Report on

Elliptic Stable Envelopes and R-matrices for Superspin Chains from 3d $\mathcal{N}$=2 Gauge Theories (23rit110)

Research in Teams, BIRS

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1 Overview of the Field

The complicated nature of physical systems makes it both difficult and desirable to find dynamics that can be completely determined. Known examples of such systems, the integrable models, are usually characterized by rich physical and mathematical structures. Integrable structures have been discovered in many a priori distinct and unrelated physical systems, such as different quantum field theories (QFTs) and string theory. These discoveries have proved exceptionally useful for our understanding of nonperturbative dynamics in QFTs and string theory but remained mostly mysterious in origin. This begs for the development of a unified picture of integrability in physical systems.

The simplest and earliest-known examples of integrable systems are one-dimensional lattice models or spin chains where the sites carry representations of a symmetry algebra $\mathfrak{g}$. These were developed as simple models for magnetic materials with nearest-neighbor interactions. Bethe showed in his pioneering work [5] that these systems are exactly solvable. The Schrödinger equation for the system can be solved by making an ansatz about the eigenvalues of the Hamiltonian and then solving certain constraints on them called the Bethe ansatz equations. It turns out that these algebraic equations define the critical locus of a potential function called the Yang-Yang functional Y studied in the context of one-dimensional Schrödinger operator [22]. Within this framework, known as the coordinate Bethe ansatz, the origin of integrability is obscure. In an attempt to understand this origin, the algebraic Bethe ansatz was formulated in which crucial roles are played by certain operators characterizing the spin chains such as the Lax operator and the R-matrix [10]. Integrability of these spin chains is encoded in certain algebraic braiding relations satisfied by the R-matrix called the Yang-Baxter equations [20, 21, 4, 3]. Another example of integrability appears in supersymmetric gauge theories with 4 or 8 supercharges. When all matters of these theories are massive, the low energy dynamics are controlled by a (twisted chiral) superpotential $\widetilde{W}$. For generic $\widetilde{W}$, such theories are topological and the integrability on the Coulomb branch can be argued based on general grounds [16]. Remarkably, the spin chains and the supersymmetric QFTs encode precisely the same integrable dynamics. By identifying $\widetilde{W}$ with the Yang-Yang functional Y, the Bethe/Gauge Correspondence [17, 18] shows that the gauge theory vacua are in one to one correspondence with the excitations of the spin chains. Creation and annihilation of spin excitations correspond to certain solitonic operators in the gauge theory and the full spin chain Hilbert space can be constructed by taking a collection of gauge theories with varying gauge groups. These gauge theory vacua can be characterized by the equivariant cohomology of the Higgs branch of the theory with respect to the maximal torus of the flavor symmetry group.

Given a Lie group G, its Kac-Dynkin diagram Q, a set $\mathbf{w}$ of its representations, and a complex curve C called a spectral curve, we can define a spin chain with G-symmetry. With the additional information $\mathbf{v}$ about the ranks of guage groups, we can also define supersymmetric gauge theories with 4 or 8 supercharges. Suppose we denote the Higgs branches of these theories by $\mathcal{M}[Q(\mathbf{v},\mathbf{w})]$. Then the vacua of these gauge theories can be identified with $H_T(\mathcal{M}[Q(\mathbf{v},\mathbf{w})])$ where T is the maximal torus of the flavor symmetry of the theory. Then the Bethe/Gauge correspondence can be schematically stated as:

$$\text{Spin chain Hilbert space with } G\text{-symmetry and weights } \mathbf{w} = \bigoplus_{\mathbf{v}} H_T(\mathcal{M}[Q(\mathbf{v},\mathbf{w})])\,. \tag{1}$$

The choice of the spectral curve C determines the infinite dimensional quantum algebra that acts on the spin chain and the space-time dimension of the corresponding gauge theory. The space-time dimension in turns determine the choice of cohomology theory on the r.h.s. of 1. For a G-symmetric spin chain with Lie algebra $\mathfrak{g}$, the possible choices are as follows:

Spectral curve	Quantum algebra	Space-time dimension	Cohomology
$\mathbb{E}_\tau$	Elliptic quantum group, $E_{\hbar,\tau}(\mathfrak{g})$	3	Elliptic cohomology
$\mathbb{C}^\times$	Quantum affine, $U_\hbar(\hat{\mathfrak{g}})$	2	K-theory
$\mathbb{C}$	Yangian, $Y_\hbar(\mathfrak{g})$	1	de Rham

Table 1: Choices of spectral curves and the corresponding data in G-symmetric spin chains and gauge theories. $\hbar$ is the quantization parameter, τ is the complex moduli of the elliptic curve $\mathbb{E}_\tau$, and $\hat{\mathfrak{g}}$ is the affine Lie algebra of $\mathfrak{g}$. Depending on whether $\mathfrak{g}$ is $\mathbb{Z}_2$ graded or not, the gauge theories have 4 or 8 supercharges.

All these algebras can be defined via the R-matrix solving the Yang-Baxter equations. The Bethe/Gauge correspondence therefore implies that the same R-matrices must be computable from the gauge theory side as well. From the mathematical studies on the cohomology theories [13, 2], it was discovered that a more fundamental quantity than the R-matrix is the stable envelope. The stable envelopes are triangular matrices acting on the spin chain Hilbert space which provide a Gauss decomposition of the R-matrix. It is thus natural to try to compute the stable envelopes using the gauge theories. In the hierarchy of spin chains from Table 1, at the top sits the elliptic spin chain. In that sense, the elliptic stable envelopes for superspin chains are the most general quantities that we can compute, because all other stable envelopes can be derived from them by taking various limits.

2 Recent Developments and Open Problems

A suggestion for the construction of cohomological stable envelopes from the Bethe/Gauge Correspondence has been laid out by Nekrasov [15, 14], and has been implemented for $\mathfrak{sl}_2$ spin chains in [6]. The notion of cohomological stable envelopes was formalized by Maulik and Okounkov [13]. Later this was generalized to the elliptic version by Aganagic and Okounkov [2, 1]. Recently, a gauge-theoretic construction of elliptic stable envelopes has been given in [9, 7] for SL_n symmetric spin chains. All these constructions concern elliptic cohomology of (complex) symplectic manifolds which appear in physics as Higgs branches of the gauge dual of bosonic spin chains via the Bethe/Gauge correspondence. Extending these constructions to the superspin chase was the main goal of our project. The Bethe/Gauge correspondence assigns to a superspin chain a gauge theory whose Higgs branch is non-symplectic. Our project can thus be decomposed into two main parts:

1. Give a mathematical definition of the elliptic stable envelopes for non-symplectic quiver varieties.
2. Identify proper gauge theoretic quantities corresponding to these stable envelopes and compute them.

To address the second point we rely on a string theoretic perspective on the Bethe/Gauge correspondence. For bosonic spin chains this was perspective was developed by Costello and Yagi [8] and this was later generalized to the superspin case in [11]. The string theoretic construction works for A-type Lie algebras so far, we thus focused on these $\mathrm{SL}_{m|n}$ symmetric Lie algebras for the current project.

3 Research Project Highlights

On the mathematical side, we have developed the mathematical foundation of the theory of elliptic stable envelopes for certain non-symplectic varieties. Our main example of such varieties were classical Higgs branches of 3d $\mathcal{N} = 2$ supersymmetric gauge theories. In particular,

- We have shown that such Higgs branches always admit a mathematical notion which we dubbed "partial polarization", replacing the usual notion of holomorphic polarization which is a key ingredient in the construction of [2];
- Furthermore, we have proven that for any variety (in particular Higgs branches of 3d $\mathcal{N} = 2$ supersymmetric gauge theories) admitting partial polarization, elliptic stable envelope exists, and it is unique;
- Finally, we have proven that certain limits of (properly normalized) elliptic stable envelopes produces the corresponding K-theoretic and cohomological stable envelopes.

On the physical front, we show that ellitptic stable envelopes of superspin chains correspond to mass Janus partition functions of 3d $\mathcal{N} = 2$ gauge theories interpolating between Higgs branch vacua. This directly generalizes the work of Dedusenko and Nekrasov [9] who showed similar results for bosonic spin chains corresponding to 3d $\mathcal{N} = 4$ gauge theories. As an illustrative example, we study in detail the simplest situation of an integrable spin chain with a Lie superalgebra symmetry, i.e. $\mathfrak{sl}_{1|1}$. To construct elliptic stable envelopes, we start with 3d $\mathrm{U}(N)$ gauge theories with $\mathcal{N} = 2$ supersymmetry on $I \times \mathbb{E}_\tau$, where I is a finite interval and $\mathbb{E}_\tau$ is an elliptic curve with complex moduli τ. These gauge theories are coupled to L fundamental and L anti-fundamental chiral multiples. After identifying the Higgs branches of these gauge theories, we use supersymmetric localization to compute the relevant Janus partition function. Furthermore, we reduce our result to 2d and 1d by taking appropriate limits to construct K-theoretic and cohomological stable envelopes, respectively. In particular, we recover the cohomological stable envelopes constructed recently by Rimańyi and Rozansky using a different method [19].

4 Workflow at BIRS and Outcomes

Based on online discussions between the participants of the program, various aspects of the project were outlined prior to our meeting at Banff. Most importantly, we had figured out the gauge theory quantities needed to be computed to make sure that our conjectural relation between gauge theory and superspin chains was true. At Banff we went through all the computations in detail during our regular meetings and started writing down the defining properties of elliptic stable envelopes for non-symplectic varieties. Near the end of our stay we finalized the structure of the paper to be written with details to be filled in.

We finished writing down the draft a few weeks after our meeting and the preprint was put on ArXiv in August 2023 [12]. We have received very positive feedbacks from the main figures of the field including some of the authors of [2, 9].

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