A New Time-series Model Class Amenable to Multitaper Spectral Analysis for Cyclostationary and Stationary Processes

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Outline

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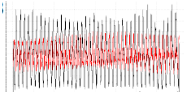
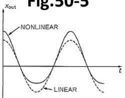
Harmonic Residuals and Nonlinearity

Voltage, Harmonic mean signal (volts)



Marshall et al. (2019)

Fig.50-5



Feynman et al. (2011)

Fig.5.2

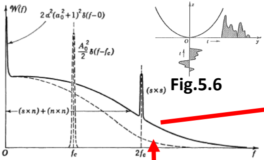
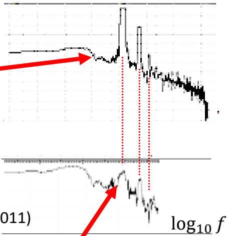


Fig.5.6

Middleton (1960)

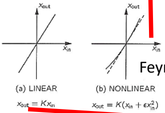


Residual voltage, Confidence interval (volts)

Kolmogorov, 1941

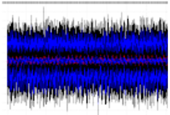
Burr (2012)
Marshall et al. (2018)

Fig.50-4



Feynman et al. (2011)

Stationary voltage, Mean, Median, 5% and 95% Quantiles (volts)



Time

Yaglom (1962)

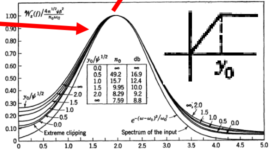


Fig.13.18

Middleton (1960)

Spectral Quantities

- Δt-discretization, $\mathcal{X}$, of X:
$$\mathcal{X} = \{\, X(t_n)\,\}_{n\in\mathbb{Z}}\,.$$
- **Discrete-time Fourier transform**, $\tilde{X}$, of $\mathcal{X}$:
$$\tilde{X}(f) = \sum_{n\in\mathbb{Z}} e^{-i2\pi fn} X(t_n).$$
- Spectral representation of $\mathcal{X}$:
$$X(t_n) = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{i2\pi fn}\,\tilde{Z}_\mu(df)$$
  - **Spectral measure**, $\tilde{Z}_\mu$, of $\mathcal{X}$:
$$\tilde{Z}_\mu(\,(f, f+\Delta f]\,) = \tilde{Z}(f+\Delta f) - \tilde{Z}(f).$$
  - $\tilde{Z}$ **the integrated spectrum** of $\mathcal{X}$.
- $\check{Z}$ **the normalized integrated spectrum** of $\mathcal{X}$.
$$\check{Z}(f) = \frac{1}{2}\lim_{\boldsymbol{\delta}\to\mathbf{0}^+_{2\times 1}} \left\{\, \tilde{Z}(f-\boldsymbol{\delta}[1]) + \tilde{Z}(f+\boldsymbol{\delta}[2])\,\right\}$$

Spectral Analysis,: Koopmans (1995); Percival & Walden (1993)

- **Desire**: Signals characterizing the $\tilde{X}$ FDD's.
-
- **Options**:
    -
    - Assumption: $\tilde{Z} = \tilde{X}$.
    -
    - Approximation: $\tilde{Z} \neq \tilde{X}. \Rightarrow \check{Z} \approx \tilde{Z}, \check{Z} \approx \tilde{X}$.
-
- **Reconstructions**: $\{\boldsymbol{Y_k}\}_{\boldsymbol{k=0}}^{\boldsymbol{K_N-1}}$**, the DFT-eigencoefficient processes**.
    -
    - Y_k reconstructs $\tilde{X}$.
    -
    - Y_k reconstructs $\check{Z}$.
-
- $\boldsymbol{X}$ **specification**: Y_k asymptotically complex-normal; $K_N = 2NW$.

Outline

1. A Physical Model
   - Motivation
   - A Design from Physical First Principals

2. Robust Modelling
   - Eigencoefficient Distribution Theory
   - DTFT: A Direct Characterization
   - DTFT: An Indirect Characterization

3. Conclusion

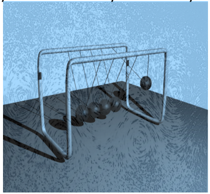
System Effects

Feynman et al. (2011)
Thornton & Marion (2004)

Model class

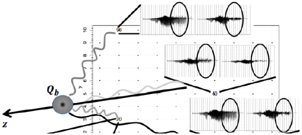
Coupled oscillations in a damping medium with turbulent driving $\mathcal{L}_{ODE}^{-1}$

http://convozine.com/blackcircle/25778



LTI-filtering $\mathcal{L}_{LTI}$

Prince & Links (2006)



Model component

$X^{(OUT)}(t_n)$

$\mathcal{L}_{LTI}$

$X^{(SLN)}(t_n)$

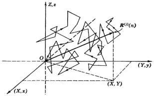
$\mathcal{L}_{ODE}^{-1}$

$X^{(DRV)}(t_n)$

$g^{(NLN)}$

$X^{(INP)}(t_n)$

Fig.7.1



NSS

Middleton (1960)

Standardized displacement

$X_0^{(STD)}$ $X_1^{(STD)}$ .... $X_u^{(STD)}$ ⇨ $X^{(GSN)}(t_n)$ + $X^{(INP)}(t_n)$

$X^{(CMP)}(t_n)$

Outline

Yaglom/Brockwell-Davis/Brillinger Stationary Noise

Kay et al. (1981)
Kolmogorov et al. (1960)
Martin et al. (1982)
Percival et al. (1993)
Rozanov (1990)
Slutsky (1937)
Stoica et al. (1999)

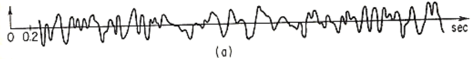
<u>**Model class**</u>

<u>**Yaglom (1962), Figs.1-3**</u>

NSS

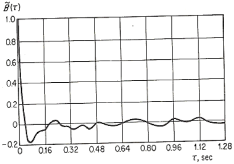
<u>**Fading radio intensity**</u>

Time series

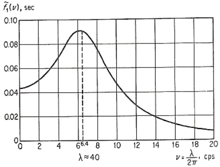


NSS

Autocorrelation



PSD



AR(1)

IID

<u>**Model component**</u>

~ ARMA(p,q) integrator
Brillinger (1981)
Brockwell & Davis (1991)
Corollaries 4.4.1, 4.4.2

~ AR(2) integrator
Shumway et al. (2017)
Wodeyar et al. (2021)

~ quadratic
~ AR(1) integrator
Andrews (1983)
Brockwell & Davis, 1991
Gray (1999)
Kiusalaas (2010)
Middleton (1960)
Marshall (2020)

$X^{(OUT)}(t_n)$

$\mathcal{L}_{LTI}$

$X^{(SLN)}(t_n)$

$\mathcal{L}_{ODE}^{-1}$

$X^{(DRV)}(t_n)$

$g^{(NLN)}$

$X^{(INP)}(t_n)$

Normality of the DFT-eigencoefficient Processes

Marshall (2020), Chapter 4

Brillinger (1981), Mallows (1967)
Rosenblatt (1961), Andrews (1983),
Somerset (2017), Springford (2017)

Model class

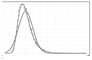
Innovations PDF (standard-normal overlay)

$$\boldsymbol{Y_k^{(OUT)}(f_m) \Rightarrow N^{(OUT)} \sim \textbf{Complex Normal}}$$

Model component

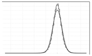
LAPTV+NSS

Gumbel



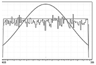
$$\mathrm{RMS}_{2K_N}\left\{\hat{\gamma}_{\mu}^{(STD)}\left({A_m}^2\right)\right\} = \left[1 - o\left(N^{-\alpha}\right)\right]\left|\sigma^{(INP)}\right|^2$$

Logistic



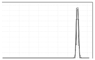
LAPTV+ARMA(1,q)

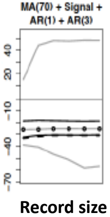
Uniform



ACS-AP+MA(q)

Student





$X^{(OUT)}(t_n)$

$\mathcal{L}_{LTI}$

$X^{(SLN)}(t_n)$

$\mathcal{L}_{ODE}^{-1}$

$X^{(DRV)}(t_n)$

$g^{(NLN)}$

$X^{(INP)}(t_n)$

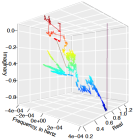
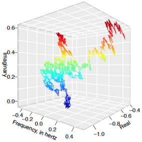
IID

Gumbel (G), Laplacian (L), Normal (N), Student (S), Uniform (U)

Outline

State-model Specification

Napolitano (2012)
Krishnan (1984)
Thomson (2000)

Layer	**State point**	**Assumption**	$\tilde{Z}^{(\#)}$ **trace**
Observable layer	$X^{(OUT)}$		
Hidden output layer	$\tilde{Z}^{(OUT)}$	$\tilde{Z}^{(OUT)} = \tilde{X}^{(OUT)}$	
Hidden driver layer	$\tilde{Z}^{(DRV)}$	$\tilde{Z}^{(DRV)} = \tilde{X}^{(DRV)}$	

Model Class

Model class

Regularity conditions

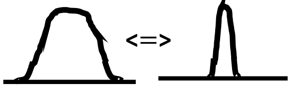
Reconstructed spectral process

Model component

ACS-AP

$\tilde{Z}^{(OUT)} = \tilde{X}^{(OUT)}$

$\mathcal{L}_{CSC}$ H_{CSC}



ACS-AP

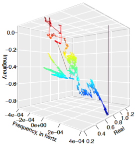
$\tilde{Z}^{(OUT)}$

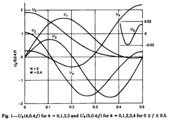
Slepian expansion error, $N \to \infty$

$O_p(1)$

Reconstruction composite FDD's , $N \to \infty$

Multivariate Gaussian





$X^{(OUT)}(t_n)$

$\mathcal{L}_{CSC}$

$X^{(DRV)}(t_n)$

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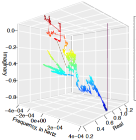
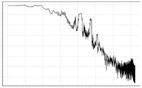
State-model Specification

Loeve (1963)
Thomson (1990)

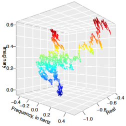
Layer	**State point**	**Assumption**
Observable layer	$X^{(OUT)}$	
Hidden output layer	$\check{Z}^{(OUT)}$	$\tilde{Z}^{(OUT)} \neq \tilde{X}^{(OUT)}$
Hidden driver layer	$\check{Z}^{(DRV)}$	$\tilde{Z}^{(DRV)} \neq \tilde{X}^{(DRV)}$



Spectral power

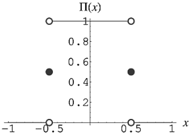


Frequency



Model Class

Model class	Regularity conditions	Reconstructed spectral process	Model component
LAPTV+NSS	$W \in (0, 0.5\pi^{-1})$	$\check{Z}^{(OUT)}$	$X^{(OUT)}(t_n)$
		Slepian expansion error, $N \to \infty$, $W \to 0$	$\mathcal{L}_{LTI}$
		$O_p(1)$	
LAPTV+ARMA(1,q)		**Reconstruction composite FDD's , $N \to \infty$**	$X^{(SLN)}(t_n)$
		Multivariate Gaussian	$\mathcal{L}_{ODE}^{-1}$
ACS-AP+MA(q)			$X^{(DRV)}(t_n)$
			$g^{(NLN)}$
IID GLNSU			$X^{(INP)}(t_n)$



https://mathworld.wolfram.com/RectangleFunction.html

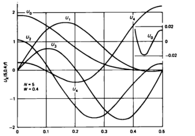


Fig. 1—$U_k(4,0.4;f)$ for $k = 0,1,2,3$ and $U_k(5,0.4;f)$ for $k = 0,1,2,3,4$ for $0 \le f \le 0.5$.

Conclusions - The Model Class

Convention

Model component	**Model class**
$X^{(OUT)}(t_n)$	**ACS-AP**
$\mathcal{L}_{LTI}$	$\mathcal{L}_{LTI}$
$X^{(SLN)}(t_n)$	**ACS-AP**
$\mathcal{L}_{ODE}^{-1}$	$\mathcal{L}_{ODE}^{-1}$
$X^{(DRV)}(t_n)$	**ACS-AP**

Proposed

Model component	**Model class**
$X^{(OUT)}(t_n)$	**LAPTV+NSS**
$\mathcal{L}_{LTI}$	
$X^{(SLN)}(t_n)$	**LAPTV+ARMA(1,q)**
$\mathcal{L}_{ODE}^{-1}$	
$X^{(DRV)}(t_n)$	**ACS-AP+MA(q)**
$g^{(NLN)}$	
$X^{(INP)}(t_n)$	**IID GLNSU**

References

- Andrews, D. W. (1983). First order autoregressive processes and strong mixing. Cowles Foundation Discussion Paper No. 664. Cowles Foundation for Research in Economics at Yale University.
- Babadi, B., & Brown, E. N. (2014). A review of multitaper spectral analysis. IEEE Transactions on Biomedical Engineering, 61(5), 1555-1564.
- Blackman, R. B., & Tukey, J. W. (1958). The measurement of power spectra Dover Publications. Inc, New York.
- Browne, S., Hargreaves, J. K., & Honary, B. (1995). An imaging riometer for ionospheric studies. Electronics & communication engineering journal, 7(5), 209-217.
- Burr, W. S. (2012). Air pollution and health: Time series tools and analysis. Doctoral dissertation. Queen's University.
- Corduneanu, C. (1989). Almost Periodic Functions, Chelsea Publ. Co., New York.
- Feynman, R. P., Leighton, R. B., & Sands, M. (2011). The Feynman lectures on physics, Vol. II: The new millennium edition (Vol. 1). Basic books.
- Gray, T. D. (1999). Analysis and Simulation of Ambient Noise Time Series. M.Sc. dissertation. Dalhousie University.
- Hecht, E. (2002). Optics 4th edition. Addison Wesley. San Fransisco.
- Kay, S. M., & Marple, S. L. (1981). Spectrum analysis—A modern perspective. Proceedings of the IEEE, 69(11), 1380-1419.
- Kelly, M., & Gráda, C. Ó. (2014). Change points and temporal dependence in reconstructions of annual temperature: did Europe experience a Little Ice Age?. The Annals of Applied Statistics, 8(3), 1372-1394.
- Kiusalaas, J. Numerical Methods in Engineering with Python (2010). Cambridge University Press.
- Kolmogorov, A. N. (1941). The local structure of turbulence in incompressible viscous fluid for very large Reynolds numbers. Cr Acad. Sci. URSS, 30, 301-305.
- Kolmogorov, A. N., & Rozanov, Y. A. (1960). On strong mixing conditions for stationary Gaussian processes. Theory of Probability & Its Applications, 5(2), 204-208.
- Koopmans, L. H. (1974). The spectral analysis of time series. Academic Press, Inc.
- Krishnan, V. (1984). Nonlinear filtering and smoothing: An introduction to martingales, stochastic integrals and estimation. Dover Publications. New York.
- Lea, S. (2004). Mathematics for physicists. Belmont, CA: Brooks/Cole-Thomson Learning.
- Lepage, K. (2009). Some advances in the multitaper method of spectrum estimation. Doctoral dissertation. Queen's University.
- Loève, M. (1963). Probability theory. D. van Nostrand Co.
- Mallows, C. L. (1967). Linear processes are nearly Gaussian. Journal of Applied Probability, 4(2), 313-329.

References

- Marshall, F. A., Takahara, G., & Thomson, D. J. (2018, June). A Multitaper Test For The Detection of Non-Stationary Processes Using Canonical Correlation Analysis. In 2018 IEEE Statistical Signal Processing Workshop (SSP) (pp. 702-706). IEEE.
- Marshall, F. (2020) Advances in the Detection and Characterization of Nonstationary Processes: An Application to Riometers. Doctoral dissertation. Queen's University.
- Marshall, F. A., Thomson, D. J., Takahara, G., & Fiori, R. A. (2019, November). A Multitaper Model for Quiet Voltage in Relative Ionospheric Opacity Meters. In 2019 IEEE Global Conference on Signal and Information Processing (GlobalSIP) (pp. 1-5). IEEE.
- Marshall, F. A., Thomson, D. J., Chave, A. D., Fiori, R. A. D., & Danskin, D. W. (2020). A characterization of periodicity in the voltage time series of a riometer. Journal of Geophysical Research: Space Physics, 125(7), e2019JA027160.
- Martin, R. D., & Thomson, D. J. (1982). Robust-resistant spectrum estimation. Proceedings of the IEEE, 70(9), 1097-1115.
- Mellors, R. J., Vernon, F. L., & Thomson, D. J. (1998). Detection of dispersive signals using multitaper dual-frequency coherence. Geophysical Journal International, 135(1), 146-154.
- Middleton, D., & Institute of Electrical and Electronics Engineers. (1960). An introduction to statistical communication theory (Vol. 960). New York: McGraw-Hill.
- Moghtaderi, A. (2009). Multitaper Methods for Time-Frequency Spectrum Estimation and Unaliasing of Harmonic Frequencies. Doctoral dissertation. Queen's University.
- Moghtaderi, A., Takahara, G., & Thomson, D. J. (2009). Evolutionary spectrum estimation for uniformly modulated processes with improved boundary performance. In 2009 IEEE International Conference on Acoustics, Speech and Signal Processing (pp. 2993-2996). IEEE.
- Napolitano, A. (2012). Generalizations of cyclostationary signal processing: spectral analysis and applications. John Wiley & Sons.
- Øigård, T. A., Hanssen, A., & Scharf, L. L. (2006). Spectral correlations of fractional Brownian motion. Physical Review E, 74(3), 031114.
- Percival, D. B., & Walden, A. T. (1993). Spectral analysis for physical applications. cambridge university press.
- Prince, J. L., & Links, J. M. (2006). Medical imaging signals and systems (Vol. 37). Upper Saddle River: Pearson Prentice Hall.
- Rosenblatt, M. (1961). Some comments on narrow band-pass filters. Quarterly of Applied Mathematics, 18(4), 387-393.
- Rozanov, Y. A. (1990). On the theory of generalized random functions. Theory of Probability & Its Applications, 34(1), 197-200.
- Schevon, C. A., Weiss, S. A., McKhann, G., Goodman, R. R., Yuste, R., Emerson, R. G., & Trevelyan, A. J. (2012).

References

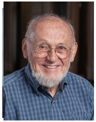
- Shumway, R. H., & Stoffer, D. S. (2017). ARIMA models. In Time series analysis and its applications (pp. 75-163). Springer, Cham.
- Slepian, D. (1976). On bandwidth. Proceedings of the IEEE, 64(3), 292-300.
- Slepian, D. (1978). Prolate spheroidal wave functions, Fourier analysis, and uncertainty—V: The discrete case. Bell System Technical Journal, 57(5), 1371-1430.
- Slutsky, E. (1937). The Summation of Random. Causes as the Source of Cyclic Processes. Econometrica 5105–146.
- Smith, E. H., Liou, J. Y., Davis, T. S., Merricks, E. M., Kellis, S. S., Weiss, S. A., ... & Schevon, C. A. (2016). The ictal wavefront is the spatiotemporal source of discharges during spontaneous human seizures. Nature communications, 7(1), 1-12.
- Springford, A. (2017). Spectral analysis of time series with latent and irregular times. Doctoral dissertation. Queen's University.
- Somerset, E. (2017). Multitaper methods for cyclostationary feature detection in time series data: application to ACE interplanetary magnetic field data. Doctoral dissertation. Queen's University.
- Stoica, P., & Sundin, T. (1999). On nonparametric spectral estimation. Circuits, Systems and Signal Processing, 18(2), 169-181.
- Thomson, D. J. (1982). Spectrum estimation and harmonic analysis. Proceedings of the IEEE, 70(9), 1055-1096.
- Thomson, D. J., Maclennan, C. G., & Lanzerotti, L. J. (1995). Propagation of solar oscillations through the interplanetary medium. Nature, 376(6536), 139-144.
- Thomson, D. J. (2000). Multitaper analysis of nonstationary and nonlinear time series data. Nonlinear and nonstationary signal processing.
- Thomson, D. J., Lanzerotti, L. J., & Maclennan, C. G. (2001). Interplanetary magnetic field: Statistical properties and discrete modes. Journal of Geophysical Research: Space Physics, 106(A8), 15941-15962.
- Thomson, D.J. (2013). Scientific spectrum estimation: Advanced multitaper methods of time-series data analysis. Course notes. Queen's University. Kingston, Canada.

References

- Thompson, K. (2018). A modified form of Loève spectrum with environmental applications. CANSSI project closing workshop: modern spectral methods in time series analysis: applications in physical science, environmental science, and computer modeling. Queen's University, Kingston, Canada.
- Thornton, S., & Marion, J. B. (2004). Classical Dynamics of Particles and Systems Brooks. Cole, New York.
- von Neumann, J. (1934). Almost periodic functions in a group. I. Transactions of the American Mathematical Society, 36(3), 445-492.
- Weiss, S. A., Banks, G. P., McKhann Jr, G. M., Goodman, R. R., Emerson, R. G., Trevelyan, A. J., & Schevon, C. A. (2013). Ictal high frequency oscillations distinguish two types of seizure territories in humans. Brain, 136(12), 3796-3808.
- Wodeyar, A., Schatza, M., Widge, A. S., Eden, U. T., & Kramer, M. A. (2021). A state space modeling approach to real-time phase estimation. Elife, 10, e68803.
- Yaglom, A. M. (1962). Stationary Random Functions (translated and edited by RA Silverman. Dover Publications.
- Yousefi, A., Fard, R. S., Eden, U. T., & Brown, E. N. (2019). State-space global coherence to estimate the spatio-temporal dynamics of the coordinated brain activity. In 2019 41st Annual International Conference of the IEEE Engineering in Medicine and Biology Society (EMBC) (pp. 5794-5798). IEEE.

Thank you





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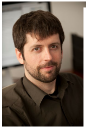


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Thank you

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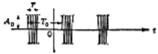
Nonlinear Processes

Middleton (1960)

TABLE 5.1. SOME REPRESENTATIVE SIGNAL WAVEFORMS

$\{u\}$; description $(c = 0)$	$T_T\{u\} = x(t) = S(t)$ Analytical expression	Wave structure of typical member
1. Sinusoid $(a = 1, b = 0)$; $A(t) = A_0$	$S(t) = A_0 \cos(\omega_c t + \phi)$	
2. Sinusoidally modulated carrier (amplitude modulation) $(a = 1, b = 0)$	$S(t) = A_0(1 + \lambda \cos \omega_a t) \cos(\omega_c t + \phi)$	
3. Pulse-modulated carrier (simple radar) $(a = 1, b = 0)$ $(\tau \ll T_0)$	$S(t) = \left[\sum_k A_0 \Delta_k(t)\right] \cos(\omega_c t + \phi)$ $\Delta_k(t) = \begin{cases} 1 & T_0 k - \frac{\tau}{2} < t < T_0 k + \frac{\tau}{2} \\ 0 & \text{elsewhere} \end{cases}$	
4. Random amplitude modulation of a carrier $(a = 1, b = 0)$; $A(t)$ random	$S(t) = \begin{cases} A(t) \cos(\omega_c t + \phi) & A > 0 \\ 0, & A < 0 \end{cases}$ $(S = 0$; overmodulation)	
5. Simple sinusoidal angle modulation $(a = 1, b = 1)$; $A = A_0$	$S(t) = A_0 \cos(\omega_c t + \phi + \alpha_0 \cos \omega_a t)$	
6. Random angle modulation of a carrier $(a = 1, b = 1)$; $A = A_0$	$S(t) = A_0 \cos[\omega_c t + \phi + \Phi(t)]$	
7. Simultaneous amplitude and frequency modulation; general modulations $(a = b = 1)$	$S(t) = \begin{cases} A(t) \cos[\omega_c t + \phi + \Phi(t)] & A > 0 \\ 0 & A < 0 \end{cases}$ $(S = 0$; overmodulation)	

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Nonstationarity and Filtering

- **Goal**: Infer input autocorrelation structure.
- NSS input - motivation:
  - Spectral-correlation detectors for nonstationarity.
  - Distribution analysis.
  - Robustness, model validation.

Class	Model
Mellors et al. (1998): Seismic surface waves Input: NSS Output: Contaminated narrowband, period-$M\Delta t$ signal	$X^{(OUT)}(t) = \sum_{m=1}^{0.5M-1} x_m^{(INP)}\left(t + [2\pi f_m]^{-1}\cdot\phi_m(t)\right) + X^{(INP)}(t).$
Thompson (2018): La Nina Gulf Stream eddies Input: NSS Output: Narrowband, ACS, period-$M\Delta t$ AP	$X^{(OUT)}(t) = \sum_{m=1}^{0.5M-1} X_m^{(INP)}\left(t + [2\pi f_m]^{-1}\cdot\Phi_1(t)\right).$
Brillinger (1993), Schevon et al. (2012), Smith et al. (2016), Weiss et al. (2013): California earthquake mag. 1932-1992 Multiunit neuron activity in seizures Input: NSS Output: Stationary-increment process	$X^{(OUT)}(t) = \int_0^t X^{(INP)}(s)ds$

Why Stationary Input?

- Model validation: Segregates the base and all upper layers by autocorrelation functionality. Discrete NSS processes are functions of IID sequences (Brillinger, 1981).
-
- Nonstationarity detectors: Specifies the null hypothesis.
-
- Distribution analysis: Specifies the stationary approximation.
-
- Robustness: $\mathrm{ARIMA}(0,1,0) \sim \mathrm{AR}(1) \sim \mathrm{ARMA}(1)$. The lower down the layers the inference is to be conducted, the fewer the parameters (model complexity inversely-proportional to layer height).
- Physical foundations: Thermal, scatter noise: normal, first-order Markovian processes (Middleton (1960), Chapter 7).
-
- Model/test performance: Computationally-inexpensive simulations.

Stationary-input/Nonstationary-output Systems

Class	Model
Thompson (2018): La Nina Gulf Stream eddies Input: NSS Output: Narrowband, ACS, period-$M\Delta t$ AP	$X_m^{(INP)}(t) = A_m \cdot \cos\left(2\pi m \cdot [M\Delta t]^{-1} t\right).$ $X^{(OUT)}(t) = \sum_{m=1}^{0.5M-1} X_m^{(INP)}\left(t + [2\pi f_m]^{-1} \cdot \Phi_1(t)\right).$
Moghtaderi et al., 2009a,b Input: NSS Output: Uniformly-modulated	$X^{(OUT)}(t) = \int_{-\infty}^{\infty} e^{i2\pi f \cdot \|\Delta t\|^{-1} t} \cdot H^{(IOS)}(t)\, \tilde{Z}_\mu^{(INP)}(df)$ $\mathrm{argmax}_{\xi \in \mathbb{R}} \left\| \mathscr{F}\left\{ H^{(IOS)}(t) \right\}(\xi) \right\| = 0$
Øigård (2006), Napolitano (2012) Section 4.2.4.4: # earthquakes mag.$>$ 7.0 Input: Gaussian NSS Output: fBM	$X^{(OUT)}(a\tau) = \int_{-\infty}^{\infty} e^{i2\pi f \cdot \|\Delta t\|^{-1} a\tau} H^{(IOS)}(f,t) Z_\mu^{(INP)}(df)$ $H^{(IOS)}(f,t) = a^{-H} \cdot e^{-i2\pi f \|\Delta t\|^{-1} t}$ Self-similarity: $X^{(OUT)}(t_n) = m^{-H} X^{(INP)}(t_n; t_m)$ $X^{(INP)}(t_n; t_m) = X^{(OUT)}(t_n) - X^{(OUT)}(t_n - t_m)$

Stationary-input/Nonstationary-output Systems

Class	Model
Mellors et al. (1998): Seismic surface waves Input: NSS Output: Contaminated narrowband, period-$M\Delta t$ signal	$x_m^{(DSP)}(t) = a_m \cdot \cos\left(2\pi m \cdot [M\Delta t]^{-1} t\right).$ $X^{(OUT)}(t) = \sum_{m=1}^{0.5M-1} x_m^{(INP)}\left(t + [2\pi f_m]^{-1} \cdot \phi_m(t)\right) + X^{(INP)}(t).$
Brillinger (1993), Schevon et al. (2012), Smith et al. (2016), Weiss et al. (2013): California earthquake mag. 1932-1992 Multiunit neuron activity in seizures Input: NSS Output: Stationary-increment process	$X^{(OUT)}(t) = \int_0^t X^{(INP)}(s)ds$

- $X^{(INP)}$ unifrequency spectral mass: Linear-algebra approach.
-
- $X^{(OUT)}$ bifrequency spectral mass:
-
-
    - $Z_{\mu}^{(INP)} = Z^{(INP)} d\lambda_1.$
    -
    - $Z^{(INP)}$: Integrated spectrum of $X^{(INP)}$ (Napolitano, 2012).
    -
    - $Z^{(INP)}(f + f_m)$ Slepian expansion.

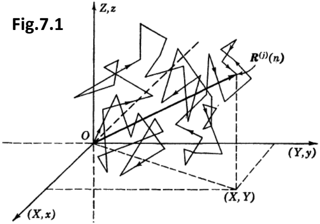
Multitaper Cyclostationary Analysis - Overcoming Limitations

- $Z^{(INP)}$ is not a derivative. Require:
  1. Spectral quantity under analysis.
  2. Slepian-expansion theory for that quantity (e.g., $2NW$-theorem).
  3. Errors to compare old and new models.
-
- $X^{(OUT)}$ DFT-eigencoefficient processes: justify normality. Require:
  1. ACS-model having accurate NSS-approximation.
  2. NSS-approximation must satisfy the Marshall (2022) conditions.
-
- Robustness. Require:
  1. $X^{(OUT)} \sim X^{(INP)}$ spectral associations invariant to linear effects.
  2. $X^{(OUT)}$ bifrequency distribution $\rightarrow X^{(INP)}$ low-dimensional subspace.

Model class | **Sources of nonstationarity** | **Model component**

Standardized displacement

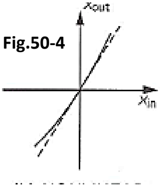
Fig.7.1



Middleton (1960)

$g^{(NLN)}$ nonlinear functional

Fig.50-4



Feynman et al. (2011)

NSS $X_0^{(STD)}$ $X_1^{(STD)}$ .... $X_u^{(STD)}$ ⇒ $X^{(GSN)}(t_n)$ + $X^{(INP)}(t_n)$

$X^{(CMP)}(t_n)$

$X^{(DRV)}(t_n)$ — $g^{(NLN)}$ — $X^{(INP)}(t_n)$

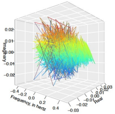
Estimating AP-model

Model class	Estimating stochastic AP function	Model component
LAPTV+NSS	$Z_{\mu}^{(OUT)}(A_{-0.5M+1}) \oplus Z_{\mu}^{(OUT)}(A_{-0.5M+2}) \oplus \cdots \oplus Z_{\mu}^{(OUT)}(A_{0.5M-1})$	$X^{(OUT)}(t_n)$
		$\mathcal{L}_{LTI}$
LAPTV+ARMA(1,q)	$Z_{\mu}^{(SLN)}(A_{-0.5M+1}) \oplus Z_{\mu}^{(SLN)}(A_{-0.5M+2}) \oplus \cdots \oplus Z_{\mu}^{(SLN)}(A_{0.5M-1})$	$X^{(SLN)}(t_n)$
		$\mathcal{L}_{ODE}^{-1}$
ACS-AP+MA(q)	$Z_{\mu}^{(DRV)}(A_{-0.5M+1}) \oplus Z_{\mu}^{(DRV)}(A_{-0.5M+2}) \oplus \cdots \oplus Z_{\mu}^{(DRV)}(A_{0.5M-1})$	$X^{(DRV)}(t_n)$
		$g^{(NLN)}$
IID GLNSU	$Z_{\mu}^{(INP)}(A_{-0.5M+1}) \oplus Z_{\mu}^{(INP)}(A_{-0.5M+2}) \oplus \cdots \oplus Z_{\mu}^{(INP)}(A_{0.5M-1})$	$X^{(INP)}(t_n)$

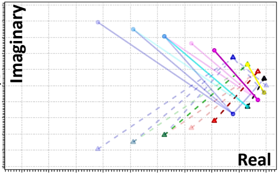
A State Model

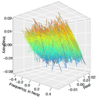
Yousefi et al. (2019)
Babadi et al. (2014)

Layer	**State points**			
Observable layer	$X^{(OUT)}(t_0)$	$X^{(OUT)}(t_1)$	....	$X^{(OUT)}(t_{N-1})$
Hidden output layer	$Z_\mu^{(OUT)}(A_{-0.5M+1})$	$Z_\mu^{(OUT)}(A_{-0.5M+2})$	....	$Z_\mu^{(OUT)}(A_{0.5M-1})$
Hidden driver layer	$Z_\mu^{(DRV)}(A_{-0.5M+1})$	$Z_\mu^{(DRV)}(A_{-0.5M+2})$	....	$Z_\mu^{(DRV)}(A_{0.5M-1})$
Base layer	$Z_\mu^{(INP)}(A_{-0.5M+1})$	$Z_\mu^{(INP)}(A_{-0.5M+2})$	....	$Z_\mu^{(INP)}(A_{0.5M-1})$



GLM Interactions
Cyclic connectivity map





Cyclostationarity from Nonlinearity

Model class | **Estimating stochastic AP function** | Napolitano (2012) Theorem 1.2.24, Corduneanu (1989) Chapter VII, Von Neumann (1934) | **Model component**

Infinite series, only finitely many coefficients nonzero.

GLM: $\check{X}^{(DRV)}(t_n) = \sum_{u=1}^{2} \sum_{l} \alpha_{ul} Z_{\mu}^{(DRV,u)}(A_l)$

ACS-AP+MA(q)

$Z_{\mu}^{(DRV,1)}(A_l)$: One of the $Z_{\mu}^{(INP)}(A_m)$.

$Z_{\mu}^{(DRV,2)}(A_l)$: $Z_{\mu}^{(DRV,1)}(A_l) \times \{$One of the $Z_{\mu}^{(INP)}(A_m)\}$.

$X^{(DRV)}(t_n)$

$g^{(NLN)}$

Uncorrelated jumps, Koopmans (1974)

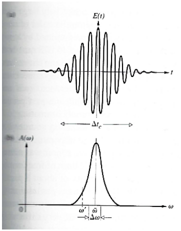
IID
GLNSU

$Z_{\mu}^{(INP)}(A_{-0.5M+1}) \oplus Z_{\mu}^{(INP)}(A_{-0.5M+2}) \oplus \cdots \oplus Z_{\mu}^{(INP)}(A_{0.5M-1})$

$X^{(INP)}(t_n)$

Hecht (2002)

Model class	**Estimating stochastic AP function**	**Model component**
		$X^{(SLN)}(t_n)$
		$\mathcal{L}_{ODE}^{-1}$
ACS-AP+MA(q)	$Z_\mu^{(DRV)}(A_{-0.5M+1}) \oplus Z_\mu^{(DRV)}(A_{-0.5M+2}) \oplus \cdots \oplus Z_\mu^{(DRV)}(A_{0.5M-1})$	$X^{(DRV)}(t_n)$
		$g^{(NLN)}$
IID GLNSU	$Z_\mu^{(INP)}(A_{-0.5M+1}) \oplus Z_\mu^{(INP)}(A_{-0.5M+2}) \oplus \cdots \oplus Z_\mu^{(INP)}(A_{0.5M-1})$	$X^{(INP)}(t_n)$



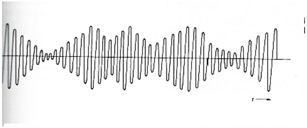


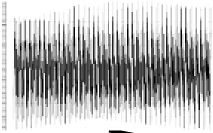
Figure 7.38 A quasimonochromatic lightwave.

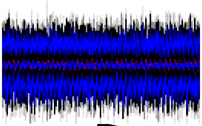
The waves cannot be too dispersive, and Hecht uses the term, "quasimonochromatic".

Particular and Characteristic Solutions

Thornton & Marion (2004)

Model class | **Decomposition** | **Model component**





LAPTV+ARMA(1,q)

$$X^{(PRT)}(t_n) \oplus X^{(CHR)}(t_n) = X^{(SLN)}(t_n)$$

$$\mathcal{L}_{ODE}^{-1}$$

ACS-AP+MA(q)

$$Z_\mu^{(DRV)}(A_{-0.5M+1}) \oplus Z_\mu^{(DRV)}(A_{-0.5M+2}) \oplus \cdots \oplus Z_\mu^{(DRV)}(A_{0.5M-1}) \quad X^{(DRV)}(t_n)$$

$$g^{(NLN)}$$

IID GLNSU

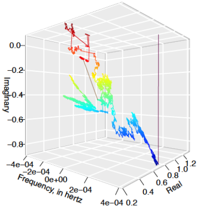
$$Z_\mu^{(INP)}(A_{-0.5M+1}) \oplus Z_\mu^{(INP)}(A_{-0.5M+2}) \oplus \cdots \oplus Z_\mu^{(INP)}(A_{0.5M-1}) \quad X^{(INP)}(t_n)$$

Assumption - Asymptotic Errors

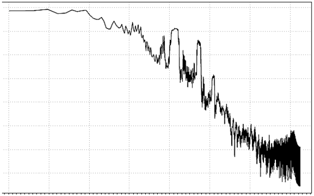
Napolitano (2012)
Lea (2004)

Layer **Aliased bifrequency density**

Hidden output layer $E_\theta \left|\tilde{Z}^{(OUT)}(f_m)\right|^2 = \tilde{\gamma}_Z^{(OUT)}(f_m, f_m)$

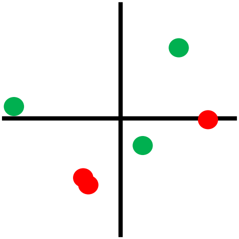


Zero-offset bifrequency mass



Frequency

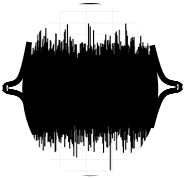
$Im\{\tilde{\gamma}_Z(f_m, f_m)\}$



$Re\{\tilde{\gamma}_Z(f_m, f_m)\}$

Condition - Slepian Expansion of the Integrated Spectrum

$\boldsymbol{X}^{(OUT)}$ $\mathcal{L}_{CSC}$ $\boldsymbol{X}^{(DRV)}$ Slepian (1976)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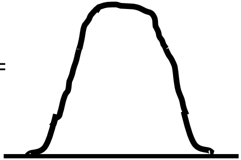
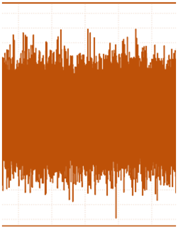
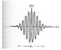



Slepian's errors:

$\boldsymbol{X}^{(OUT)}$ **is timelimited level** $\boldsymbol{E_{TML}}$ **:**

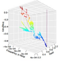
$$\int_{-\infty}^{\infty} E_\theta \left| \left\{ 1 - \Pi\left(\frac{t}{N}\right) \right\} X^{(OUT)}\left(\frac{N}{2} t\right) \right|^2 dt \leq E_{TML}$$



Hecht (2002) Fig.7.37a

$\boldsymbol{X}^{(OUT)}$ **is bandlimited level** $\boldsymbol{E_{DMN}}$ **:**

$$\int_{-\frac{N}{2}}^{\frac{N}{2}} E_\theta \left| \sum_{k=K_N}^{\infty} \Psi_k(\zeta)\, X_k^{(OUT)} \right|^2 d\zeta \leq E_{DMN}$$



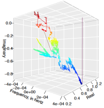
Condition - Slepian Expansion of the Integrated Spectrum

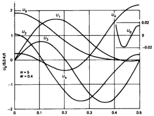
Slepian (1976)

Layer

Observable layer $\underbrace{Y_k^{(OUT)}(f_m)} = \int_{f_m-W}^{f_m+W} \underbrace{V_k(f-f_m)}\, \underbrace{\tilde{Z}^{(OUT)}(f)}df + O_p(1) \quad \boldsymbol{N \to \infty}$

Hidden output layer $\underbrace{\tilde{Z}^{(OUT)}(f+f_m)} = \sum_{k=0}^{K_N} \epsilon_k^{-1}\, \underbrace{U_k(f)}\, \underbrace{\tilde{Z}_k^{(OUT)}(f_m)} + o_p\left(N^{-\frac{1}{2}}\right)$



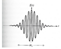


Slepian (1978), Fig.1

<u>Slepian's errors:</u>

$\boldsymbol{X^{(OUT)}}$ **is timelimited level** $\boldsymbol{E_{TML}}$ **:**

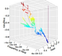
$$\int_{-\infty}^{\infty} E_\theta \left|\left\{1 - \Pi\left(\frac{t}{N}\right)\right\} X^{(OUT)}\left(\frac{N}{2}t\right)\right|^2 dt \le E_{TML}$$



Hecht (2002) Fig.7.37a

$\boldsymbol{X^{(OUT)}}$ **is bandlimited level** $\boldsymbol{E_{DMN}}$ **:**

$$\int_{-\frac{N}{2}}^{\frac{N}{2}} E_\theta \left|\sum_{k=K_N}^{\infty} \Psi_k(\zeta)\, X_k^{(OUT)}\right|^2 d\zeta \le E_{DMN}$$



Expansion Estimates - Asymptotic Theory

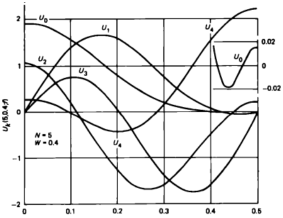
Slepian (1978)

Layer

Aliased, integrated-spectrum process

Hidden output layer $$\tilde{Z}^{(OUT)}(f + f_m) = \sum_{k=0}^{K_N} \epsilon_k^{-1}\, U_k(f)\, \tilde{Z}_k^{(OUT)}(f_m) + o_p\left(N^{-\frac{1}{2}}\right)$$

Estimating element
$N \to \infty$



Slepian (1978), Fig.1

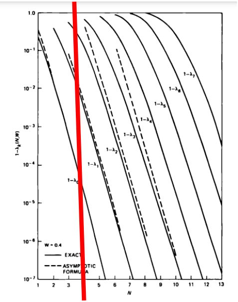


Fig. 5—1 — $\lambda_k(N,0.4)$ for $k = 0, \ldots, 7$ and $N = 1, 2, \ldots, 13$.

Asymptotic Expansion Errors

Slepian, (1976, 1978)
Thomson (1982, 2000, 2013)

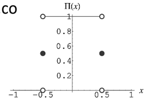
Layer

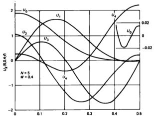
Observable layer

$$Y_k^{(OUT)}(f_m) = \int_{-\infty}^{\infty} \Pi\left(\frac{f}{2W}\right) V_k(f)\, \tilde{Z}_\mu^{(OUT)}(df + f_m) + O_p(1) \qquad \boldsymbol{N \to \infty, W \to 0}$$

$$\Pi\left(\frac{f}{2W}\right) = \sum_{k=0}^{K_N - 1} \beta_k\, U_k(f) + o_p\left(N^{-\frac{1}{2}}\right)$$

https://mathworld.wolfram.com/RectangleFunction.html



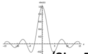


Slepian (1978), Fig.1

<u>Slepian's errors:</u>

$\mathrm{sinc}(2s)$ is frequency-limited level E_{TML} :

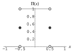
$$4 \int_{-\infty}^{\infty} E_\theta \left| \{1 - \Pi(t)\}\, \mathrm{sinc}(2t) \right|^2 dt \le E_{TML}$$



https://mathworld.wolfram.com/SincFunction.html

$\mathrm{sinc}(2s)$ is bandlimited level E_{DMN} :

$$\int_{-\frac{K_N}{4}}^{\frac{K_N}{4}} E_\theta \left| \Pi\left(\frac{f}{2W}\right) - \sum_{k=0}^{K_N - 1} \beta_k\, U_k(f) \right|^2 df \le E_{DMN}$$



Expansion Errors, $N = O\left(10^4\right)$, $NW = 5$

E_{DMN} is 5% of max.

h relative to [0,1]	No. test abscissa in [0,0.5W]	$E^{(TML)}$ estimate
0.08	12	7.99314129
0.04	25	7.99314129
0.02	50	7.99314129
0.01	100	7.99314131
0.005	200	7.99314131
0.0025	400	8.00000000e
0.00125	800	8.00000000e

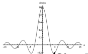
Slepian, (1976)
Kiusalaas (2010)

Step	Nodes multiple	Eigenvalue index	Eigenvalue	Asymptotic eigenvalue	% difference	% bandlimitation energy
10	1	10	0.49	0.5	2.8	17.5
	10	12	0.05	0	100	9.4
	30	12	0.07	0	100	9.6
	50	12	0.07	0	100	9.7
4	1	10	0.49	0.5	2.8	11.1
	10	12	0.05	0	100	5.9
	30	12	0.07	0	100	6.1
	50	12	0.07	0	100	6.1
2	1	10	0.49	0.5	2.8	7.8
	10	12	0.05	0	100	4.2
	30	12	0.07	0	100	4.3
	50	12	0.07	0	100	4.3
1	1	10	0.49	0.5	2.8	5.5
	10	12	0.05	0	100	3
	30	12	0.07	0	100	3
	50	12	0.07	0	100	3.1

Slepian's errors:

$\mathrm{sinc}(2s)$ is frequency-limited level E_{TML} :

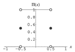
$$4\int_{-\infty}^{\infty} E_\theta \left| \{1 - \Pi(t)\}\, \mathrm{sinc}(2t) \right|^2 dt \le E_{TML}$$



https://mathworld.wolfram.com/SincFunction.html

$\mathrm{sinc}(2s)$ is bandlimited level E_{DMN} :

$$\int_{-\frac{K_N}{4}}^{\frac{K_N}{4}} E_\theta \left| \Pi\left(\frac{f}{2W}\right) - \sum_{k=0}^{K_N-1} \beta_k\, U_k(f) \right|^2 df \le E_{DMN}$$



Conclusions

- Stationary modelling does not preclude multitaper cyclostationary analysis.
- Normalized, integrated spectrum - not the integrated spectrum itself.
- $2NW$ theorem for the normalized, integrated spectrum.
- State model: $\mathrm{AR}(1) \Rightarrow$ nonlinearity, Gaussian statistics, reduced dimension.