Siblings of Countable *NE*-Free Posets

Davoud Abdi

University of Calgary

Alberta-Montana Combinatorics and Algorithms Days
Banff International Research Station

June 5, 2022

Embedding and Sibling

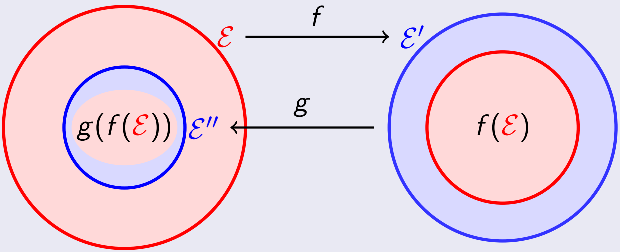
Embedding

An injective map preserving the structure.

Sibling

Two structures $\mathcal{E}$ and $\mathcal{E}'$ are called *siblings* (or equimorphic), denoted by $\mathcal{E} \approx \mathcal{E}'$, when there are mutual embeddings between them.

$\mathcal{E} \approx \mathcal{E}' \cong \mathcal{E}''$, $g(\mathcal{E}') = \mathcal{E}'' \supseteq (g \circ f)(\mathcal{E}) \cong \mathcal{E}$.



Siblings in Some Categories

Cantor-Schröder-Bernstein Theorem (Sets)

If there exist injective maps $f : A \to B$ and $g : B \to A$ between two sets A and B, then there exists a bijection (isomorphism) $h : A \to B$.

Vector Spaces

If there are injective linear transformations between two vector spaces over a fixed field, then they are isomorphic.

Rational Numbers

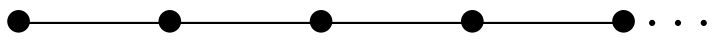
$\mathbb{Q}$ as a chain: there are mutual injective and order preserving maps between $\mathbb{Q}$ and $\mathbb{Q} + \infty$, nonetheless, $\mathbb{Q} \ncong \mathbb{Q} + \infty$.

Thomassé's Conjecture

Sibling Number

The number of isomorphism classes of siblings of a structure $\mathcal{R}$, denoted by $Sib(\mathcal{R})$.

If R is a ray, $Sib(R) = 1$ in the category of trees,



and $Sib(R) = \aleph_0$ in the category of graphs.

Thomassé's Conjecture (2000)

For a countable relation $\mathcal{R}$, $Sib(\mathcal{R}) = 1$ or $\aleph_0$ or $2^{\aleph_0}$.

The Alternate Thomassé Conjecture

For a relation $\mathcal{R}$ of any cardinality, $Sib(\mathcal{R}) = 1$ or ∞.

The Bonato-Tardif Conjecture, Positive Results

The Bonato-Tardif (BT) Conjecture

If T is a tree, then $Sib(T) = 1$ or ∞ in the category of trees.

The BT conjecture holds for:

- rayless trees [Bonato, Tardif] (2006)
- rooted trees [Tyomkyn] (2009)
- scattered trees [Laflamme, Pouzet, Sauer] (2017)

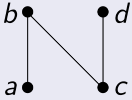
The Alternate Thomassé conjecture holds for:

- rayless graphs [Bonato, Bruhn, Diestel, Sprüssel] (2011)
- chains [Laflamme, Pouzet, Woodrow] (2017)
- countable $\aleph_0$-categorical structures [Laflamme, Pouzet, Sauer, Woodrow] (2021)
- countable cographs [Hahn, Pouzet, Woodrow] (2021)
- countable universal theories [Braunfeld, Laskowski] (2021)

NE-Free Posets

N

'*N*' is the following poset on four elements $\{a, b, c, d\}$: $a < b$, $c < b$, $c < d$, $a \perp c$, $b \perp d$ and $a \perp d$.



NE-Free Poset

An *NE-free poset* is a poset which does not embed an induced N.

Simple Examples

Chains, Antichains, Antichains Substituted with Chains (direct sums of chains)

Poset Substitution

Poset Substitution

Let Q be a poset and $\{P_u\}_{u \in Q}$ a pairwise disjoint family of posets. The poset obtained by replacing each $u \in Q$ with a poset P_u is called *poset substitution*, denoted by $P := Q[P_u/u : u \in Q]$.

Direct Sum and Linear Sum

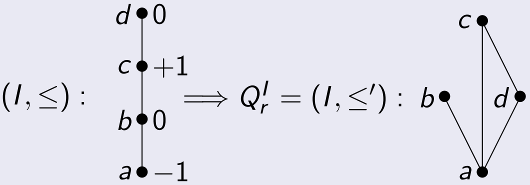
Q antichain $\Longrightarrow$ P is called a *direct sum*, denoted by $P = \bigoplus_{u \in Q} P_u$, each P_u is called a *component*.
Q chain $\Longrightarrow$ P is called a *linear sum*, denoted by $P = +_{u \in Q} P_u$, and each P_u is called a *summand*.

Context Poset of Poset Labelled Sum

Let $(I, \leq)$ be a chain and $r : I \to \{-1, 0, +1\}$. Define $Q_r^I = (I, \leq')$ as follows: for $i < j$,

- $i \perp j$ if $r(i) = 0$,
- $i <' j$ if $r(i) = -1$,
- $j <' i$ if $r(i) = +1$.

$I \Longrightarrow Q_r^I$



Proposition

For any map r, $Q_r^I = (I, \leq')$ is an NE-free poset.

Poset Labelled Sum

Poset Labelled Sum

I a chain,
$r : I \to \{-1, 0, +1\}$ a map,
$\{(P_i, \leq_i)\}_{i \in I}$ a pairwise disjoint family of non-empty posets.
The poset substitution $P = Q_r^I[P_i/i : i \in I]$ is called the *poset labelled sum* of the P_i.

Proposition

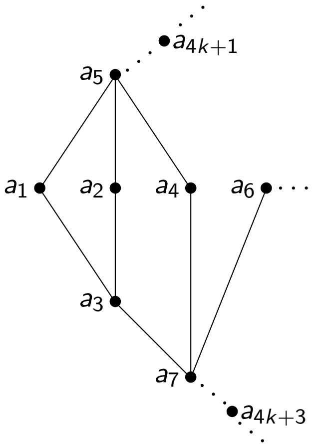
A poset substitution $P = Q[P_i/i : i \in I]$ is NE-free if and only if Q and each P_i are NE-free.

Thus, the poset labelled sum of NE-free posets is NE-free.

An Example of Poset Labelled Sum

$(I, \leq)$	r	P_i
i_1	$+1$	$\{a_1\}$
i_2	0	$\{a_2\}$
i_3	-1	$\{a_3\}$
i_4	0	$\{a_4\}$
i_5	$+1$	$\{a_5\}$
i_6	0	$\{a_6\}$
$\vdots$	$\vdots$	$\vdots$

$$P = Q_r^I[P_i/i : i \in I]$$



Classification of *NE*-Free Posets

Dense Mapping

$r : I \to \{-1, 0, +1\}$ takes 0 and ± 1 *densely* if the following holds: for $i < k$ there is j with $i < j \leq k$ such that $|r(i)| \neq |r(j)|$.
For instance, suppose $I = \mathbb{Z}$, $r(i) = 0$ for $i = 2k$, $r(i) = -1$ for $i = 4k + 1$ and $r(i) = +1$ for $i = 4k + 3$.

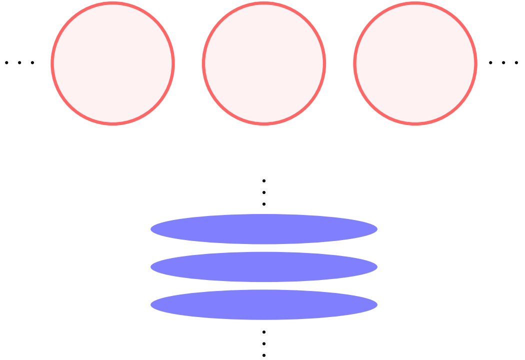
Theorem

Let P be an NE-free poset with more than one element. Then either

1. *a direct sum i.e.* $P = \bigoplus_i P_i$; *or*
2. *a linear sum i.e.* $P = +_i P_i$; *or*
3. $P = Q_r^I[P_i/i : i \in I]$ *where* $(I, \leq)$ *is a chain with no first element and the* P_i *are NE-free and r is a mapping on the chain* $(I, \leq)$ *taking 0 and* ± 1 *densely.*

Proposition (1)

If P is a countable direct, resp linear, sum of NE-free posets, then $Sib(P) = 1$ or ∞ on condition that this property holds for each component, resp summand, of P.



Siblings of Poset Labelled Sums

Theorem (2)

Let $P = Q^I_r[P_i/i : i \in I]$ be countable where $(I, \leq)$ is a chain with no first element, the P_i are non-empty NE-free posets and $r : I \to \{-1, 0, +1\}$ takes 0 and ± 1 densely. Then $Sib(P) = 2^{\aleph_0}$.

How to obtain?

P can be represented as $P = \sum C$ where $C = (I, \leq, \ell)$ such that $\ell(i) = (P_i, r(i))$.
For each $f \in \{0, 1\}^{\mathbb{N}}$, we construct a labelled chain C_f such that $\sum C \approx \sum C_f$, and it is proven that there are continuum many functions $f \in \{0, 1\}^{\mathbb{N}}$ such that $\sum C_f \not\approx \sum C_g$ for $f \neq g$.

Main Result

Theorem (Abdi)

If P is a countable NE-free poset, then $Sib(P) = 1$ or ∞.

Sketch of Proof

We can use induction because embeddability is a well-founded relation by Thomassé's theorem (The class of countable NE-free posets is wqo under embeddability).

- If $P = Q_r^I[P_i/i : i \in I]$, then $Sib(P) = 2^{\aleph_0}$ by theorem (2).
- Otherwise, $P = \bigoplus_i P_i$ or $P = +_i P_i$. In either case, if $P \hookrightarrow P_i$ for some i, then $Sib(P) = \infty$. If $P_i \hookrightarrow P$ strictly, then by $Sib(P) = 1$ or ∞ by induction hypothesis and proposition (1).

Counterexamples

Counterexample to the Bonato-Tardif Conjecture [Claimed by Tateno, Rigorous Exposition by Abdi, Laflamme, Tateno, Woodrow, preprint]

There are locally finite trees having arbitrary finite number of siblings.

Counterexample to Thomassé's Conjecture [Abdi, Laflamme, Tateno, Woodrow, preprint]

Tateno's example can be adapted to construct posets contradicting Thomassé's conjecture.

`https://arxiv.org/abs/2205.14679`

Future Directions

Thomassé's Conjecture for *NE*-Free Posets

For a countable *NE*-free poset P, is it true that $Sib(P) = 1$ or $\aleph_0$ or $2^{\aleph_0}$?

Boundaries

For which classes of relations the conjectures of Bonato-Tardif and Thomassé (both the original and the alternate form) are true and for which ones they are false?

Thank You for Your Attention