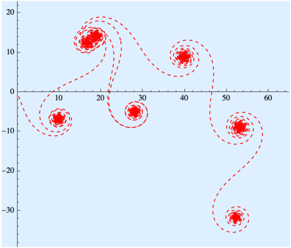
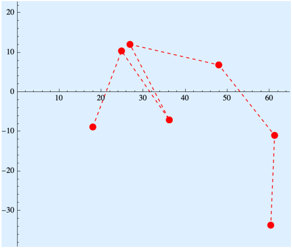
The Riemann zeta function, explicit bounds, and exponent pairs





Ghaith Hiary

The Ohio State University

Exponential sums in analytic number theory

The Riemann zeta function. Let $s = \sigma + it$.

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \cdots \qquad (\sigma > 1).$$

Zeta has an analytic continuation to $\mathbb{C} \setminus \{1\}$. An important problem is to bound $|\zeta(\sigma + it)|$ along vertical lines.

For example, on the 1-line, we have the theoretical result

$$\frac{1}{(\log t)^{2/3}(\log\log t)^{1/3}} \ll \zeta(1 + it) \ll \log^{2/3} t.$$

On the 1/2-line, for any positive ϵ, we have

$$\zeta(1/2 + it) \ll_\epsilon t^{13/84+\epsilon}.$$

Explicit bounds

The growth-rate of the Riemann zeta function is of great interest in analytic number theory.

It represents progress towards the Lindelöf hypothesis!

This motivates obtaining explicit bounds on zeta, with applications to explicit zero-free region and bounds related to $S(t)$.

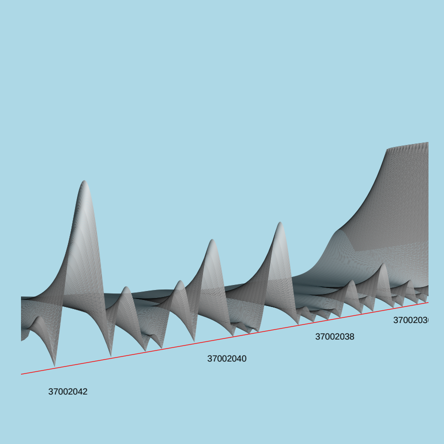
To obtain an explicit bound on $|\zeta(1/2 + it)|$ means to find a nonnegative function f, and positive constants c and t_0, such that

$$|\zeta(1/2 + it)| \leq c\, f(t) \qquad \text{for} \quad t \geq t_0.$$

One typically would like f to have a simple form for $t \geq t_0$. This helps simplify subsequent calculations.

Recent explicit unconditional results

Most recent results give the following bounds, applicable for $t \geq 3$.



$$|\zeta(1/2+it)| \leq 0.618\, t^{1/6} \log t,$$

$$|\zeta(1/2+it)| \leq 66.7\, t^{27/164},$$

$$|\zeta(1+it)| \leq \frac{1.721 \log t}{\log\log t},$$

$$\frac{1}{|\zeta(1+it)|} \leq \frac{430.5 \log t}{\log\log t}.$$

Results rely on *explicit processes from the method of exponent pairs*.

Another example

Explicit bounds for Dirichlet L-functions are also available in the literature. Here is a sample bound with a particularly nice form.

Let χ be a Dirichlet character modulo q. Let $L(1/2 + it, \chi)$ be the corresponding Dirichlet L-function on the critical line.

Let $\tau(q)$ be the number of divisors of q. If $|t| \geq 3$, say, we define the analytic conductor of $L(1/2 + it, \chi)$ to be $\mathfrak{q} := q|t|$.

We have the following hybrid bound on $L(1/2 + it, \chi)$; i.e. a bound uniform in t and q.

<u>H.</u>: Let χ be a primitive Dirichlet character modulo q. If q is a sixth power, then

$$|L(1/2 + it, \chi)| \leq 9.05\tau(q)\mathfrak{q}^{1/6} \log^{3/2} \mathfrak{q}, \qquad (|t| \geq 200).$$

Deriving explicit bounds

Derivation of explicit bounds usually involves treating several ranges of t (or t and q, or ...) separately.

A rough outline (often refined in practice) is the following.

(i) For small t, can use numerical computation.
  - Aided by rigorous error analysis or interval arithmetic.

(ii) For moderate size t, can use simple bounds that are effective in that range.
  - For example, the Euler–Maclaurin bound or Riemann–Siegel bound (a convexity bound).

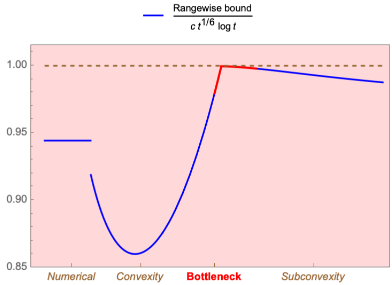
(iii) For large t, the complicated but asymptotically sharper bounds are incorporated.
  - Usually based on processes from the method of exponent pairs.

Improvements in explicit bounds come from better handling of the overlap of regions (ii) and (iii). This represents a "finite problem."

The main task is to estimate sums of the form

$$\sum_{a<n\leq b} n^{it} = \sum_{a\leq n\leq b} e(f(n)),$$

where $\quad f(u) = \frac{t}{2\pi} \log u, \qquad e(u) = e^{2\pi i u}, \qquad a, b \ll t.$



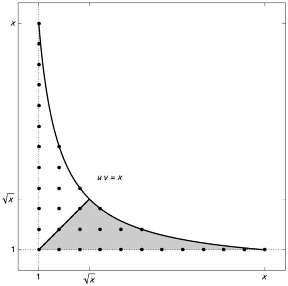
More examples of exponential sums in ANT

The Dirichlet divisor problem. Let

$$d(n) = \sum_{\substack{rd=n \\ r,d\geq 1}} 1.$$

Using the Dirichlet hyperbola method, we can get a formula for

$$\sum_{1\leq n\leq x} d(n) = \sum_{\substack{rd\leq n \\ r,d\geq 1}} 1.$$



This gives the formula

$$\sum_{1\le n\le x} d(n) = x\log x + (2\gamma - 1)x + \Delta(x),$$

where $\gamma = 0.577216\ldots$ and

$$\Delta(x) = -2\sum_{1\le d\le\sqrt{x}} \left\{\frac{x}{d}\right\} + \sqrt{x} + O(1).$$

We would like to estimate $\Delta(x)$. By well-known Fourier expansion,

$$\{w\} - 1/2 = -\sum_{k\neq 0} \frac{e(kw)}{2\pi ik}, \qquad w \in \mathbb{R}\setminus\mathbb{Z}.$$

This motivates studying exponential sums of the form

$$\sum_{a<n\le b} e(g(n)), \qquad g(u) = \frac{c}{u}.$$

More examples of exponential sums in ANT

Distribution of squarefree numbers. Let μ be the Mobius function.

So, $\mu^2(\cdot)$ detects squarefree numbers.

Montgomery and Vaughan studied the remainder term Δ_1 in the counting-function of squarefree numbers

$$\Delta_1(x) = \sum_{1 \leq n \leq x} \mu^2(n) - \frac{6}{\pi^2} x.$$

Assuming the Riemann hypothesis, they proved $\Delta_1(x) \ll_\epsilon x^{9/28+\epsilon}$.

To estimate the size of $\Delta_1(x)$, one appeals to bounds on exponential sums of the form

$$\sum_{a < n \leq b} e(h(n)), \qquad h(u) = \frac{\lambda}{u^2}.$$

Weyl sums

For motivation, let us consider the exponential sum S where the phase function $P(u)$ is a degree d polynomial with real coefficients.

$$S = \sum_{n=1}^{N} e(P(n)).$$

Case $d = 1$. In this case, $P(u) = \alpha u + \beta$, so that

$$\sum_{n=1}^{N} e(P(n)) = e\left(\frac{(N+1)\alpha}{2}\right) \frac{\sin(\pi \alpha N)}{\sin \pi \alpha}.$$

Therefore,

$$|S| \leq \frac{1}{|\sin \pi \alpha|} < \frac{1}{2\|\alpha\|},$$

where $\|\alpha\|$ is the distance from α to the nearest integer.

Case $d = 2$. In this case, $P(u) = \alpha u^2 + \beta u + \gamma$.

Unlike the sum when $d = 1$, the sum when $d = 2$ does not necessarily have a simple expression.

However, we can consider the square-modulus $|S|^2$. Rearranging,

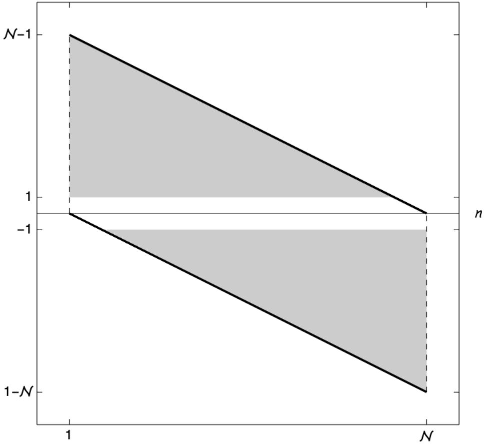
$$|S|^2 = \Big| \sum_{n=1}^{N} e(P(n)) \Big|^2$$
$$= \sum_{n=1}^{N} \sum_{m=1}^{N} e(P(m) - P(n))$$
$$= \sum_{n=1}^{N} \sum_{h=1-n}^{N-n} e(P(n+h) - P(n)).$$

Let's split the inner summation into three disjoint ranges.

$$1 - n \le h \le -1, \qquad -1 < h < 1, \qquad 1 \le h \le N - n.$$

This gives

$$|S|^2 = N + 2\,\mathrm{Re}\sum_{h=1}^{N-1}\sum_{n=1}^{N-h} e(P(n+h) - P(n)).$$



Hence, we obtain the bound

$$|S|^2 \le N + 2 \sum_{h=1}^{N-1} \left| \sum_{n=1}^{N-h} e(P(n+h) - P(n)) \right| .$$

Since $P(u) = \alpha u^2 + \beta u + \gamma$, the differenced phase function

$$P(u+h) - P(u) = c_h u + d_h$$

is linear in u. Here, $c_h = 2h\alpha$ and $d_h = \alpha h^2 + \beta h$. So the inner sum is a geometric sum.

So, we can fall back on the $d = 1$ case, bounding the inner sum by

$$\left| \sum_{n=1}^{N-h} e(P(n+h) - P(n)) \right| \le \frac{1}{|\sin \pi c_h|}.$$

If c_h is bounded away from integers, this produces savings.

In summary, when $d = 2$, we have

$$|S|^2 \le N + 2 \sum_{h=1}^{N-1} \min\left(N - h, \frac{1}{|\sin \pi 2h\alpha|}\right).$$

In view of the sine term on RHS, this bound is sensitive to the nature of rational approximations to α.

Example. Suppose $\alpha = a/q$, where a and q are integers such that $(a, q) = 1$, and q is an odd positive integer. Then,

$$2 \sum_{h=1}^{q} \min\left(N, \frac{1}{|\sin(\pi 2ha/q)|}\right) \le 2N + \sum_{h=1}^{q-1} \frac{1}{\|2ha/q\|}.$$

Noting that we are summing over a full residue system,

$$\sum_{h=1}^{q-1} \frac{1}{\|2ha/q\|} \le \sum_{m=1}^{q-1} \frac{q}{m} \le q\left(\log q + \gamma\right).$$

Therefore, applying the last estimates to at most $N/q + 1$ blocks of length $\leq q$, we obtain

$$|S|^2 \leq 3N + \frac{2N^2}{q} + (N + q)(\log q + \gamma). \qquad (*)$$

Using this, we can detect substantial cancellation in some important special cases.

For example, if q is about the size of N (an approximately complete sum), we obtain

$$|S| \ll \sqrt{N \log N},$$

demonstrating square-root cancellation for such q.

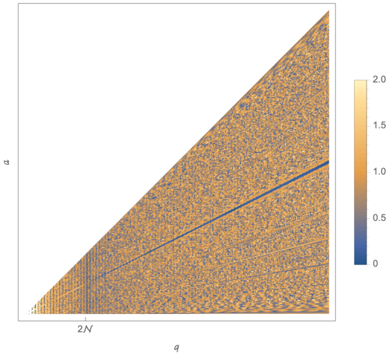
On the other hand, if q is very small or very large, then $(*)$ may be no better than the triangle inequality bound, $|S| \leq N$.

The true size of $|S|$ in such cases could be very small or very large.

Here is a color-coded plot of $|S|/\sqrt{N}$ for some $N = N_0$ when the phase function

$$P(u) = \frac{au^2}{q}.$$

The horizontal axis is q and the vertical axis is a. The vertical blue lines when q is around N is where a priori the bound $(*)$ does well.



Higher degree polynomials

The differencing method used to bound $|S|$ applies more generally, not only when the phase function P is a quadratic polynomial. Let

$$P(u) = \alpha u^d + \beta u^{d-1} + \cdots .$$

be a polynomial of degree d with real coefficients.

In this case, the differencing method applied to S produces an inner sum with a phase function of degree $d-1$.

This can be done repeatedly, applying the differencing method $d-1$ times.

Letting $D = 2^{d-1}$, one obtains (with the aid of Holder's inequality)

$$|S|^D \leq 2^{2D} N^{D-1} + 2^D N^{D-d} \sum_{\substack{1 \leq h_j < N \\ 1 \leq j < d}} \min\left(N, \frac{1}{|\sin(\pi \alpha d! h_1 \cdots h_{d-1})|}\right).$$

In fact, the last bound on $|S|^D$ does not depend not on the specific interval of summation for S, only on the interval length.

For at the bottom-most layer of repeated applications of differencing, we arrive at a geometric sum.

And a geometric sum can be bounded independent of the specific interval of summation.

One can hope in general that if $P(u)$ is a a degree d polynomial with real coefficients, then for any positive ϵ,

$$|S| \ll_{d,\epsilon} N^{1+\epsilon} \left(\frac{1}{q} + \frac{1}{N} + \frac{q}{N} \right)^{1/d}.$$

Substantial progress is made on this using the Vinogradov method, including seminal work.

Weyl differencing: The A-process

The differencing method just outlinded extends to general phase functions, not only polynomials.

One can also use a more elaborate differencing scheme involving an extra parameter M, to ensure the resulting differenced sums are not too short (have length $\geq L - M$).

van der Corput (Cheng–Graham, Platt–Trudgian, H.): If f is a real-valued regular function, then for any positive integer M,

$$\Big|\sum_{n=N+1}^{N+L} e(f(n))\Big|^2 \leq (L+M-1)\Big(\frac{L}{M} + \frac{2}{M}\sum_{m=1}^{M}\Big(1-\frac{m}{M}\Big)\,|S'_m(L)|\Big),$$

where if $L > 1$ and $m < L$, then

$$S'_m(L) := \sum_{r=N+1}^{N+L-m} e(f(r+m) - f(r)).$$

Now, using the triangle inequality to bound the RHS of the last inequality gives, if $M < L$,

$$\Big|\sum_{n=N+1}^{N+L} e(f(n))\Big|^2 \leq L^2 + \frac{M}{3}(2L - M).$$

This is worse than the trivial bound!

So, the A-process relies on the differenced phase function $f(u+m) - f(u)$ being better-behaved than the original phase function $f(u)$.

Put differently, the transformed sum under the A-process is longer than the original sum, but may present a simpler phase function.

Example. If $f(u)$ is a quadratic polynomial, then $f(u+m) - f(u)$ is a simple linear polynomial in u. So, the differenced sum can typically be bounded independent of the length of summation.

Poisson summation: The B-process

There is a different process that could produce savings by simply applying the triangle inequality to the transformed sum.

van der Corput (Patel, Karatsuba–Korolev): Suppose f' is decreasing, and let $f'(N+L)=\alpha$ and $f'(N)=\beta$. For $\alpha < v \le \beta$, let x_v be the solution to the equation $f'(x)=v$. Suppose $\lambda_2 \le |f''(x)| \le h_2\lambda_2$ and $|f'''(x)| \le h_3\lambda_3$. Then

$$\sum_{n=N+1}^{N+L} e(f(n)) = \sum_{\alpha<v\le\beta} \frac{e(f(x_v)-vx_v-1/8)}{\sqrt{|f''(x_v)|}} + \mathcal{E}$$

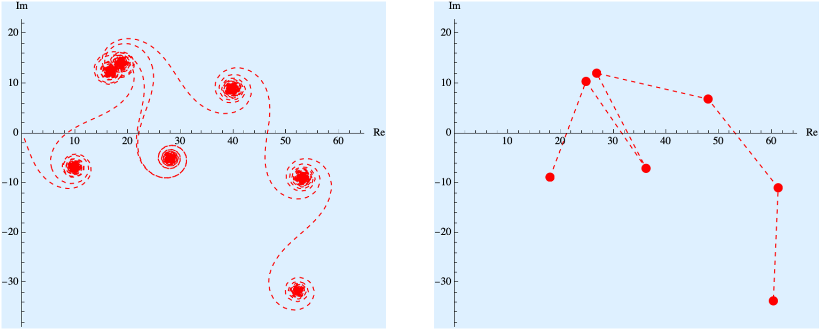
$$|\mathcal{E}| \le \frac{40}{\sqrt{\pi}}\lambda^{-1/2} + A_1\log(\beta-\alpha+4) + A_2\,L\,\lambda_2^{1/5}\lambda_3^{1/5} + A_3,$$

$$A_1 = \frac{3+2h_2}{\pi}, \quad A_2 = \frac{8}{(6\pi^3)^{1/5}}h_2h_3^{1/5}, \quad A_3 = \frac{1}{\pi}\left(4\gamma + \log 2 + \pi + \frac{20}{7}\right).$$

Note. Often, one has $(N, N+L] \subseteq (N, 2N]$, so $L \le N$.

This explicit B-process depends on the specific summation interval (rather than only the interval length).

Here is an example of the original and transformed sums (ignoring the remainder $\mathcal{E}$), visualized as walks in the complex plane.



The transformed sum under the B-process is intended to be a good enough caricature of the original sum that skips over cancellation.

Another B-process

Cheng–Graham: Suppose that

$$\frac{1}{W} \le |f''(x)| \le \frac{1}{V}.$$

Then (∗)

$$\Big|\sum_{n=N+1}^{N+L} e(f(n))\Big| \le \frac{1}{5}\left(\frac{L}{V}+1\right)(8W^{1/2}+15).$$

∗ Revised bound:

$$\Big|\sum_{n=N+1}^{N+L} e(f(n))\Big| \le 2\left(\frac{L}{V}+2\right)\left(2\sqrt{\frac{W}{\pi}}+1\right).$$

Note. The constant multiplying $L\,W^{1/2}/V$ in revised version (typically the leading constant) is larger by a factor of about $\sqrt{2}$.

The Kusmin–Landau lemma

This alternative B-process provides a final bound, rather than give a transformed sum. It is derived with the aid of a Kusmin–Landau lemma.

Note. The reason for revising the original bound of Cheng–Graham is that their B-process relied on an incorrect version of the Kusmin–Landau lemma.

A Kusmin–Landau lemma is used to bound the general sum

$$S_0 = \sum_{a \le n < b} e(f(n)),$$

where f is assumed to satisfy $f'' > 0$ (or a similar condition) over the summation interval $[a, b)$. One also supposes that for some $\theta \in (0, 1/2]$ we have for all $u \in [a, b)$,

$$\theta \le f'(u) \le 1 - \theta.$$

Landau, refining Kusmin's result, showed

$$|S_0| \leq \cot\left(\frac{\pi\theta}{2}\right) < \frac{2}{\pi\theta}.$$

Kevin Ford pointed out a later version of this bound had an incorrect constant of $1/\pi$, rather than $2/\pi$.

Namely, the bound

$$|S_0| \leq \frac{1}{\pi\theta} + 1. \qquad (**)$$

This affected many explicit bounds in the literature.

Many explicit bounds rely on a Kusmin–Landau lemma indirectly through the B process.

Exponent pairs

Let σ and c be positive constants. Define $\mathcal{F}(\sigma, c)$ to be the set of quadruples (N, I, f, y) such that:

(i) N and y are positive real numbers satisfying $yN^{-\sigma} \geq 1$;

(ii) I is a nonempty subinterval of $(N, 2N]$;

(iii) f is a real-valued function on the interval I, with derivatives of all orders p satisyfing

$$\left| f^{(p+1)}(x) - \frac{d^p}{dx^p}(yx^{-\sigma}) \right| \leq c \left| \frac{d^p}{dx^p}(yx^{-\sigma}) \right|, \qquad (p \geq 0,\ x \in I).$$

Question. For each of the following values of σ, try to give an example of a phase function f satisyfing (iii), with say $c = 1/4$. Your choice should work for any y, N, and I satisfying (i) and (ii), so that $(N, I, f, y) \in \mathcal{F}(\sigma, c)$.

$$\sigma = 1, \qquad \sigma = 2, \qquad \sigma = 3.$$

To say that (k, ℓ) is an *exponent pair* means that

(a) $0 \le k \le \frac{1}{2} \le \ell \le 1$; and

(b) for each positive constant σ, there exists a positive constant $c < 1/2$ depending on k, ℓ, and σ such that

$$\sum_{n \in I} e(f(n)) \ll_{k,\ell,\sigma} (yN^{-\sigma})^k N^\ell \qquad (***)$$

uniformly for all quadruples $(N, I, f, y) \in \mathcal{F}(\sigma, c)$.

Now, given a quadruple $(N, I, f, y) \in \mathcal{F}(\sigma, c)$, we have

$$yN^{-\sigma} \approx f'(x), \qquad (x \in I).$$

So, the bound $(***)$ is a weighted product of the magnitude of $f'(x)$ for $x \in I$, and the length of summation.

One could drop the explicit mention of the restriction $\ell \geq 1/2$ in the definition of exponent pairs.

In this case, this restriction follows anyway follow from the remaining parts of the definition.

For if one takes $\sigma = 2$, say, then no matter the positive constant $c < 1/2$, one could find a function f on $I = (N, 2N]$ and a positive number $y \approx N^2$ such that

$$\sum_{n \in I} e(f(n)) \gg N^{1/2}.$$

So, for (k, ℓ) to be an exponent pair, it must satisfy

$$(yN^{-\sigma})^k N^\ell \gg N^{1/2}.$$

Since here $yN^{-\sigma} \approx 1$, this forces $\ell \geq 1/2$.

Similarly, one could drop the explicit mention of the restriction $k \geq 0$ in the definition of exponent pairs.

For, again, taking $\sigma = 2$, then no matter the positive constant $c < 1/2$, one could find an f on $I = (N, 2N]$ and y tending fast to $+\infty$ with N such that

$$\left| \sum_{n \in I} e(f(n)) \right| \geq N - 1.$$

So, for (k, ℓ) to be an exponent pair, it must satisfy

$$(yN^{-\sigma})^k N^\ell \gg N,$$

which forces $k \geq 0$.

Example. By the triangle inequality. $(0, 1)$ is an exponent pair. By the Cheng–Graham B-process, $(1/2, 1/2)$ is an exponent pair.

Exponent pairs and the A-process

The A-process interacts well with exponent pairs.

Suppose $1 < L \leq N$. By the A-process,

$$\Big|\sum_{n=N+1}^{N+L} e(f(n))\Big|^2 \leq (L+M-1)\Big(\frac{L}{M} + \frac{2}{M}\sum_{m=1}^{M}\Big(1-\frac{m}{M}\Big)\,|S'_m(L)|\Big),$$

where $\qquad S'_m(L) := \sum_{r=N+1}^{N+L-m} e(f_m(r)), \qquad f_m(u) = f(u+m) - f(u).$

Suppose $f \in \mathcal{F}$ and $f' \approx V$. Then $f_m(u) \in \mathcal{F}$ and $f'_m \approx Vm/N$. So, if (k, ℓ) is an exponent pair, then the A-process gives

$$\Big|\sum_{n=N+1}^{N+L} e(f(n))\Big|^2 \ll \frac{L^2}{M} + M^k V^k L^{1-k+\ell}.$$

Choosing $M = L^{\frac{k-\ell+1}{k+1}}$ therefore gives

$$\Big|\sum_{n=N+1}^{N+L} e(f(n))\Big|^2 \ll V^{\frac{k}{2k+2}} N^{\frac{k+\ell+1}{2k+2}}.$$

In summary, if (k, ℓ) is an exponent pair, then the A-process gives

$$\left(\frac{k}{2k+2}, \frac{k+\ell+1}{2k+2}\right)$$

is also an exponent pair.

One can encode this information using the notation

$$A(k, \ell) = \left(\frac{k}{2k+2}, \frac{k+\ell+1}{2k+2}\right).$$

Example. $A(0, 1) = (0, 1)$ and $A(1/2, 1/2) = (1/6, 2/3)$.

Exponent pairs and the B-process

The B-process also interacts well with exponent pairs. By the B-process,

$$\sum_{n=N+1}^{N+L} e(f(n)) = e(-1/8) \sum_{\alpha < v \leq \beta} \frac{e(\phi(v))}{\sqrt{|f''(x_v)|}} + \mathcal{E}$$

where

$$\begin{aligned} \phi(v) &= f(x_v) - v x_v, \\ f'(N+L) &= \alpha, \\ f'(N) &= \beta, \\ f'(x_v) &= v. \end{aligned}$$

Suppose $f \in \mathcal{F}$ and $f' \approx V$.

Then one can show $\phi \in \mathcal{F}$ and $f''(x_v) \approx V/N$. Also, $\phi'(v) = -x_v$, so that $|\phi'(v)| \approx N$. Lastly, it is clear that $\beta - \alpha \ll V$.

Suppose (k, ℓ) is an exponent pair. Then the B-process and partial summation give

$$\Big|\sum_{n=N+1}^{N+L} e(f(n))\Big|^2 \ll \sqrt{N/V}N^k V^\ell = V^{\ell-1/2}N^{k+1/2}.$$

So, $(\ell - 1/2, k + 1/2)$ is also an exponent pair.

One can encode this information using the notation

$$B(k,\ell) = (\ell - 1/2, k + 1/2).$$

Example. $B(0,1) = (1/2, 1/2)$ and $B^2(k,\ell) = (k,\ell)$.

Lastly, one may combine the A, B processes to obtain further exponent pairs, starting with the trivial pair $(0,1)$.

Example. $AB(0,1) = (1/6, 2/3)$, $ABA^3B(0,1) = (11/82, 57/82)$.

The set of exponent pairs

Exponent pairs (k, ℓ) lie in the portion of the $k\ell$-plane satisfying

$$0 \leq k \leq 1/2 \leq \ell \leq 1.$$

Suppose (k_1, ℓ_1) and (k_2, ℓ_2) are exponent pairs. Then, keeping the same notation from before, for $0 \leq t \leq 1$,

$$\begin{aligned} |S| &= |S|^t |S|^{1-t} \\ &\ll (V^{k_1} N^{\ell_1})^t (V^{k_2} N^{\ell_2})^{1-t} \\ &= V^{tk_1+(1-t)k_2} N^{t\ell_1+(1-t)\ell_2}, \end{aligned}$$

This shows that

$$(tk_1 + (1-t)k_2, t\ell_1 + (1-t)\ell_2)$$

is also an exponent pair.

So, exponent pairs form a convex set in the $k\ell$-plane.

Building a table of explicit A, B words

This approach to obtaining explicit bounds for zeta via explicit A, B processes was pioneered by Y. F. Cheng and S. W. Graham.

A	Cheng–Graham–Kolesnik
B	Cheng–Graham
B	Patel–Karatsuba–Korolev
AB	Hiary
A^2B	Patel
A^3B	Patel
$A^{k-2}B$	Granville–Ramaré
$A^{k-2}B$	Yang
ABA^3B	Patel
$\vdots$	

Thank you!

Activity 1

Recall the following version of Kusmin–Landau lemma with the incorrect constant $1/\pi$.

Suppose f is continuously differentiable with a monotonic derivative and $\|f'\| \geq \theta$ for some $0 < \theta \leq 1/2$ throughout the interval $(a, b]$. Then

$$\sum_{a<n\leq b} e(f(n)) \leq \frac{1}{\pi\theta} + 1.$$

Although this bound does not hold in general, it might in some important special cases.

- Prove or disprove this bound holds when $f(u) = \alpha u$ (geometric sums).
- Prove or disprove this bound holds when the phase function is $f(u) = \alpha u^2$.

Activity 2

Recall that by the B-process, we can hope for the following approximation.

Suppose f' is decreasing, and let $f'(N+L) = \alpha$ and $f'(N) = \beta$. For $\alpha < v \leq \beta$, let x_v be the solution to the equation $f'(x) = v$. Then one can expect that

$$\sum_{n=N+1}^{N+L} e(f(n)) \approx \sum_{\alpha < v \leq \beta} \frac{e(f(x_v) - vx_v - 1/8)}{\sqrt{|f''(x_v)|}}.$$

Try to implement this approximation is a computer algebra system with, say

$$f(u) = t \log u, \qquad t \approx 10^6, \qquad N = 20000, \qquad L = 3000,$$

or usiong any other choices of f, N and L that interest you.

Activity 3

Question. Recall the definition of an exponent pair (k, ℓ) requires

$$0 \leq k \leq 1/2 \leq \ell \leq 1.$$

(i) Is there ever need to take $\ell > 1$?

(ii) Is there ever need to take $k > 1/2$?

Question. It was explained why the restriction $\ell \geq 1/2$ in the definition of exponent pairs would follow anyway from the remaining parts of the definition. The explanation relied on producing a phase function f satisfying several properies.

Namely, on taking $\sigma = 2$, then no matter the positive constant $c < 1/2$, one could find a function f on $I = (N, 2N]$ and a positive number y (possibly growing with N) such that $\sum_{n \in I} e(f(n)) \gg N^{1/2}$. Try to give a concrete example of such f.