Spherical Multitaper Analysis via Spatio-Spectrally Concentrated Slepian Functions: Theory and Applications

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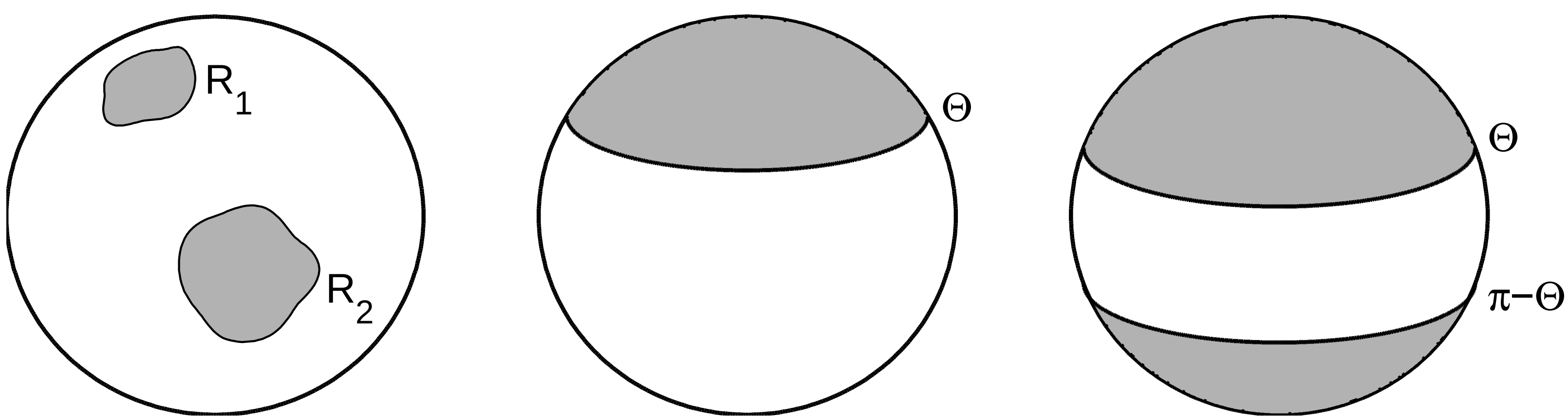
We can use these “Slepian” functions as **windows**, for spectral analysis, or we can use them as a **(sparse) basis** to represent geophysical observables—on a sphere.



A brief history of Slepian functions — 1

In the 60s Slepian *et al.* solved the problem of concentrating a **bandlimited** signal

$$g(t) = \frac{1}{2\pi} \int_{-W}^{+W} G(\omega)\, e^{i\omega t}\, d\omega, \qquad |W| < \infty, \tag{1}$$

into a **time interval** $|t| \leq T$. The "Slepian functions" optimize the **concentration**

$$\lambda = \frac{\displaystyle\int_{-T}^{+T} g^2(t)\, dt}{\displaystyle\int_{-\infty}^{+\infty} g^2(t)\, dt}, \qquad 0 < \lambda < 1. \tag{2}$$

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They are **eigenfunctions** of a Fredholm integral equation,

$$\int_{-T}^{T} \left[\frac{\sin W(t-t')}{\pi(t-t')}\right] g(t')\, dt' = \lambda g(t). \tag{3}$$

A brief history of Slepian functions — 2

Similarly, *two*-dimensional Slepian functions are bandlimited Fourier expansions

$$g(\mathbf{x}) = \frac{1}{(2\pi)^2} \int_{\mathcal{K}} G(\mathbf{k})\, e^{i\mathbf{k}\cdot\mathbf{x}}\, d\mathbf{k}, \qquad |\mathcal{K}| < \infty, \tag{4}$$

that concentrate into a finite **spatial** region $\mathcal{R} \in \mathbb{R}^2$ of area A by maximizing

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These are also **eigenfunctions** of a Fredholm integral equation,

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On a **sphere**, Slepian functions are bandlimited *spherical-harmonic* expansions

$$g(\hat{\mathbf{r}}) = \sum_{l=0}^{L} \sum_{m=-l}^{l} g_{lm} Y_{lm}(\hat{\mathbf{r}}), \qquad L < \infty, \tag{7}$$

that are concentrated within a region $R \in \Omega$ by optimizing the energy ratio

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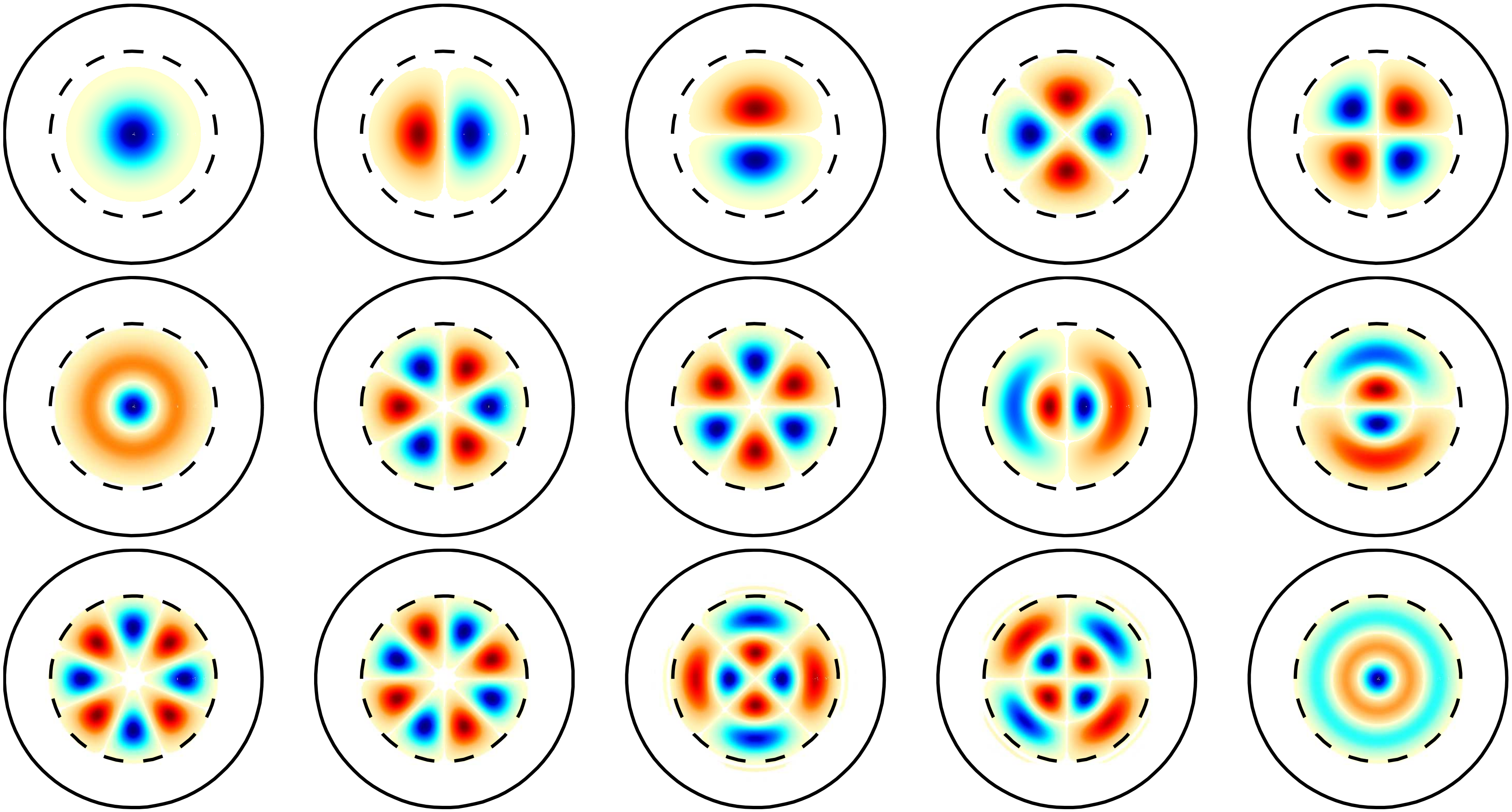
$$\int_R \left[\sum_{l=0}^{L} \left(\frac{2l+1}{4\pi} \right) P_l(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}') \right] g(\hat{\mathbf{r}}')\, d\Omega' = \lambda g(\hat{\mathbf{r}}). \tag{9}$$

Some examples of Slepian functions — 1



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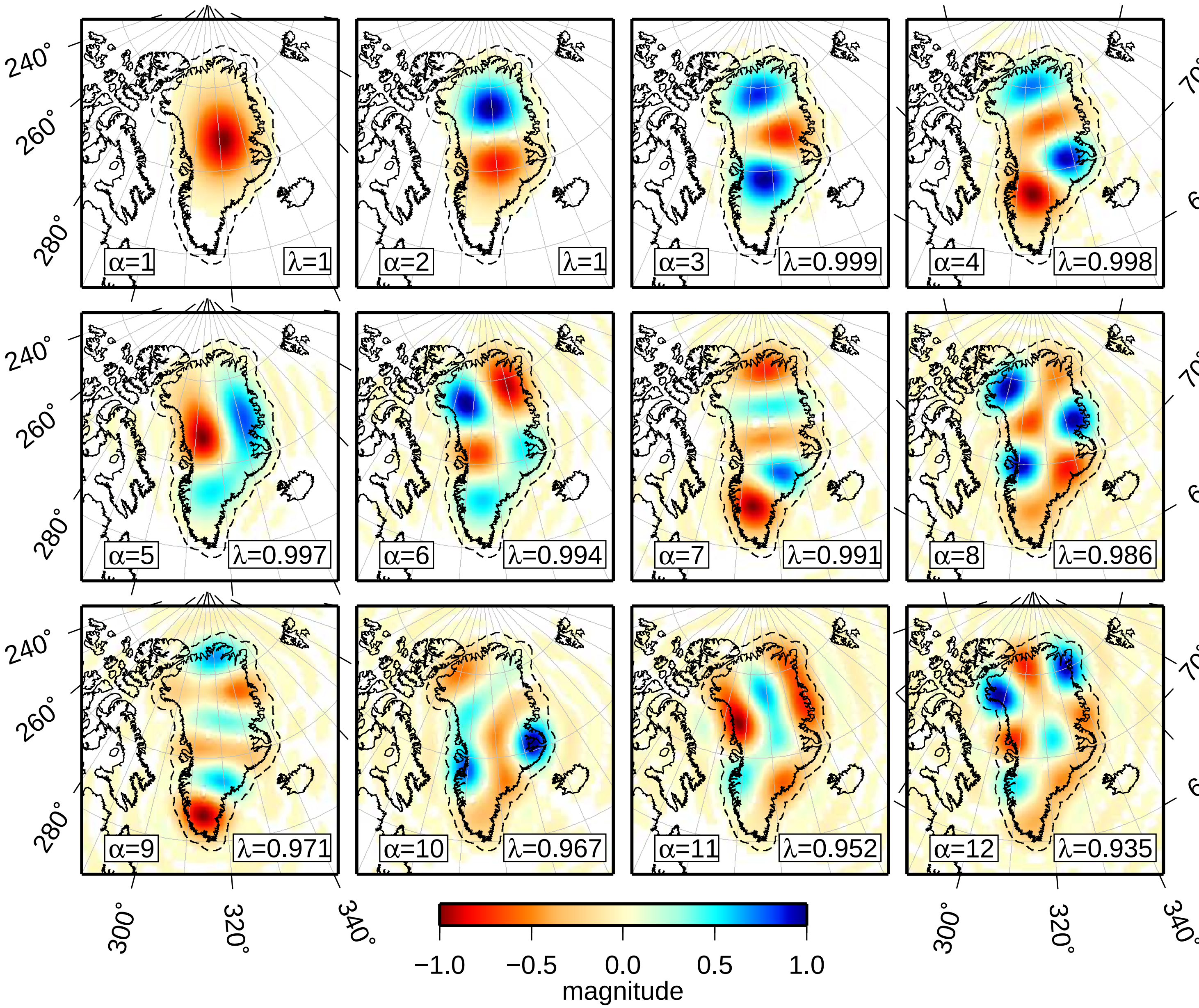

A unified framework — 1

The integral-equation kernels are all *spectrally bandlimited spatial delta functions* that are “reproducing kernels” for the bandlimited functions of the kinds considered:

$$D(t,t') \;=\; \frac{1}{2\pi}\int_{-W}^{+W} e^{i\omega\,(t-t')}\,d\omega, \qquad \mathrm{tr}\{D\} = 2\frac{TW}{\pi}, \tag{10}$$

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Thus, the Slepian functions are **bases** for **bandlimited** geophysical processes **anywhere** (not just on the domain for which they were constructed, though, there, they will be a **sparse** basis). Their trace is a space-bandwidth joint "Shannon" *area*.

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Remember that the *trace* of an operator is the *sum* of all of its eigenvalues, N.

A unified framework — 2

In the *spectral* domain, the Slepian functions are eigenfunctions of equations that have *spacelimited spectral delta functions* as kernels. On the **sphere**, we solve for the spherical harmonic expansion coefficients of the functions as

$$\sum_{l'=0}^{L}\sum_{m'=-l'}^{l'}\left[\int_R Y_{lm}Y_{l'm'}\,d\Omega\right]g_{l'm'}=\lambda g_{lm},\qquad 0<\lambda<1. \tag{13}$$

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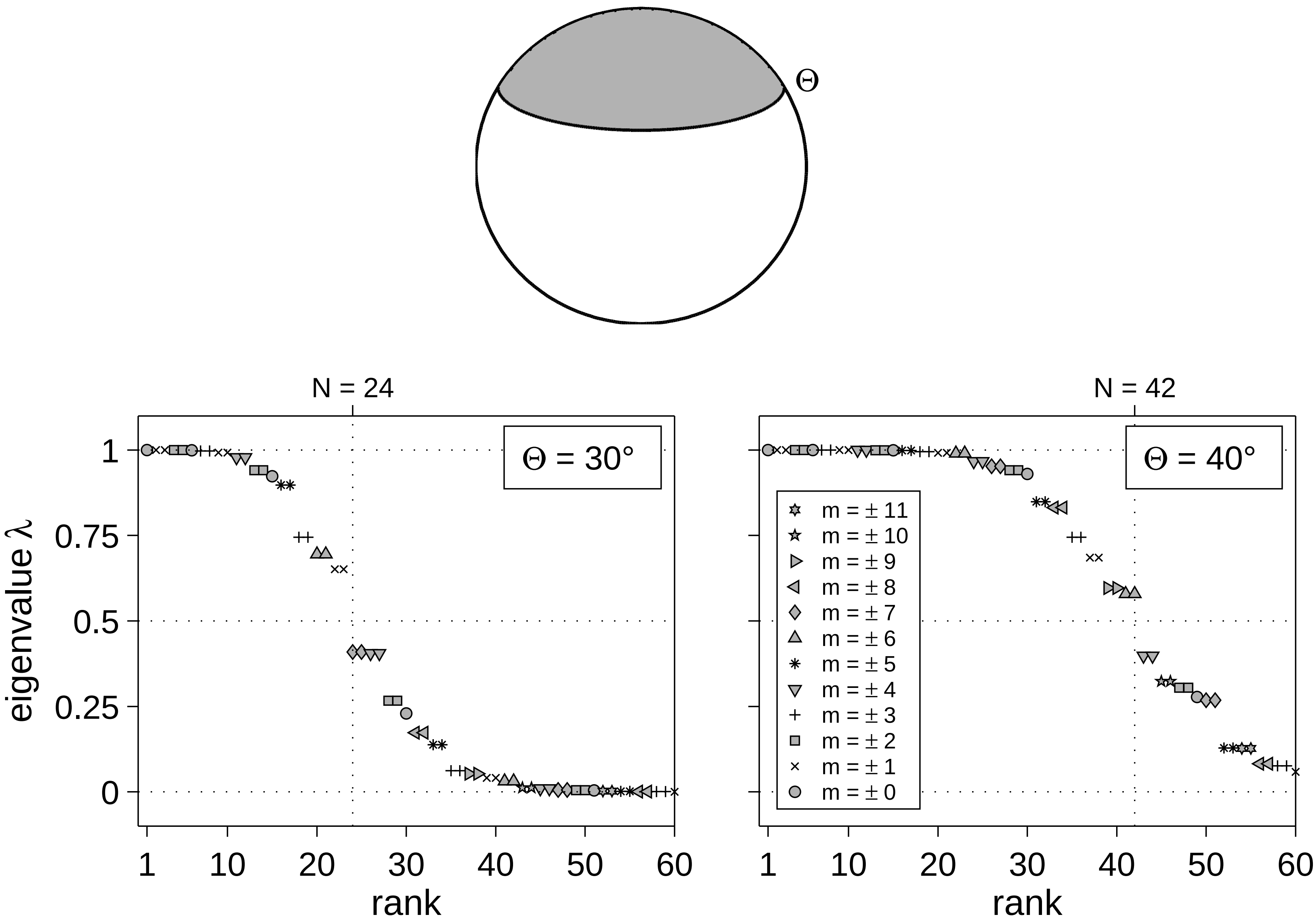
We define the **spatiospectral localization kernel**, with eigenvalues λ, as

$$D_{lm,l'm'}=\int_R Y_{lm}Y_{l'm'}\,d\Omega,\qquad \mathrm{tr}\{\mathsf{D}\}=(L+1)^2\frac{A}{4\pi}. \tag{14}$$

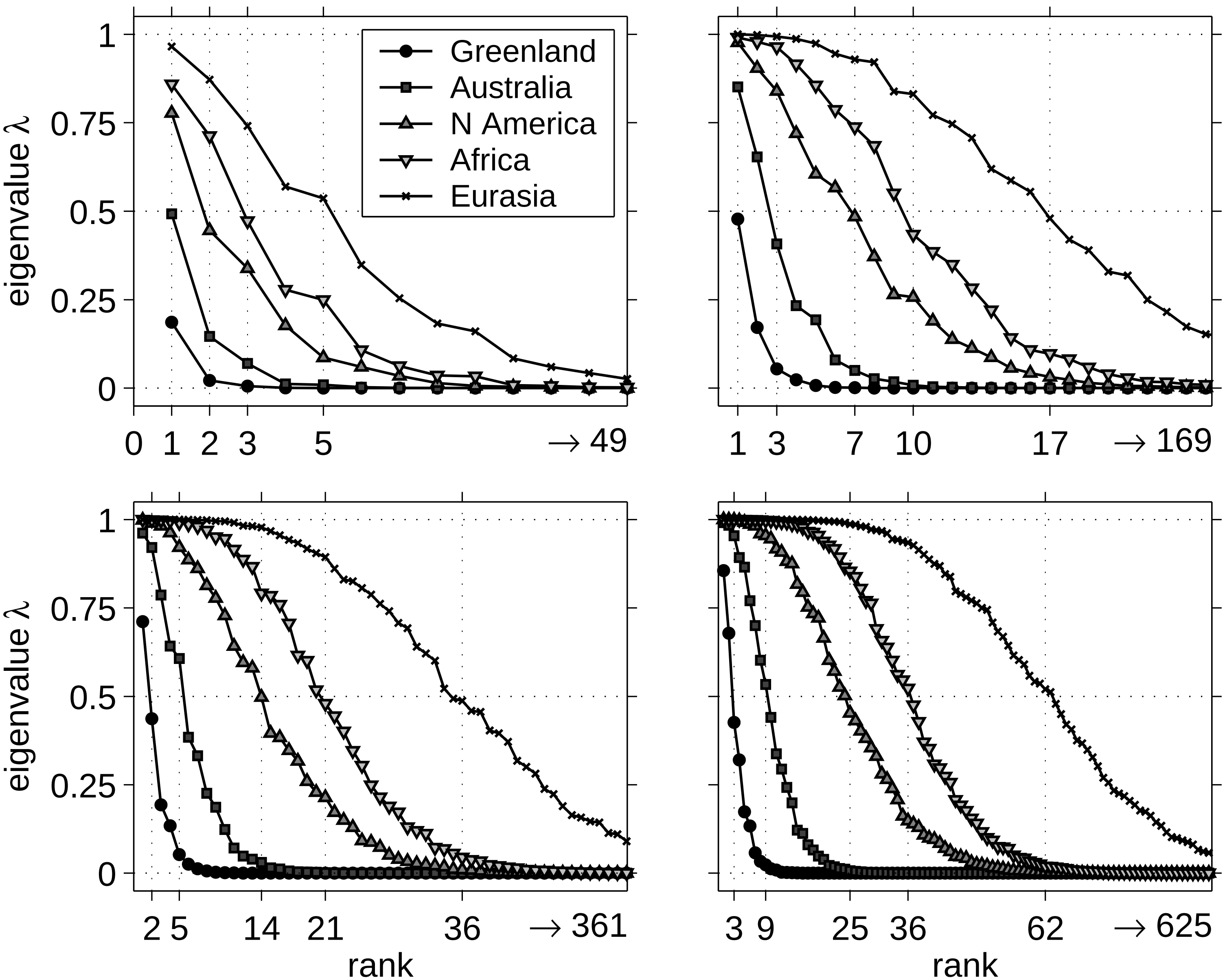
Many of the eigenvalues are very, very small. Thus, D may be hard to calculate—and even harder to invert.

And remember that the spatial region R can be completely arbitrary.

Eigenvalue behavior — 1



Eigenvalue behavior — 2



A “lucky accident”: the “magic of commutation”

Diagonalization of the operator D, with elements

$$D_{lm,l'm'} = \int_R Y_{lm} Y_{l'm'} \, d\Omega, \tag{15}$$

is often hard and sometimes impossible.

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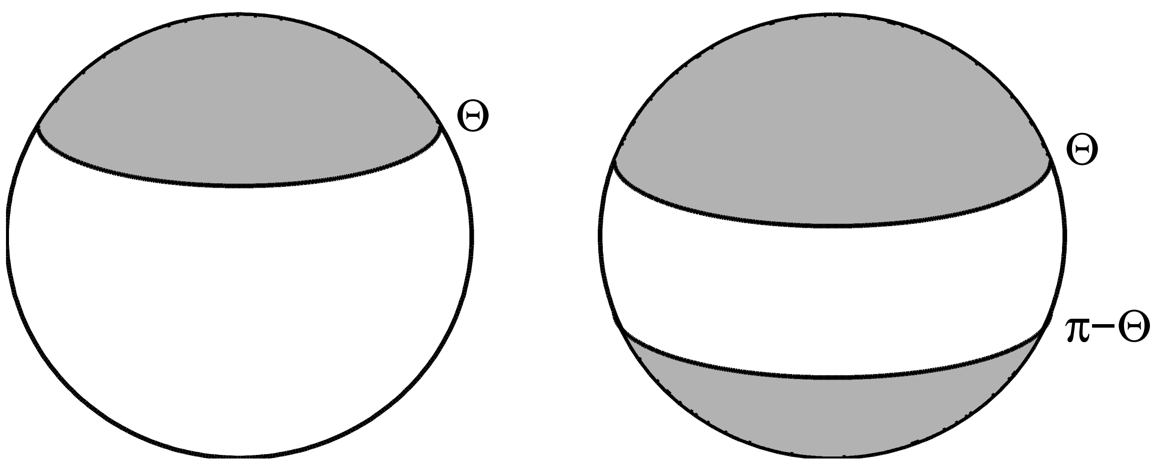
is often hard and sometimes impossible.

But if R is **axisymmetric**, i.e. a **single polar cap** or a **double polar cap**, we can find the Slepian functions as the solutions to a **different** eigenvalue problem involving a *very simple* kernel with *very well-behaved* eigenvalues.



Summary of the theory (on the sphere)

Spherical harmonics Y_{lm} form an **orthonormal** basis on Ω:

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The spherical harmonics Y_{lm} are **not orthogonal** on R:

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The eigenfunctions of D are called **Slepian functions**, $g(\hat{\mathbf{r}})$. They form a **band-limited localized basis**, *doubly* orthogonal: on R (to λ) and *also* on Ω (to 1).

The **Shannon number**, or sum of the eigenvalues, the space-bandwidth product,

$$N = (L+1)^2 \frac{A}{4\pi},$$

is the **effective dimension** of the space for which the bandlimited g are a **basis**.

Application 1 : Sparse approximation

The expansion of a bandlimited process on the sphere in *either* spherical harmonics or in Slepian functions is equal and *exact*:

$$s(\hat{\mathbf{r}}) = \sum_{l=0}^{L} \sum_{m=-l}^{l} s_{lm} Y_{lm}(\hat{\mathbf{r}}) = \sum_{\alpha=1}^{(L+1)^2} s_\alpha g_\alpha(\hat{\mathbf{r}}). \tag{19}$$

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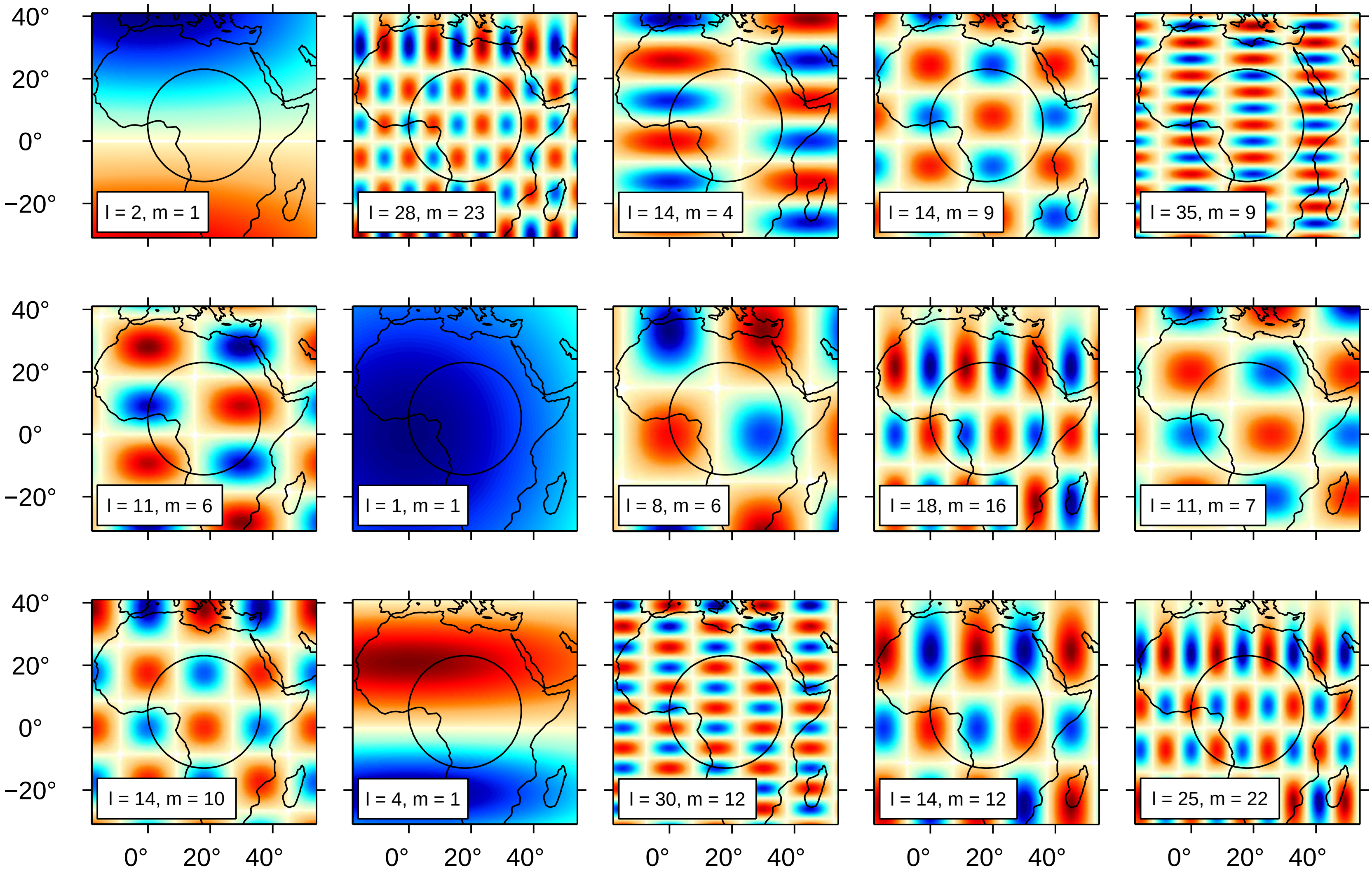
But if the signal is **regional** in nature, an expansion into Slepian functions up until the Shannon number will be **approximate but sparse**:

$$s(\hat{\mathbf{r}}) \approx \sum_{\alpha=1}^{N} s_\alpha g_\alpha(\hat{\mathbf{r}}), \qquad \hat{\mathbf{r}} \in R. \tag{20}$$

The *mean squared reconstruction error* in the noiseless case is determined by the neglected eigenvalues, which are **tiny** beyond the Shannon number.

Basis I: spherical harmonics Y_{lm}



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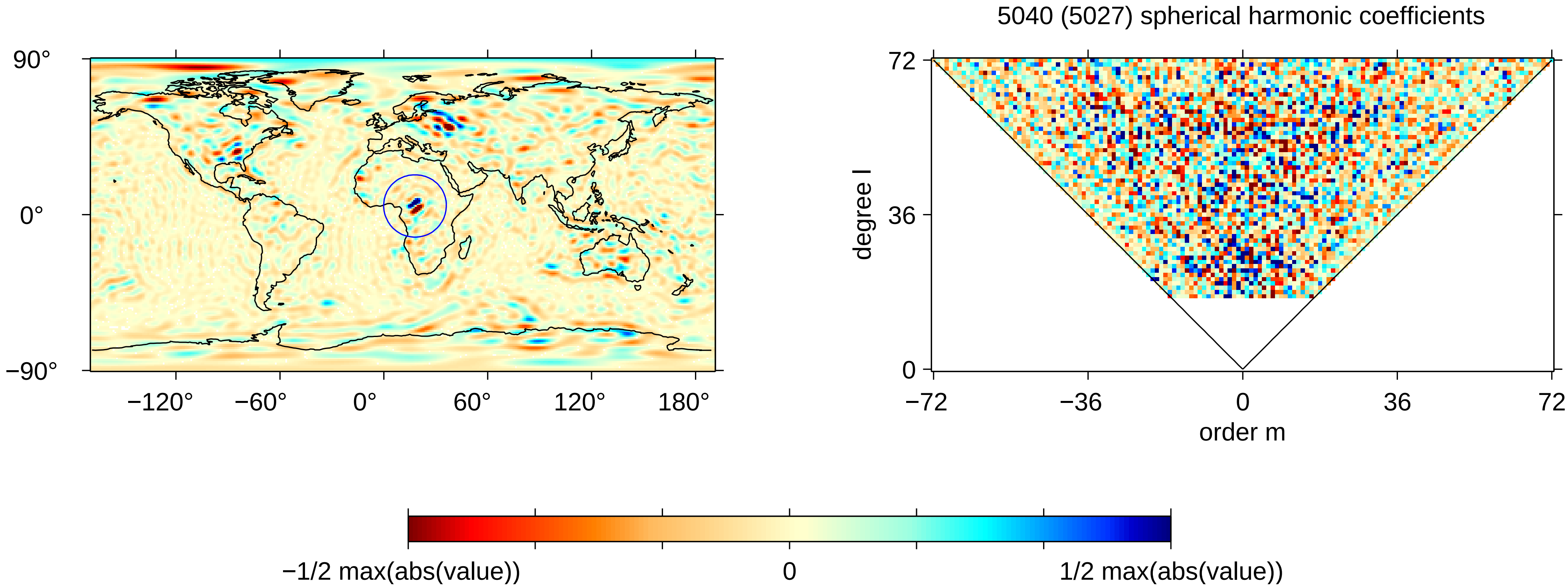

A *global* basis, **good** for *global* problems.

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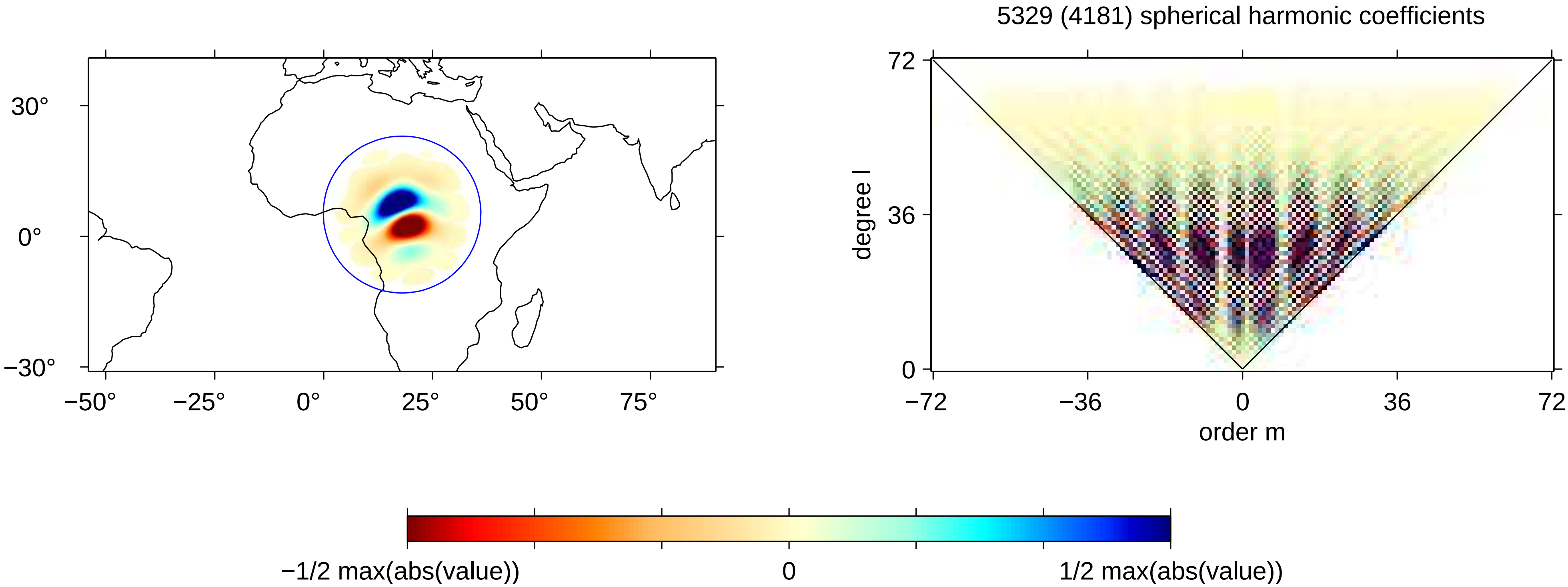
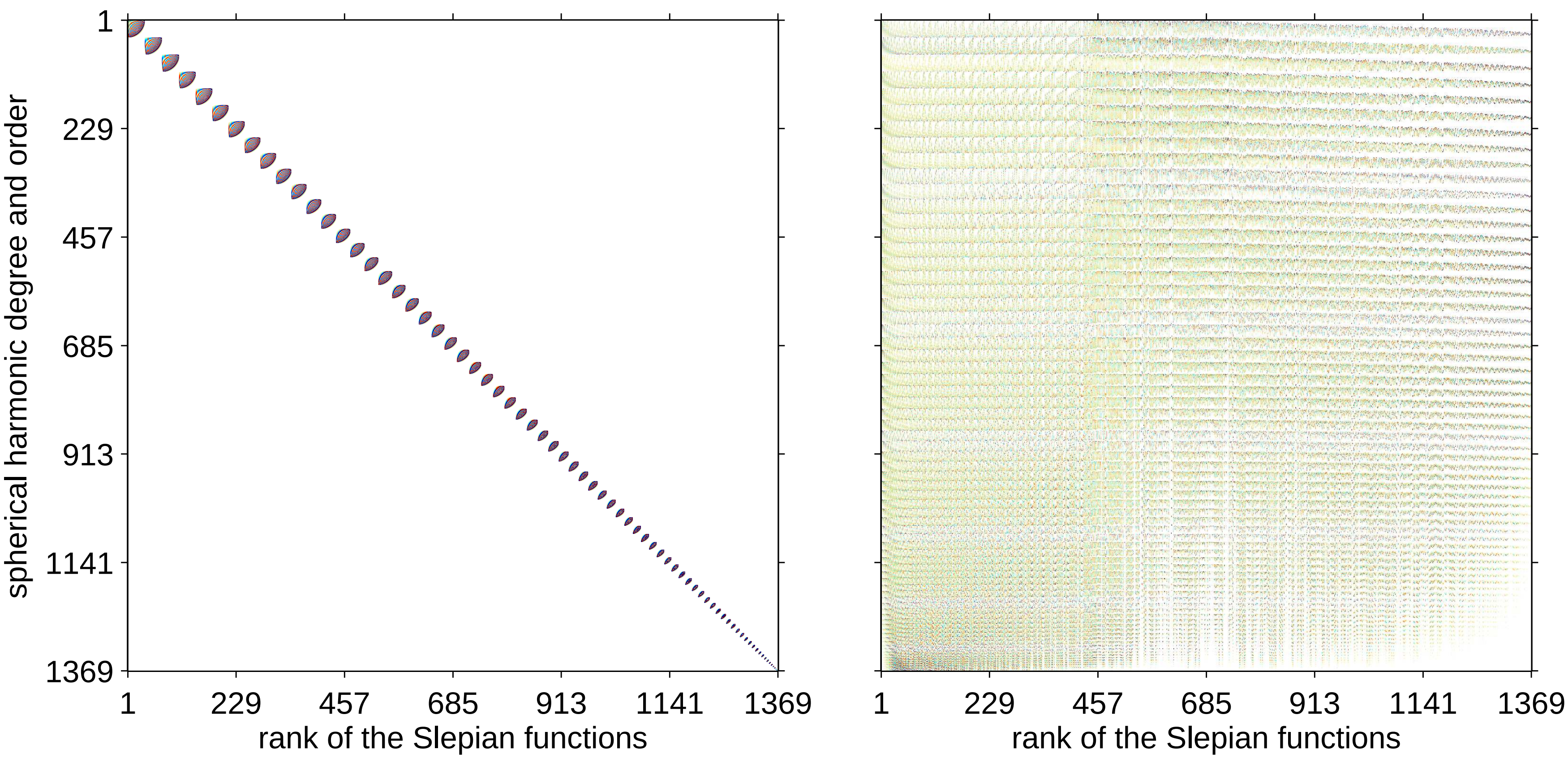


A *global* basis, **bad** for *local* problems.

Spherical harmonics $Y_{lm} \longrightarrow$ Slepian functions g

An **orthogonal transform** by the eigenmatrix of D introduces welcome sparsity.



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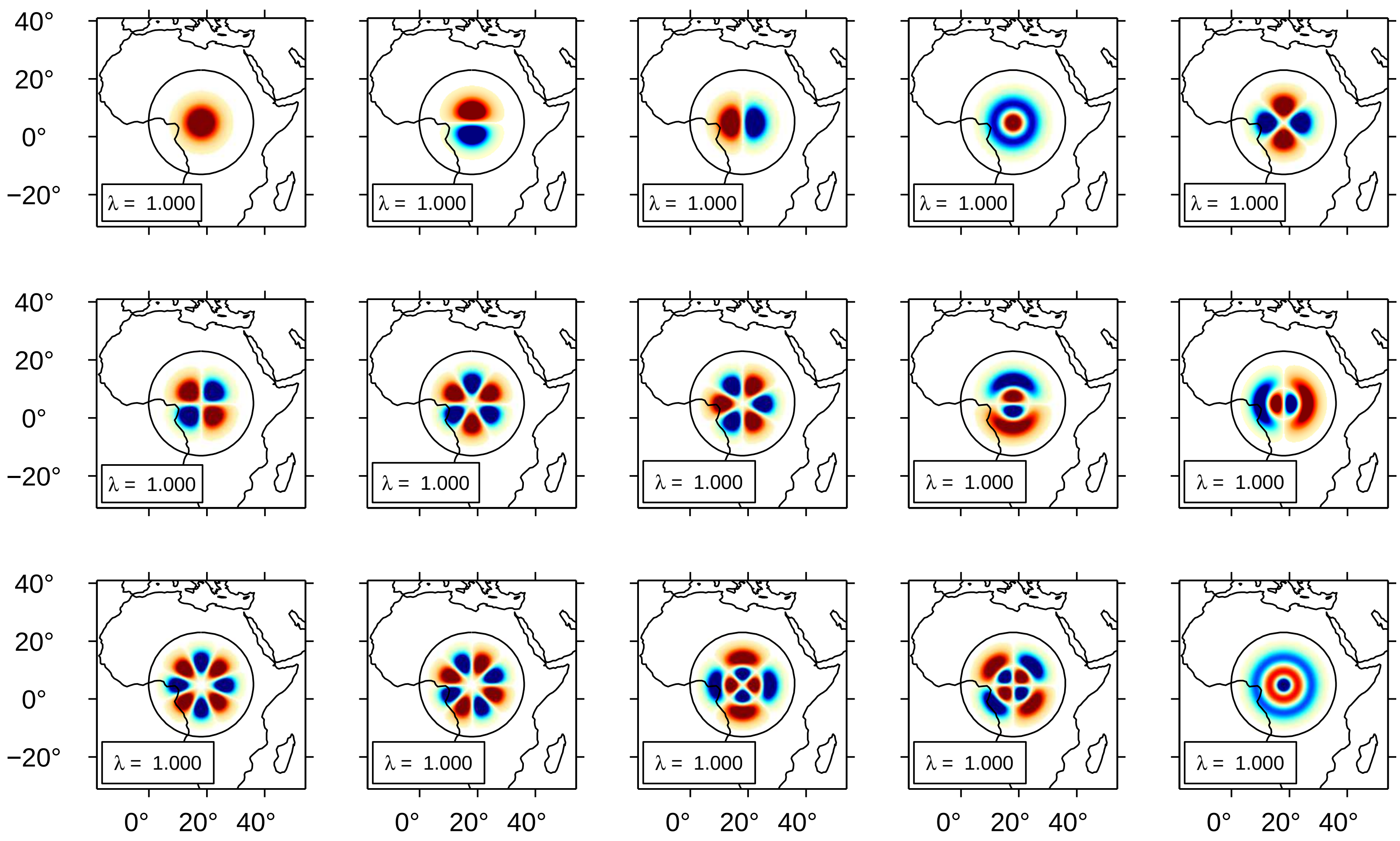

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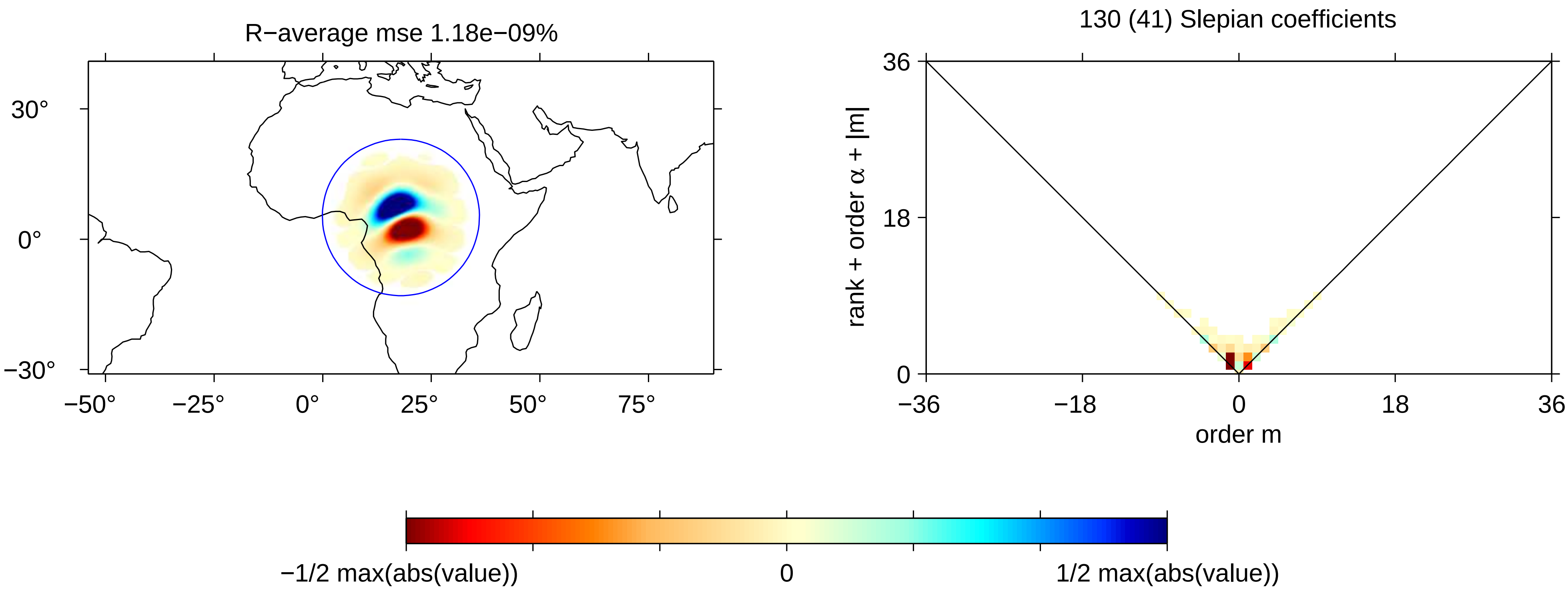
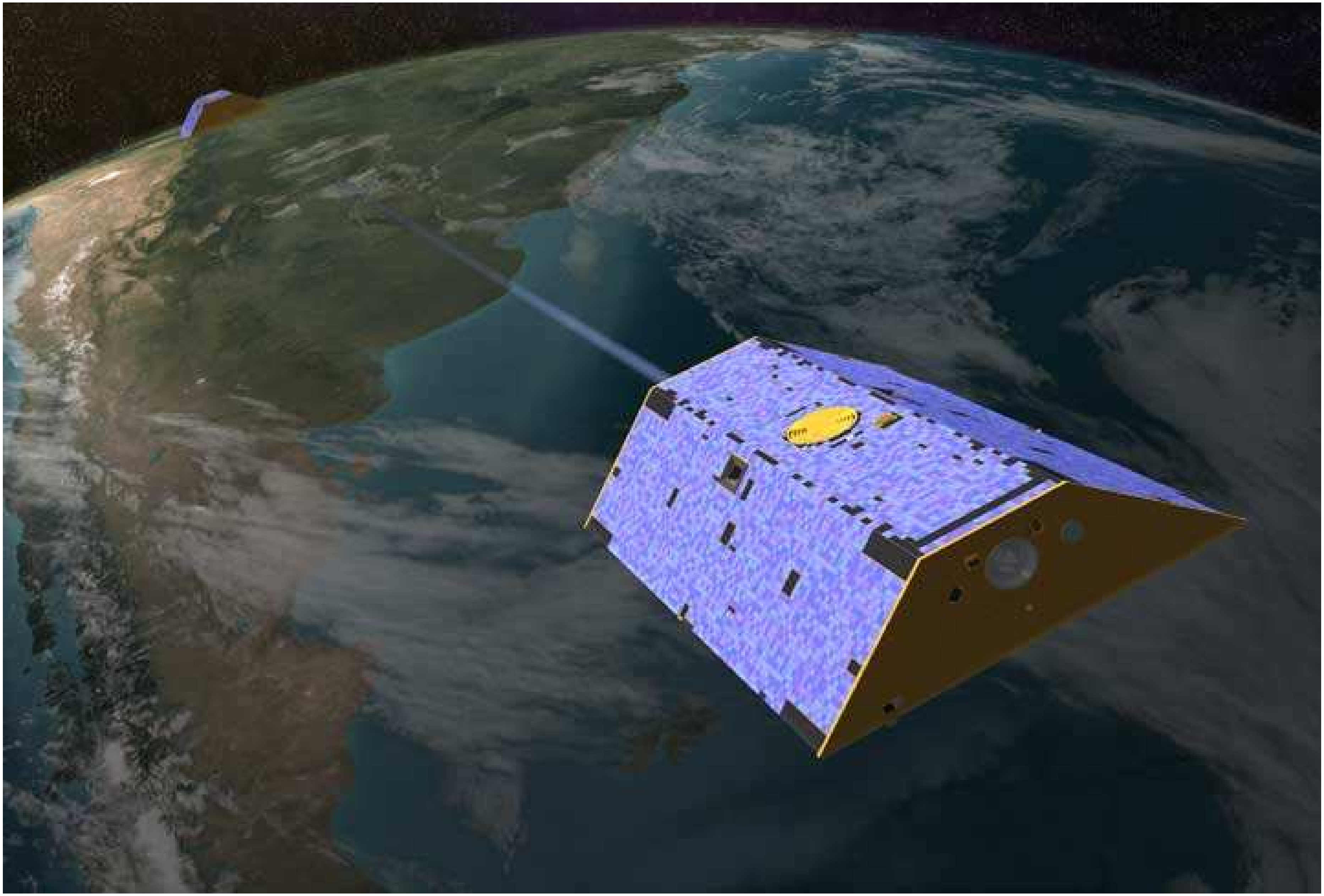
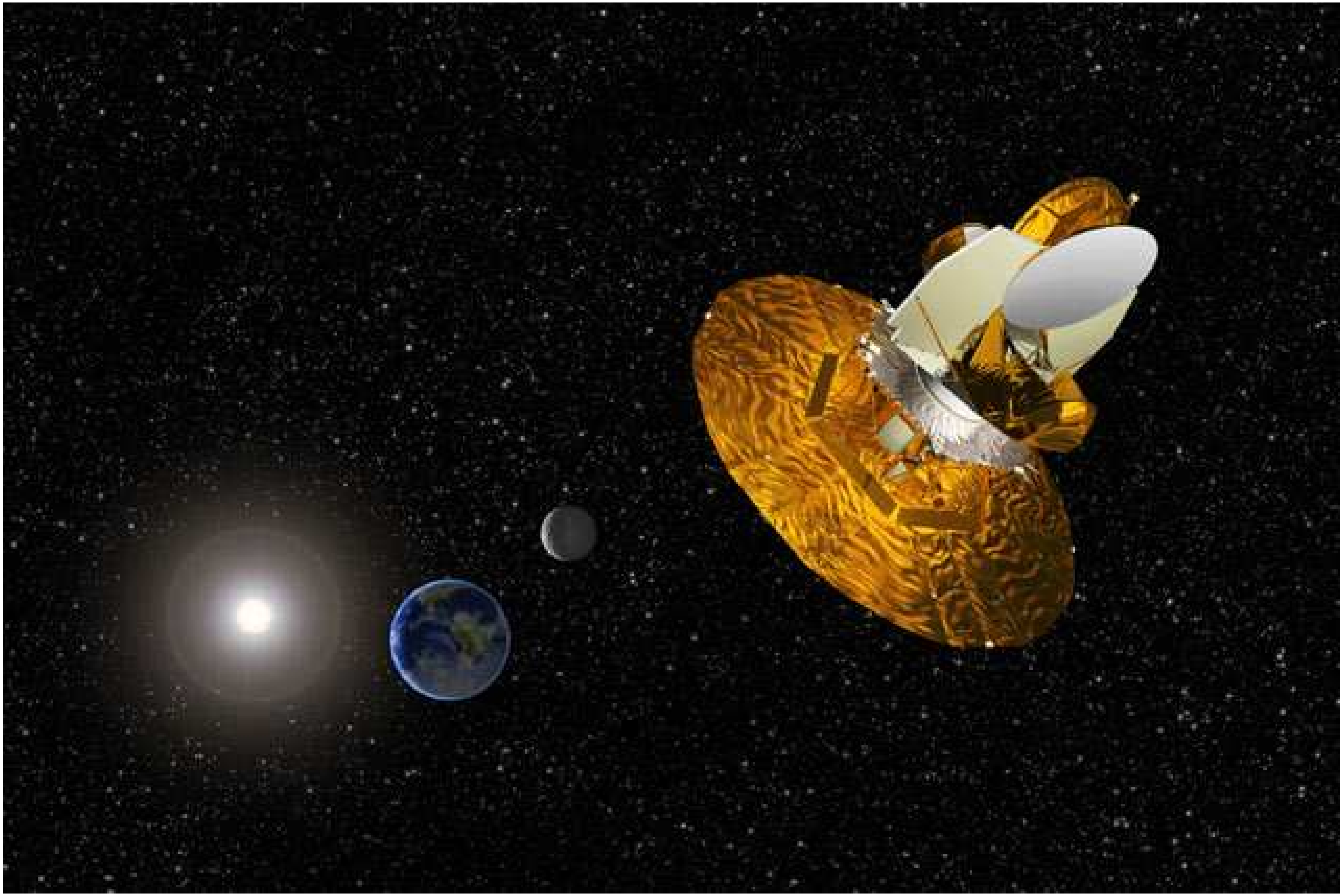


A *local* basis, **good** for *local* problems. Sparsity!

Solving problems in geophysics ...



... and cosmology



Common problems — 1

The data **collected in** or **limited to** R are **signal** plus **noise**:

We assume that $n(\mathbf{r})$ is **zero-mean** and **uncorrelated** with the signal

and consider known the **noise covariance**:

In other words: we've got **noisy** and **incomplete** data on the sphere.

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The data **collected in** or **limited to** R are **signal** plus **noise**:

$$d(\mathbf{r}) = \begin{cases} s(\mathbf{r}) + n(\mathbf{r}) & \text{if } \mathbf{r} \in R, \\ \text{unknown/undesired} & \text{if } \mathbf{r} \in \Omega - R. \end{cases}$$

We assume that $n(\mathbf{r})$ is **zero-mean** and **uncorrelated** with the signal

$$\langle n(\mathbf{r}) \rangle = 0 \qquad \text{and} \qquad \langle n(\mathbf{r}) s(\mathbf{r}') \rangle = 0,$$

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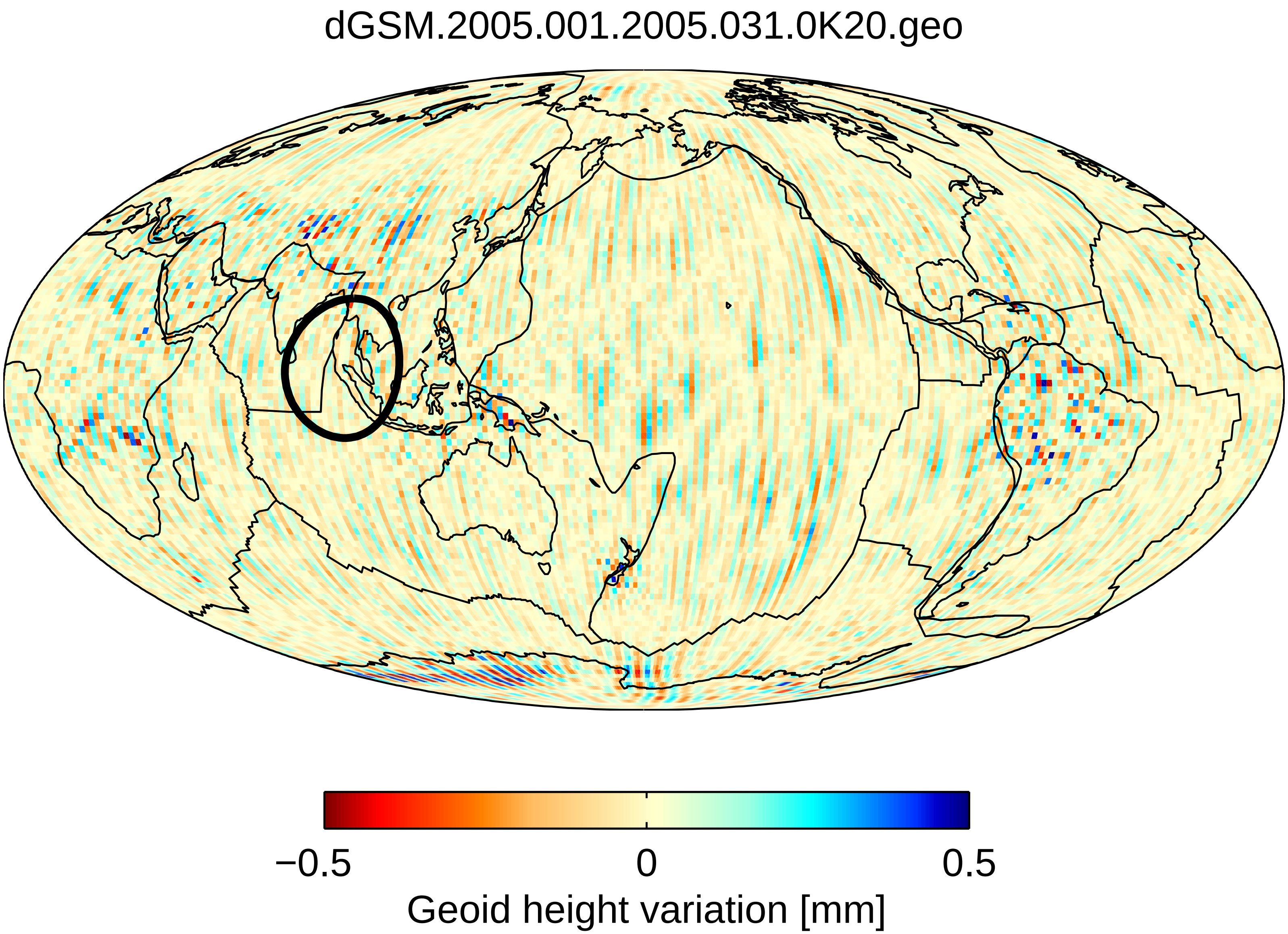
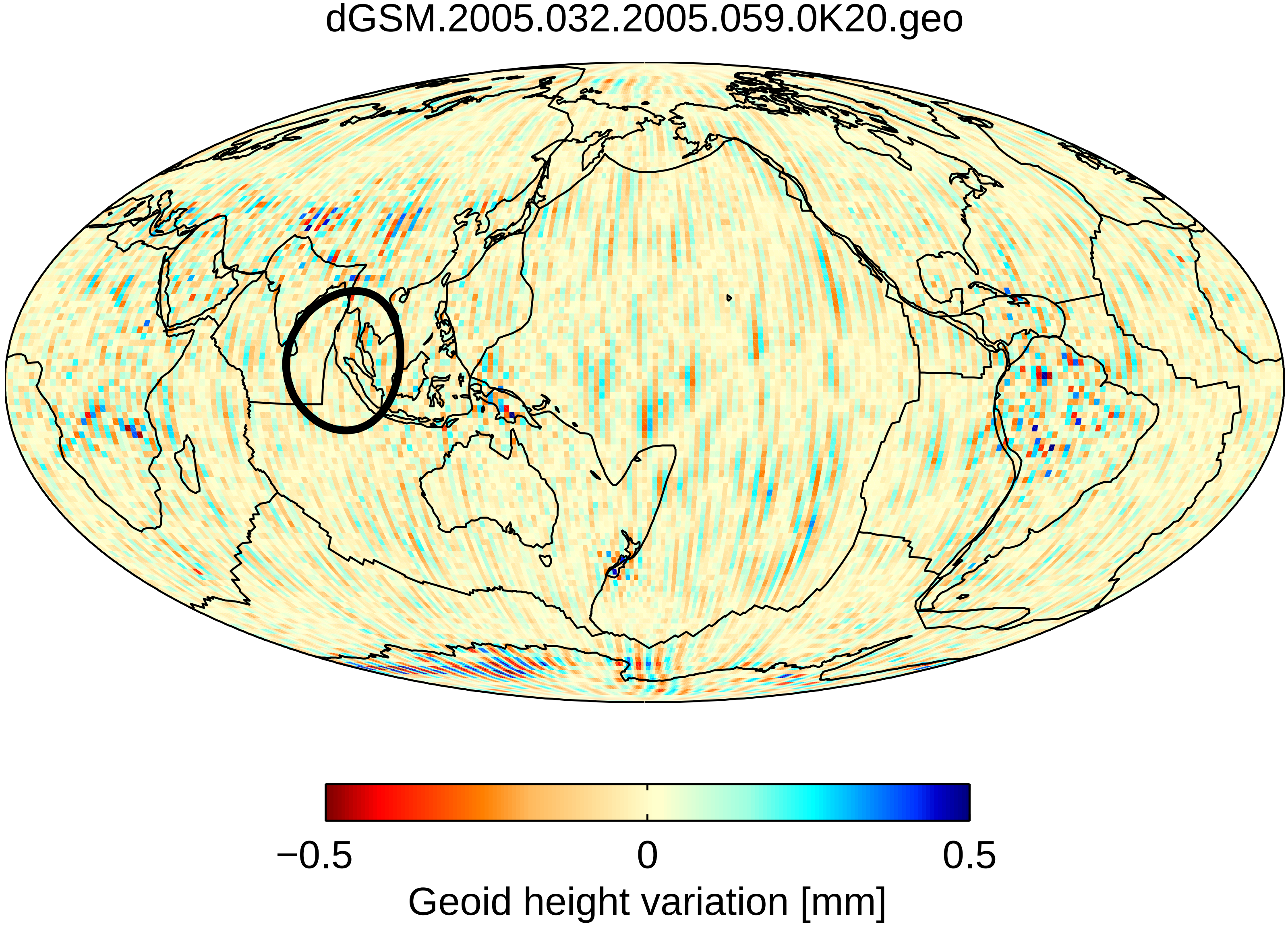
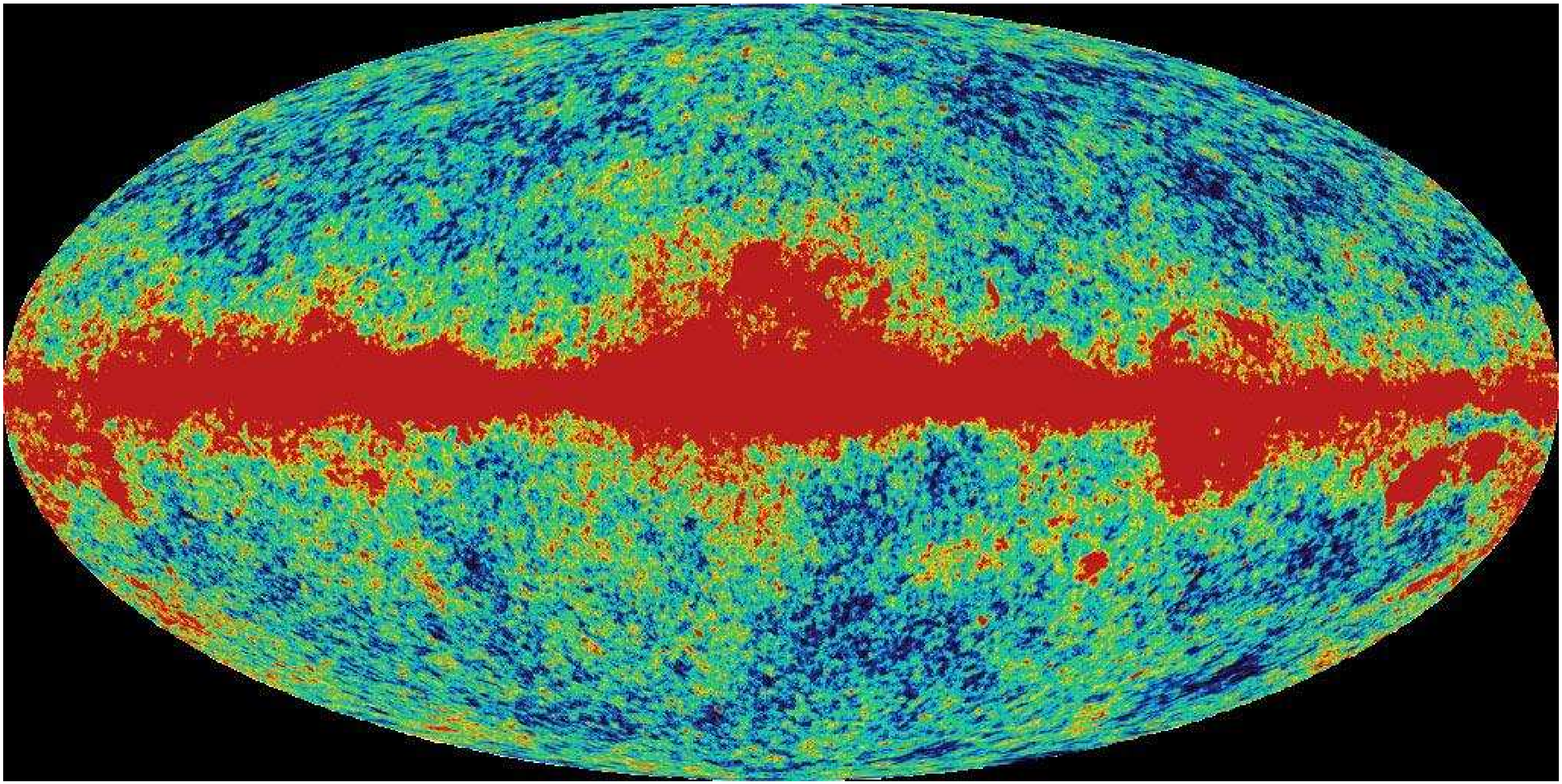
Noisy: Earth's time-variable gravity — 1



Noisy: Earth's time-variable gravity — 2



Incomplete: Cosmic Microwave Background



Common problems — 2

Consider an *unknown*, *noisily* and *incompletely observed* spherical process:

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Linear Problem: **Problem 1**

Quadratic Problem: **Problem 2**

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Linear Problem: **Problem 1**

Given $d(\mathbf{r})$ and $\langle n(\mathbf{r}) n(\mathbf{r}') \rangle$, *estimate* the signal $s(\mathbf{r})$, realizing that the estimate $\hat{s}(\mathbf{r})$ is **always bandlimited** to $0 \leq L < \infty$.

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Quadratic Problem: **Problem 2**

Given $d(\mathbf{r})$ and $\langle n(\mathbf{r}) n(\mathbf{r}') \rangle$, and assuming the field behaves as

$$\langle s_{lm} \rangle = 0 \qquad \text{and} \qquad \langle s_{lm} s_{l'm'} \rangle = S_l \, \delta_{ll'} \delta_{mm'},$$

estimate the **power spectral density** S_l, for $0 \leq l < \infty$, as $\hat{S}_l$.

Common problems — 2 (bis)

The data are **noisy** and **incomplete**.

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Problem 1

Find the **signal** that gives rise to the data.

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Find the **signal** that gives rise to the data.

Problem 2

Find the **power spectral density** of the signal.

Problem 1 — Finding the ***signal***

Construct a **bandlimited estimate** in the spherical harmonic basis by minimizing the **misfit to the data** over R. The—*linear*—optimal solution depends on D^{-1}:

$$\hat{s}_{lm} = \sum_{l'm'}^{L} D^{-1}_{lm,l'm'} \int_R d\, Y_{l'm'}\, d\Omega.$$

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Finding D^{-1} is tough, so construct a **truncated-Slepian basis** estimate instead:

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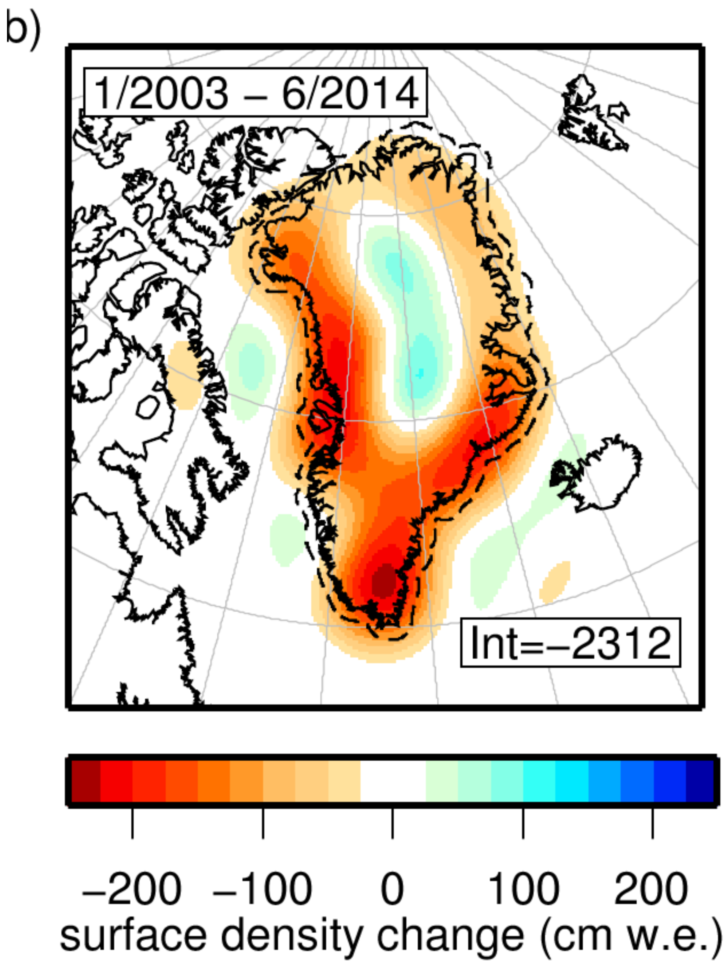
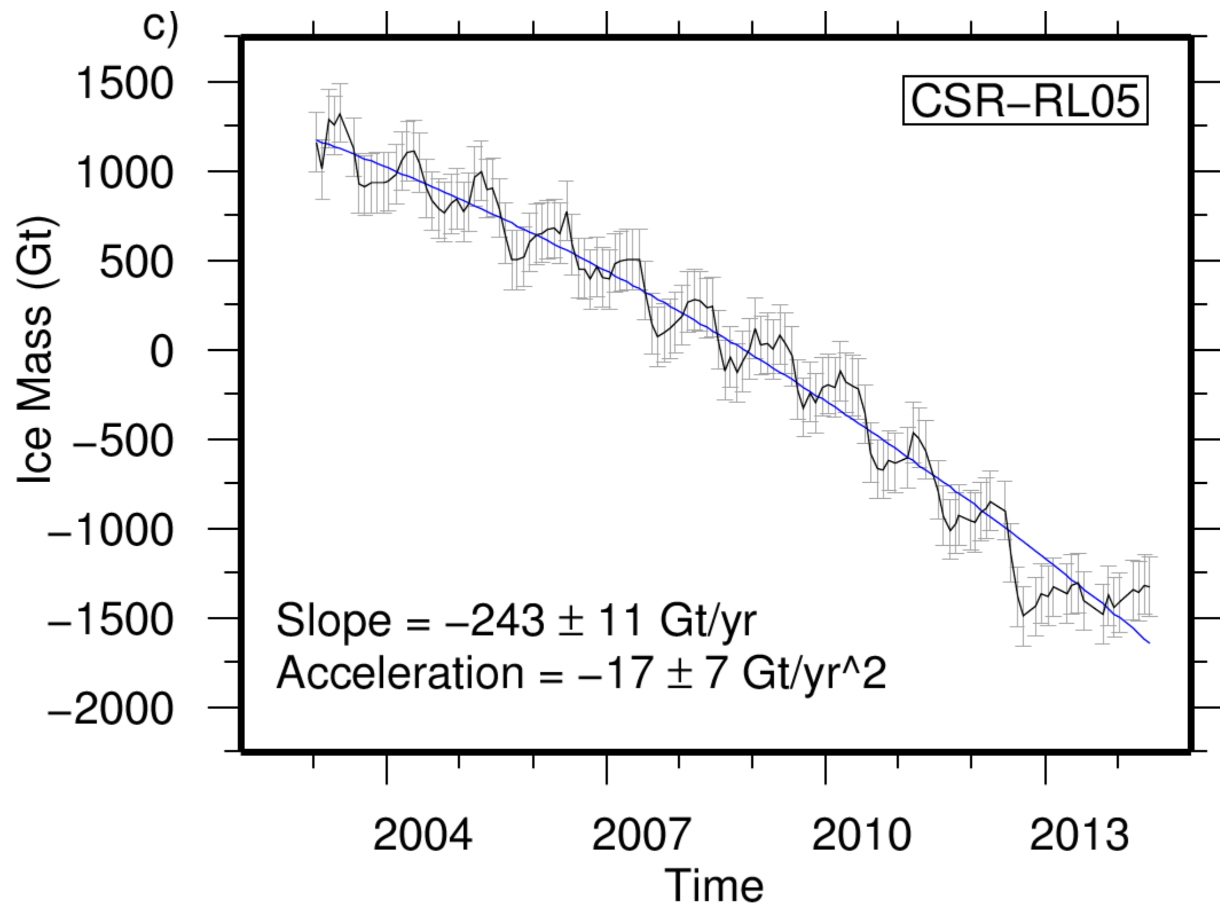
The solution depends on the localization **eigenvalue at the same rank**:

$$\hat{s}_\alpha = \lambda_\alpha^{-1} \int_R d\, g_\alpha\, d\Omega.$$

Application 2 : Time-variable gravity





Problem 2 — Finding the *spectrum*

If we simply worked with the available data we'd be using a **boxcar** window:

$$\hat{S}_l^{\mathrm{SP}} = \frac{1}{2l+1} \sum_m \left| \int_R d(\mathbf{r})\, Y_{lm}(\mathbf{r})\, d\Omega \right|^2 .$$

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This estimate is **biased** (unless $S_l = S$ or $R = \Omega$), *coupling* over the *entire* band.

Its **bias**, **variance**, and thus **mean-squared error** depend, again, on D:

$$\mathrm{mse}_l^{\mathrm{SP}} \sim \sum_{mm'} \left| D_{lm,lm'} \right|^2 .$$

Problem 2 — Finding the *spectrum*

If we simply worked with the available data we'd be using a **boxcar** window:

$$\hat{S}_l^{\mathrm{SP}} = \frac{1}{2l+1} \sum_m \left| \int_R d(\mathbf{r})\, Y_{lm}(\mathbf{r})\, d\Omega \right|^2 .$$

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The **multitaper estimate** uses a *small* L for the Slepian windows $g_\alpha(\mathbf{r})$ over R,

$$\boxed{\hat{S}_l^{\mathrm{MT}} = \sum_\alpha \lambda_\alpha \left(\frac{1}{2l+1} \sum_m \left| \int_\Omega g_\alpha(\mathbf{r})\, d(\mathbf{r})\, Y_{lm}(\mathbf{r})\, d\Omega \right|^2 \right).}$$

The *multitaper* method

It returns a **spectrally bandlimited** (to $\pm L$) average of the true spectral power while being sensitive to a **spatially localized** patch R of data.

Spectral and spatial concentration trade off via the **Shannon number**, which is the sole parameter to be chosen by the analyst:

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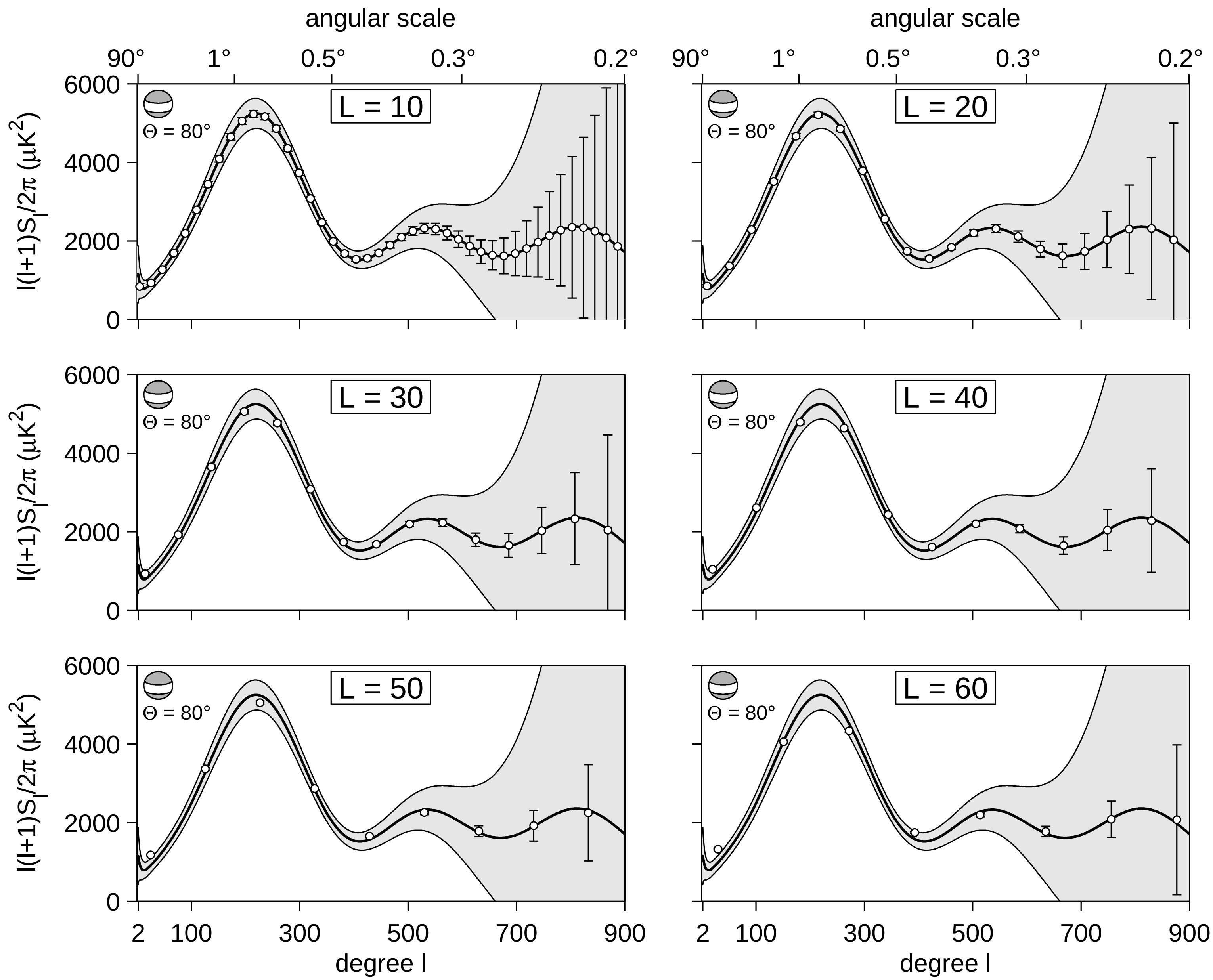
This dictates the deliberate **bias** of the estimate. More tapers $\longrightarrow$ more bias, but the **covariance** matrix of the estimates *between* tapers is almost **diagonal**.

Thus, **weighted averaging** of estimates made with many different tapers **reduces the estimation variance**. And with *eigenvalue weighting*, the bias is strictly limited to the bandwidth L, and **independent of the shape** of the region R.

Balancing *bias* and *variance*

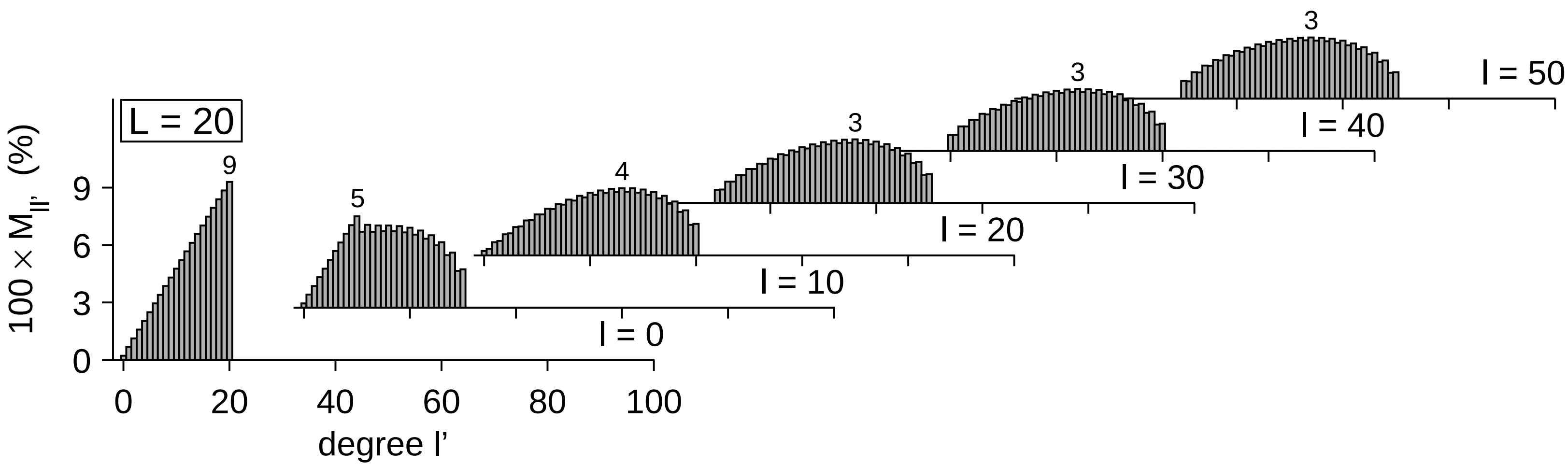


Multitaper *bias*

Using the choice of the **eigenvalues** λ of D as weights of the multitaper spectral estimate, the **multitaper coupling matrix** is

$$K_{ll'} = \frac{2l'+1}{(L+1)^2} \sum_{p}^{L} (2p+1) \begin{pmatrix} l & p & l' \\ 0 & 0 & 0 \end{pmatrix}^2 ,$$

which — amazingly — depends only upon the chosen bandwidth L.



Multitaper *variance* — 1

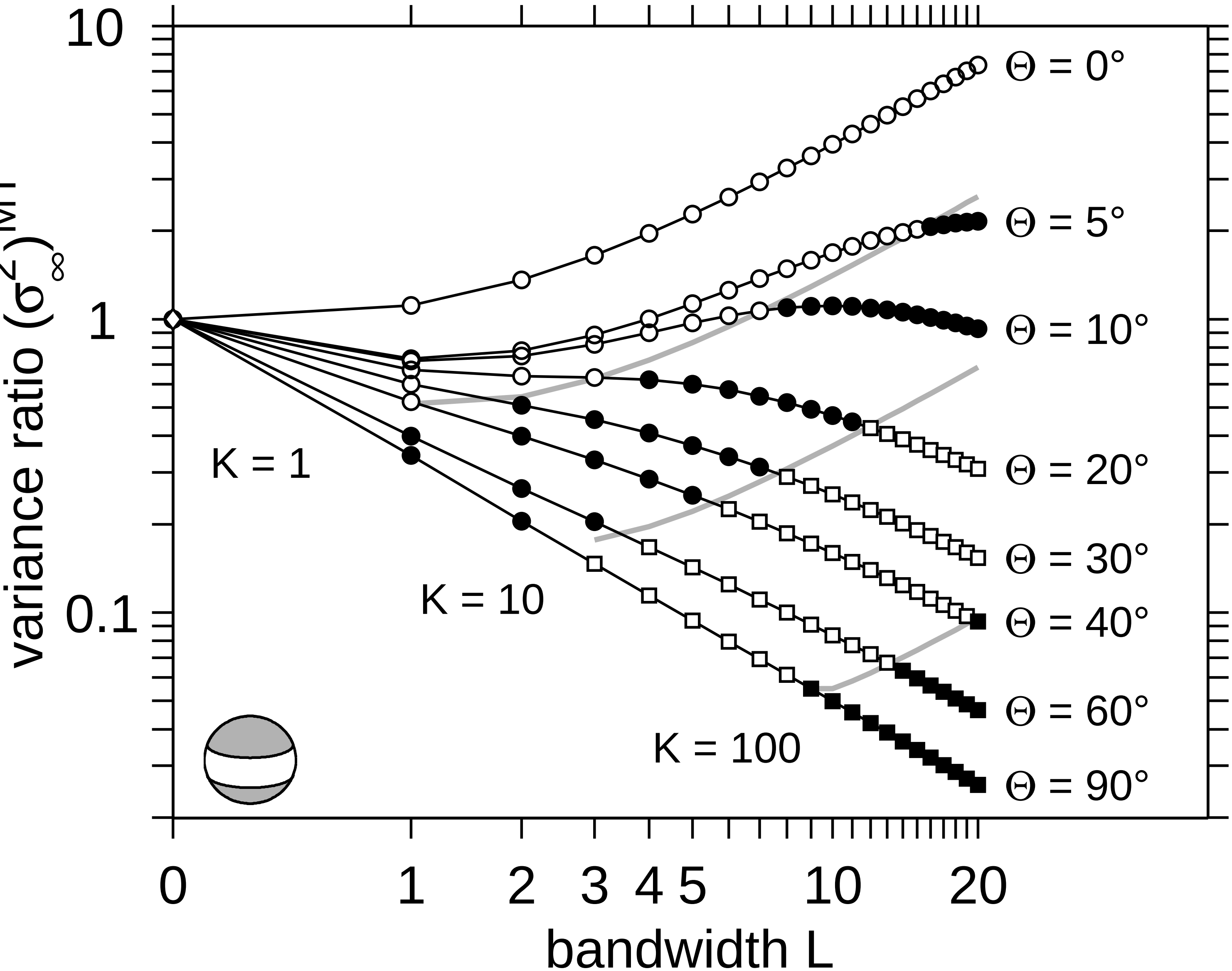
The **covariance** between the multitaper estimates is relatively simple when the spectra is **moderately colored** (compared to the bandwidth L of the estimator):

$$\Sigma_{ll'}^{\mathrm{MT}} = \frac{1}{2\pi}(S_l + N_l)(S_{l'} + N_{l'}) \sum_p (2p+1)\,\Gamma_p \begin{pmatrix} l & p & l' \\ 0 & 0 & 0 \end{pmatrix}^2, \quad (13)$$

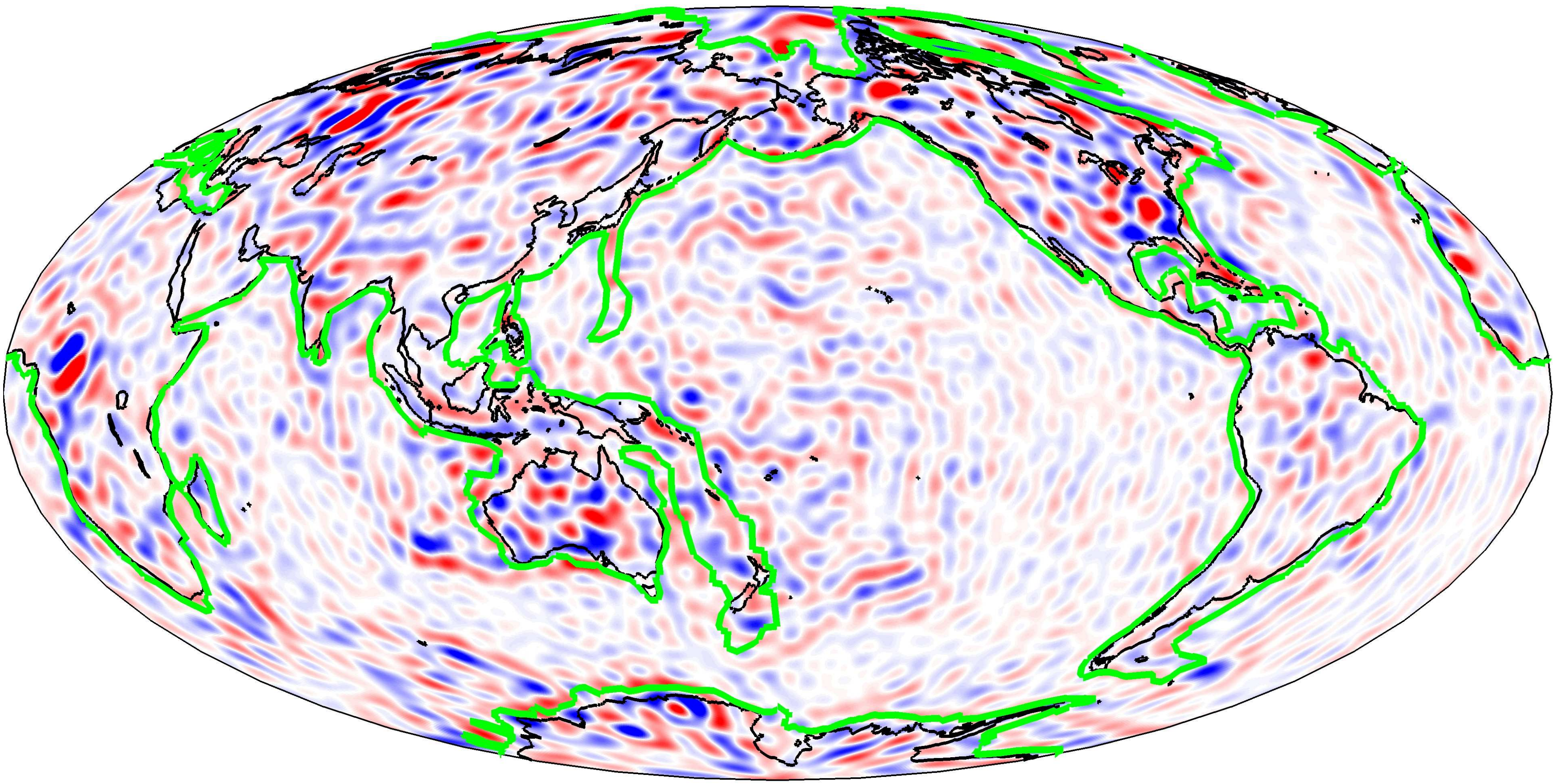
$$\Gamma_p = \frac{1}{K^2} \sum_{ss'}^{L} \sum_{uu'}^{L} (2s+1)(2s'+1)(2u+1)(2u'+1) \sum_e^{2L} (-1)^{p+e}(2e+1)B_e \times \begin{Bmatrix} s & e & s' \\ u & p & u' \end{Bmatrix} \begin{pmatrix} s & e & s' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u & e & u' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} s & p & u' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u & p & s' \\ 0 & 0 & 0 \end{pmatrix}, \quad (14)$$

with B_e the boxcar power, which depends on the **shape** of the region of interest, and the sums over angular degrees are limited by Wigner 3-j selection rules. The term in curly braces is a Wigner 6-j symbol. Ugly, but computable.

Multitaper *variance* — 2

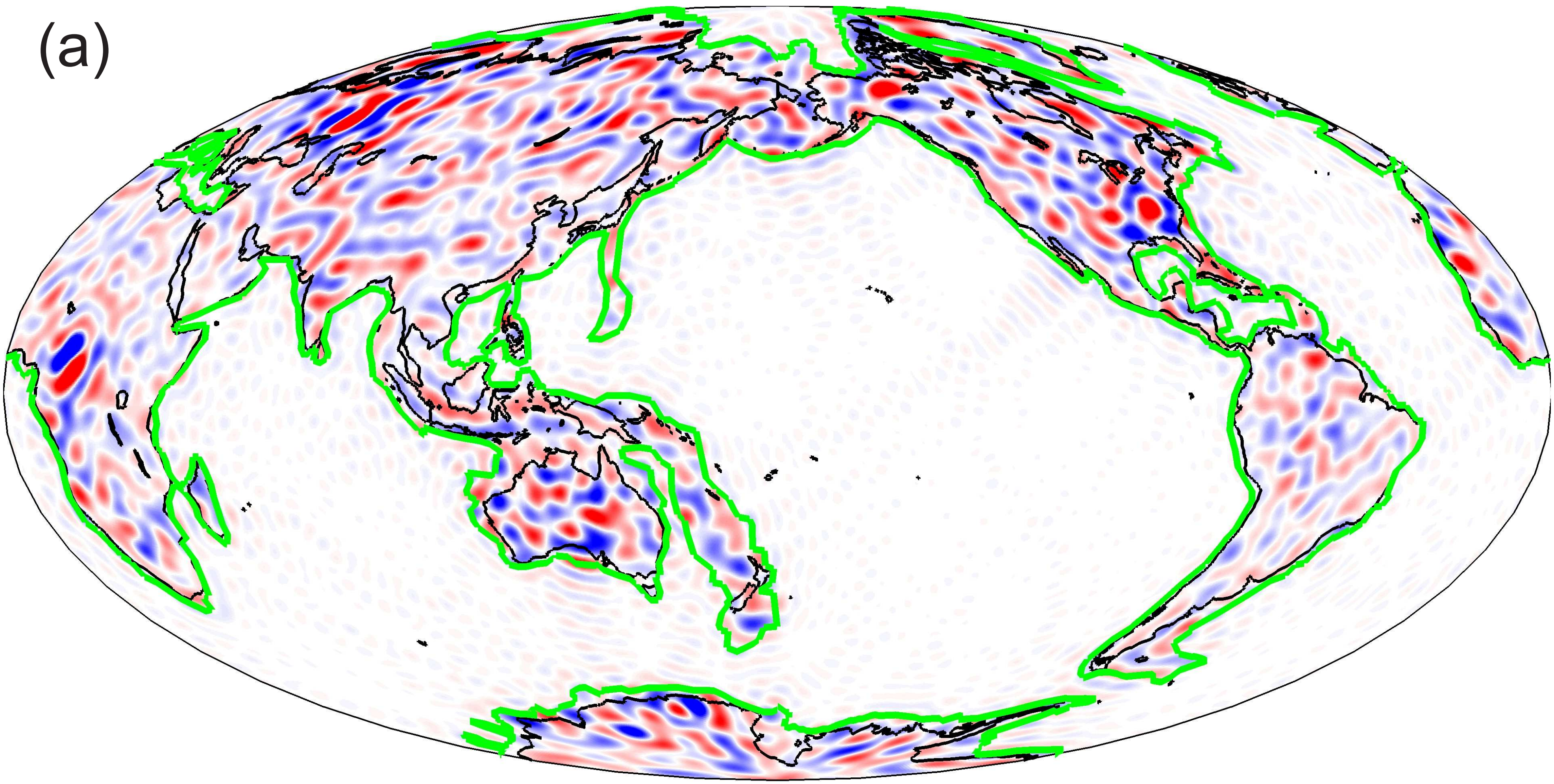


Application 3 : Magnetic field spectra



Application 3 : Magnetic field spectra

(a)



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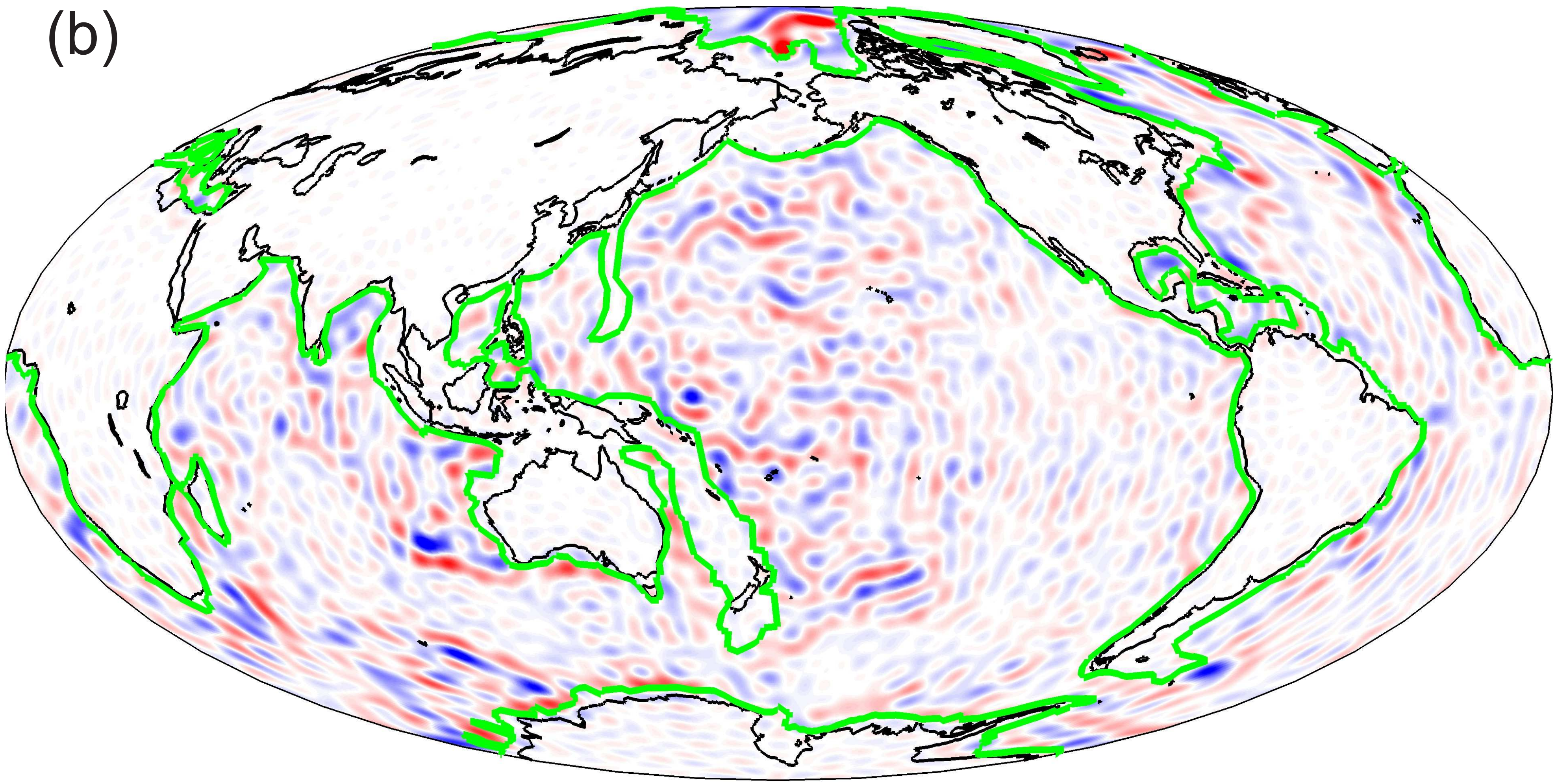

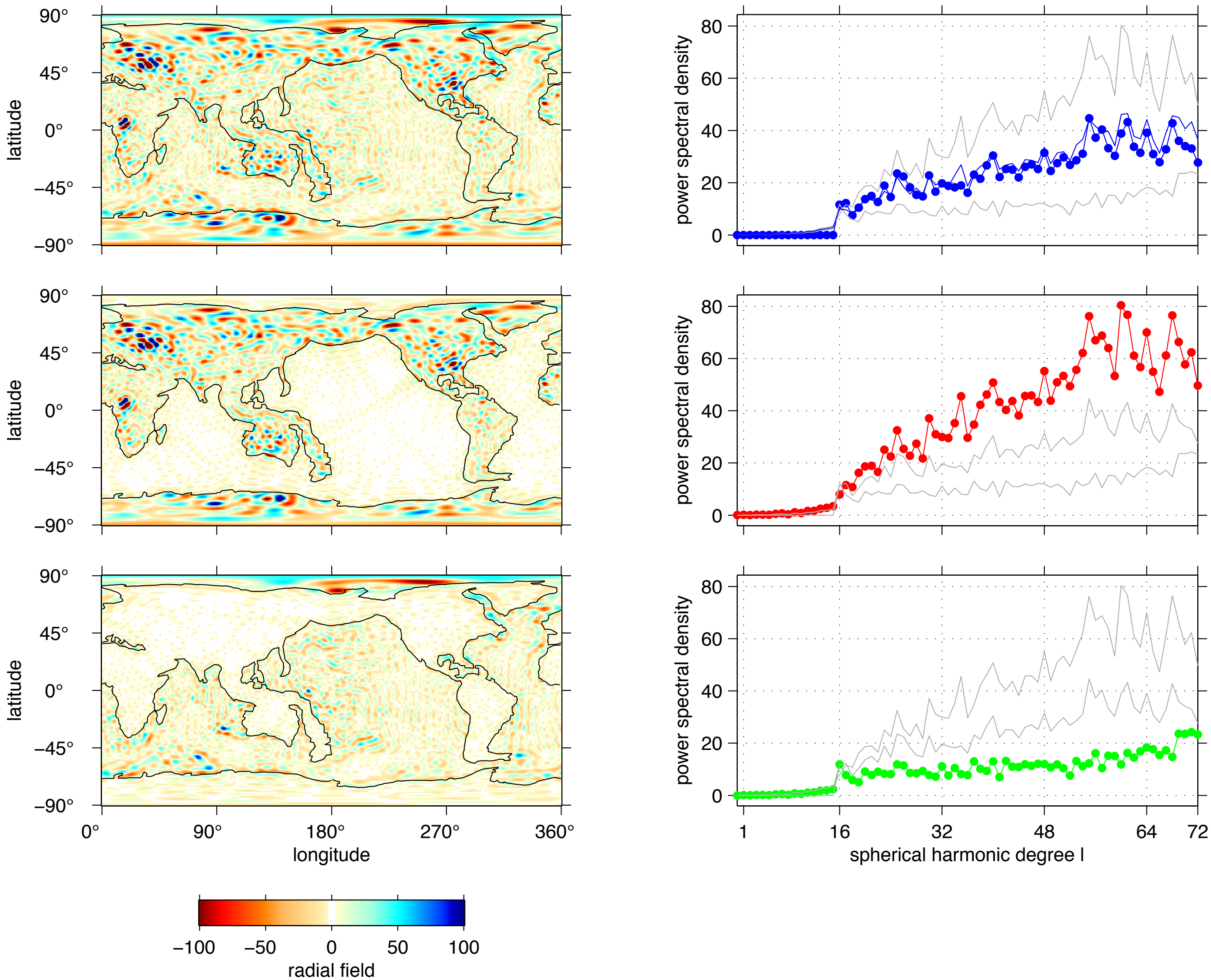
90°
45°
0°
–45°
–90°
latitude
0°
90°
180°
270°
360°
longitude
power spectral density
0
20
40
60
80
1
16
32
48
64
72
spherical harmonic degree l
–100
–50
0
50
100
radial field

Conclusions: Linear & Quadratic Problems

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- The **Slepian multitaper method** yields a smoothed and thus **biased** estimate of the spectrum, but it requires neither iteration nor large-scale matrix inversion. Its **variance is much lower** than that of any other method, and the only parameter that needs to be specified by the analyst is the **Shannon number**, or the space-bandwidth product diagnostic of the spatiospectral concentration.