory*. Springer, third edition, 2000.

[dlVP99] Ch.-J. de la Vallée Poussin. Sur la fonction $\zeta(s)$ de Riemann et le nombre des nombres premiers inférieurs a une limite donnée. *Mémoires cournnés de l'Acad. roy. des Sciences de Belgique*, 59, 1899.

[For02a] K. Ford. Vinogradov's integral and bounds for the Riemann zeta function. *Proc. London. Math. Soc.*, 85(3):565–633, 2002.

[For02b] K. Ford. Zero-free regions for the Riemann zeta function. In *Number theory for the millennium, II (Urbana, IL, 2000)*, pages 25–56. A K Peters, Natick, MA, 2002.

[HB92] D. R. Heath-Brown. Zero-free regions for Dirichlet L-functions, and the least prime in an arithmetic progression. *Proc. London Math. Soc. (3)*, 64(2):265–338, 1992.

[JK14] Woo-Jin Jang and Soun-Hi Kwon. A note on Kadiri's explicit zero free region for Riemann zeta function. *J. Korean Math. Soc.*, 51(6):1291–1304, 2014.

[Kad02] H. Kadiri. *Une région explicite sans zéro pour les fonctions L de Dirichlet*. Theses, Université des Sciences et Technologie de Lille - Lille I, December 2002.

[Kad05] H. Kadiri. Une région explicite sans zéros pour la fonction ζ de Riemann. *Acta Arith.*, 117(4):303–339, 2005.

[Kad12] H. Kadiri. Explicit zero-free regions for Dedekind zeta functions. *Int. J. Number Theory*, 8(1):125–147, 2012.

[Ke77] V. P. Kondrat ev. Some extremal properties of positive trigonometric polynomials. *Mat. Zametki*, 22(3):371–374, 1977.

[Kha22] T. Khale. An explicit vinogradov-korobov zero-free region for dirichlet l-functions. https://arxiv.org/abs/2210.06457, 2022.

[KW22] Habiba Kadiri and Peng-Jie Wong. Primes in the Chebotarev density theorem for all number fields (with an appendix by Andrew Fiori). *J. Number Theory*, 241:700–737, 2022.

[Lee21] Ethan S. Lee. On an explicit zero-free region for the Dedekind zeta-function. *J. Number Theory*, 224:307–322, 2021.

[LZ07] A. Languasco and A. Zaccagnini. A note on Mertens' formula for arithmetic progressions. *J. Number Theory*, 127(1):37–46, 2007.

[McC84] K.S. McCurley. Explicit estimates for the error term in the prime number theorem for arithmetic progressions. *Math. Comp.*, 42(165):265–285, 1984.

[MT15] M. Mossinghoff and T. Trudgian. Nonnegative trigonometric polynomials and a zero-free region for the Riemann zeta-function. *J. Number Theory*, 157:329–349, 2015.

[MTY23] M. Mossinghoff, T. Trudgian, and A. Yang. Explicit zero-free regions for the riemann zeta-function. https://arxiv.org/abs/2212.06867, 2023.

[Pla16] D. Platt. Numerical computations concerning the GRH. *Math. Comp.*, 85(302):3009–3027, 2016.

[PT21] Dave Platt and Tim Trudgian. The Riemann hypothesis is true up to $3 \cdot 10^{12}$. *Bull. Lond. Math. Soc.*, 53(3):792–797, 2021.

[RS75] J.B. Rosser and L. Schoenfeld. Sharper bounds for the Chebyshev functions ̈(x) and $\psi(x)$. *Math. Comp.*, 129(29):243–269, 1975.

[Sta74] H. M. Stark. Some effective cases of the Brauer-Siegel theorem. *Invent. Math.*, 23:135–152, 1974.

[Ste70] S. B. Stečkin. The zeros of the Riemann zeta-function. *Mat. Zametki*, 8:419–429, 1970.

[Xyl11] Triantafyllos Xylouris. On the least prime in an arithmetic progression and estimates for the zeros of Dirichlet L-functions. *Acta Arith.*, 150(1):65–91, 2011.

[Yan23] A. Yang. Explicit bounds on $\zeta(s)$ in the critical strip and a zero-free region. https://arxiv.org/abs/2301.03165, 2023.