Multitaper and Non–stationarity

David J. Thomson, Queens Univ., Kingston, Ontario
djt@queensu.ca

BIRS East (Kingston), June 24–26, 2022

Thanks to: My Students, Queens, NSERC,
(the old) Bell Labs, Murray Hill, NJ, &
Maja–Lisa Thomson

— — Stationary, Gaussian = Overutopian (JWT) — —

Warning! Do not hold your breath while you look for an example of a stationary time series!

Outline

- Background and History

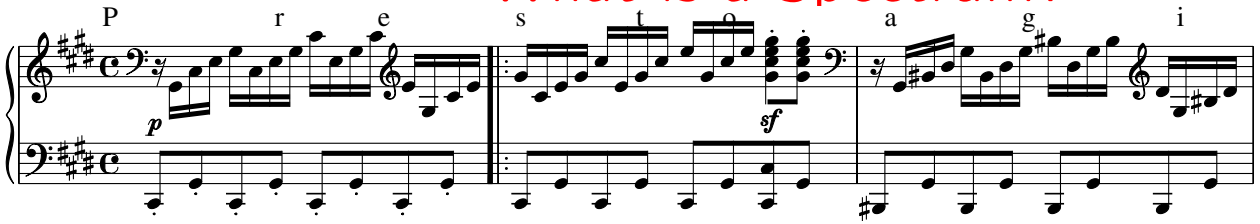
- Two major approaches:
  - Spectrogram
  - Loève spectrum

- Multitaper spectrogram

- Loève spectrum
  - Derivation
  - Examples
  - Reducind False detection rate

- Summary

What is a Spectrum?



Time: Left to Right Frequency: Bottom to Top

Notation: Volume, duration, waveform

Spectral Representation, discrete time, $t = \cdots, -1, 0, 1, \cdots$

$$x(t) = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{i2\pi ft}\, dX(f)$$

Statistics of **Non–stationary Processes**

$$\mathbf{E}\{dX(f_1)\; dX^*(f_2)\} = S(f_1, f_2)\; df_1 df_2$$

$S(f_1, f_2)$ is the *Loève spectrum.*

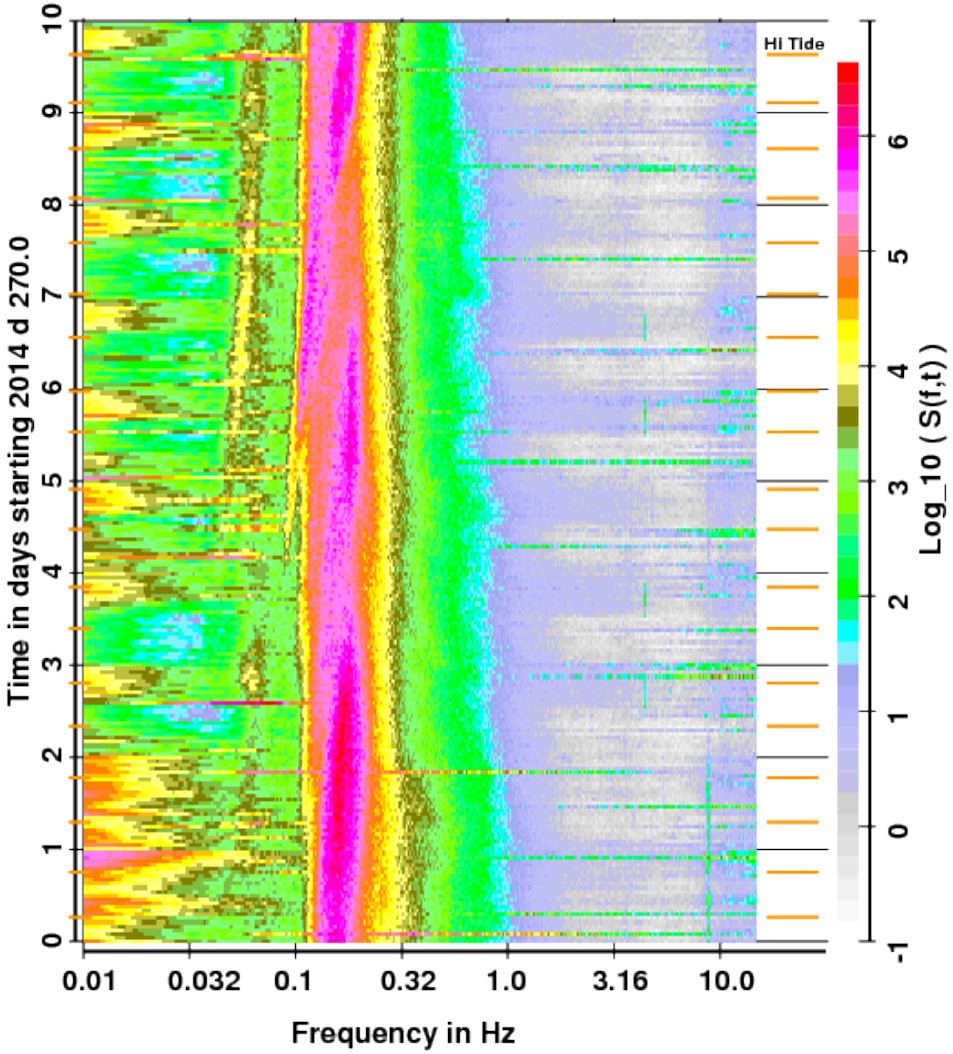
Its Fourier transform is the **Covariance**

$$R(t_1, t_2) = \mathbf{E}\{x(t_1)x(t_2)\}$$

Loève's theory: First paper 1940,

minor interruption, papers 1945–50, Thesis 1965

The Spectrogram, 1946, Koenig, Dunn, & Lacy, BTL
– Discovery date probably earlier



Data: F. Vernon

Seismic, PFO–py

North, BPH09-N

2014 d270–280

Blocks: 1 hour

144,000 samples

$NW = 5$, $K = 8$

Overlap: none

240 sections

34.6 million samples

No large quakes

Spectrogram: Advantages and Disadvantages

Depend on the problem

Simple

- Occasionally too simple!
- (Better than assuming stationary!)
- Display problems!
  - (Common)

Resolution: time, frequency, human eye

Plus: directions, one channel of an array

This is $ave\{ x\ y^* \}$, also $ave\{ x\ y \}$

Note: Single taper form obsolete, use multitaper!

Long duration, low amplitude: Sensitivity poor

Loève: Advantages and Disadvantages

Derived from "First Principles", not *ad–hoc*

But: Loève's theory is full of $\mathbf{E}\{x\}$'s.

What does this mean?

Worse:

How do you compute it ??

Two frequencies: less intuitive than time–frequency

Symmetric derivation, so T–T, T–F, F–F.

F–F better if rotated 45°

Problem: N samples, $> (2N)^2$ frequency pairs

False alarm rate $\rightarrow$ 100%.

So: Be Careful!

Multitapers — I: Fundamental Integral Equation

Fourier Transform of available data: Equivalent to data

$$y(f) = \sum_{t=0}^{N-1} x(t) e^{-i2\pi ft} \tag{1}$$

Spectral Representation, Cramér 1939, 40

$$x(t) = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{i2\pi\xi t} dX(\xi) \tag{2}$$

(1) & (2) give a convolution. Think of it as an integral equation.

$$y(f) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\sin N\pi(f-\xi)}{\sin \pi(f-\xi)} dX(\xi) \tag{3}$$

Do a least–squares solution in a series of Slepian functions, $V_k(f)$. $k = 0, 1, \cdots K-1$. Typically $K = 4$ to 20 terms.

$$\lambda_k V_k(f) = \int_{-W}^{W} \frac{\sin N\pi(f-\xi)}{\sin \pi(f-\xi)} V_k(\xi) d\xi \tag{4}$$

Multitaper Solution (1982)

For a center frequency f, do an eigenfunction solution on $(f-W, f+W)$ and compute the $K = 2NW$ *eigencoefficients*

$$y_k(f) = \frac{1}{\lambda_k} \int_{-W}^{W} y(f-\xi)\, V_k(\xi) d\xi \qquad k = 0, 1, \cdots, K \lesssim \lfloor 2NW \rfloor \quad (5)$$

$$\cdots \text{ some magic } \cdots$$

$$= \sum_{t=0}^{N-1} x(t)\ v_t^{(k)}\ e^{-i2\pi ft} \quad (6)$$

$\Rightarrow$ Orthonormal expansion of $\{$ data $\times\, e^{-i2\pi ft}\ \}$

The simplest multitaper estimate of the Loève spectrum

$$\widehat{S}(f_1, f_2) \sim \frac{1}{K} \sum_{k=0}^{K-1} y_k(f_1)\, y_k^*(f_2) \qquad \mathbf{K >> 1} \quad (7)$$

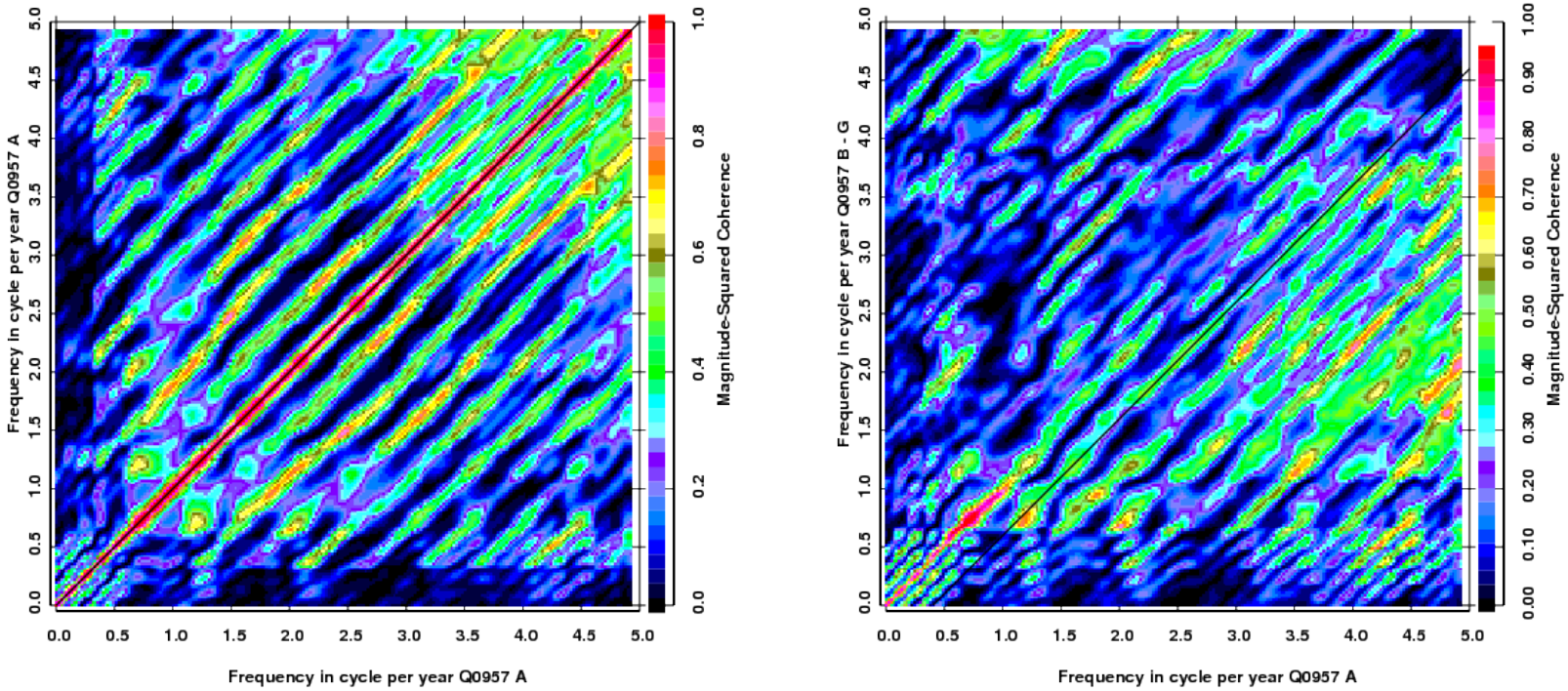
Non–stationary, so also $\widehat{S}_r(f_1, f_2) \sim ave\{\, y_k(f_1)\, y_k(f_2)\,\}$

Plot as MSC(f_1, f_2)

Best Method: Multitaper Spectrum Estimates

Q-0957+561 — Gravitationally lensed QSO, 16^{th} Mag.

Fortunately *NOT* a near neighbour!



Note curvature! What does this mean?

Loève: Some Palliatives

Rotate coordinates $-45°$,

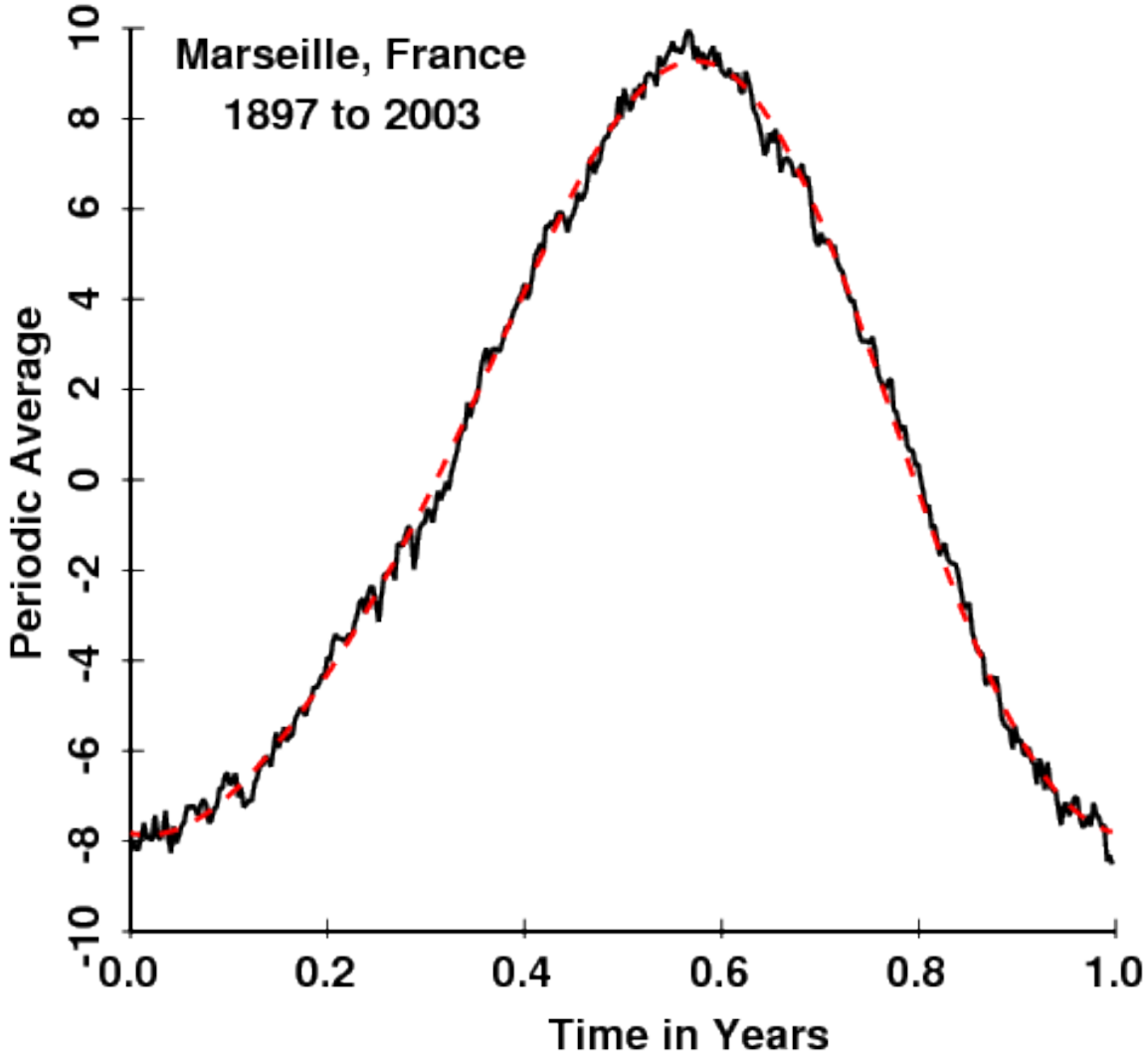
Horizontal: Center frequency

Vertical: Differentiat, $f_2 - f_1$

Convert S(f_1, f_2) to MSC$(f, \delta f)$

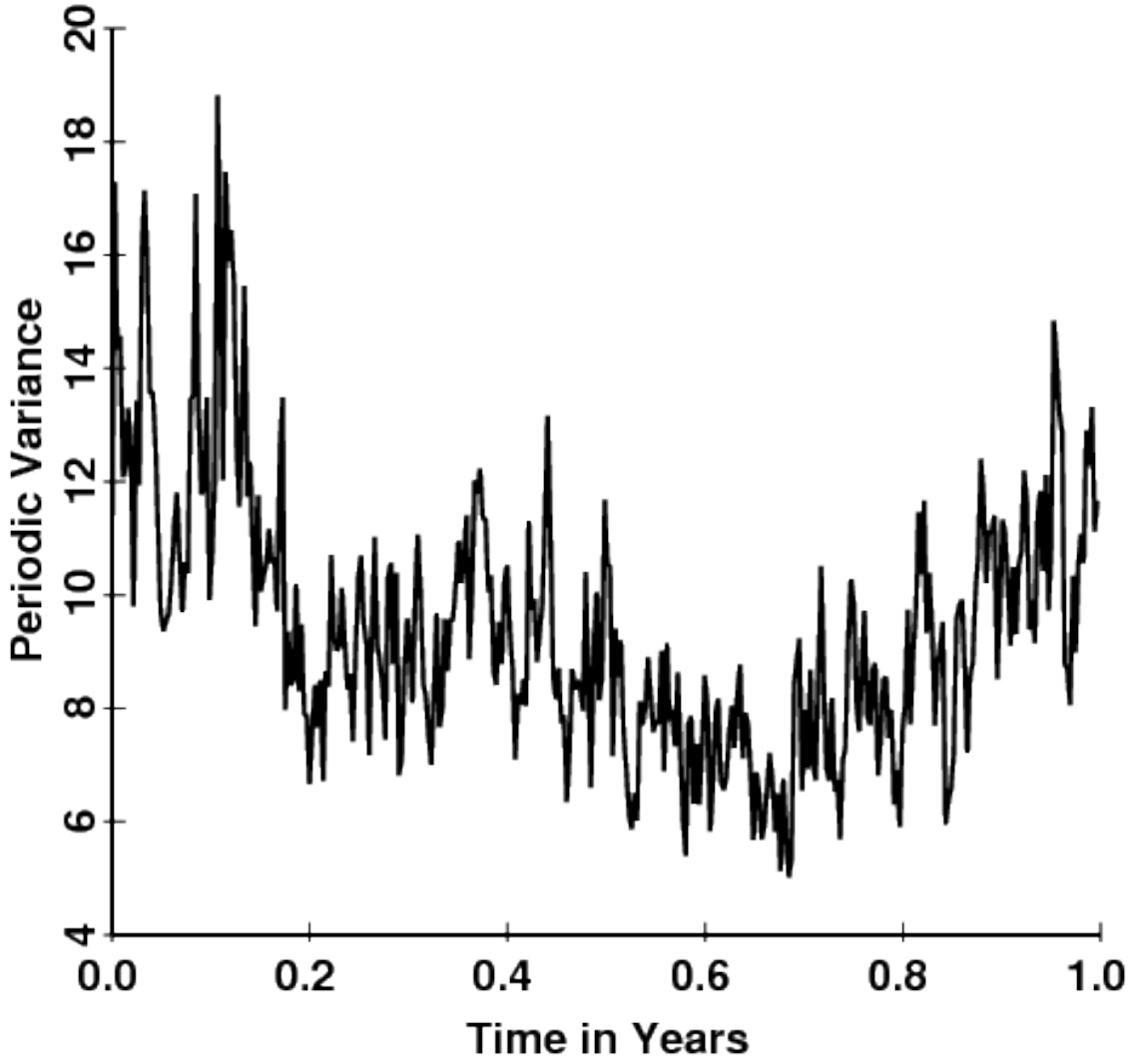
Statistica along horizontal lines

Temperature, Marseille, France, 1897 to 2003



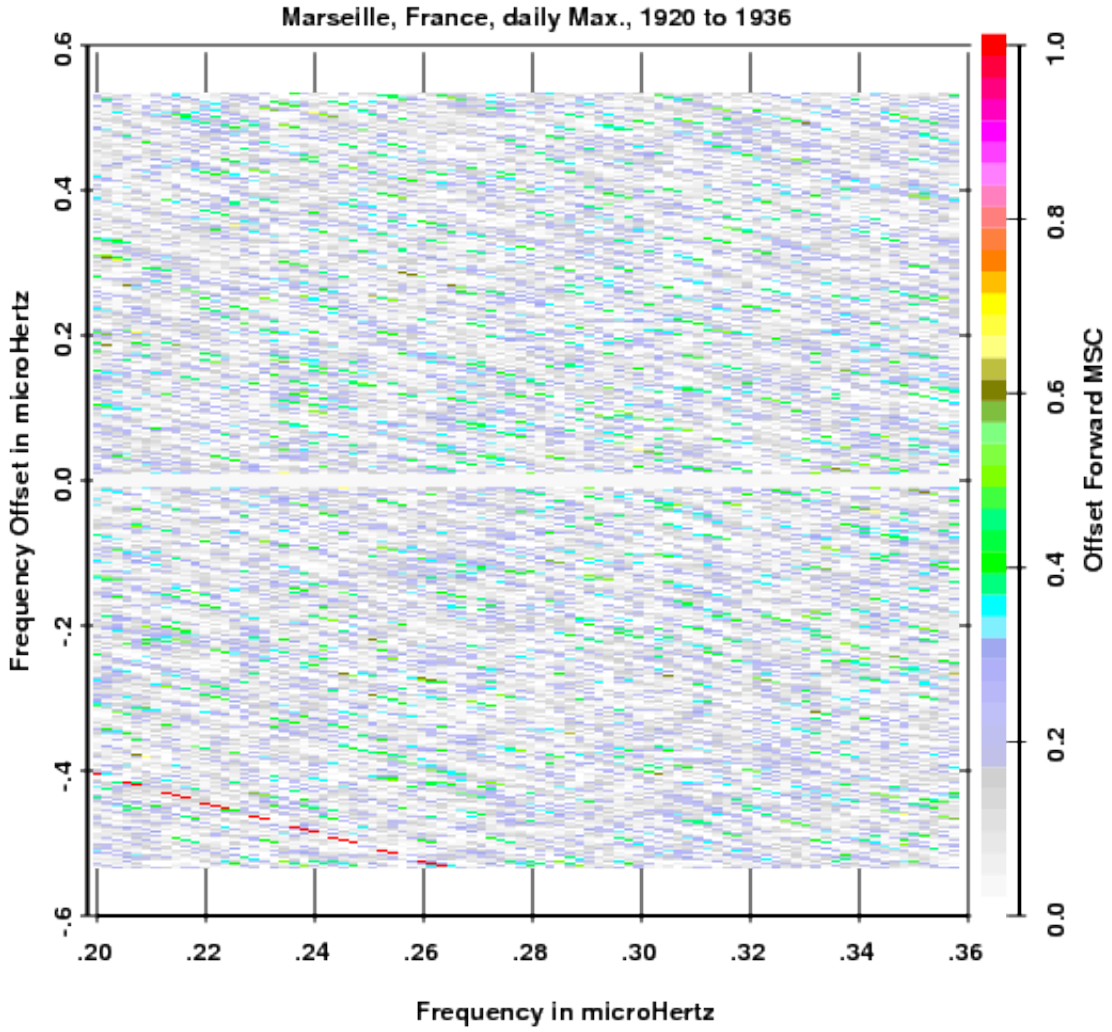
Data:
Marseille, France
daily, 1897 to 2003
(107 years)
365.25×4 day Ave.
Average and Fit
Remove It!

Temperature, Marseille, France, 1897 to 2003



Data:
Marseille, France
daily, 1897 to 2003
365.25×4 day Ave.
Daily Variance
High in WINTER!
NON–STATIONARY

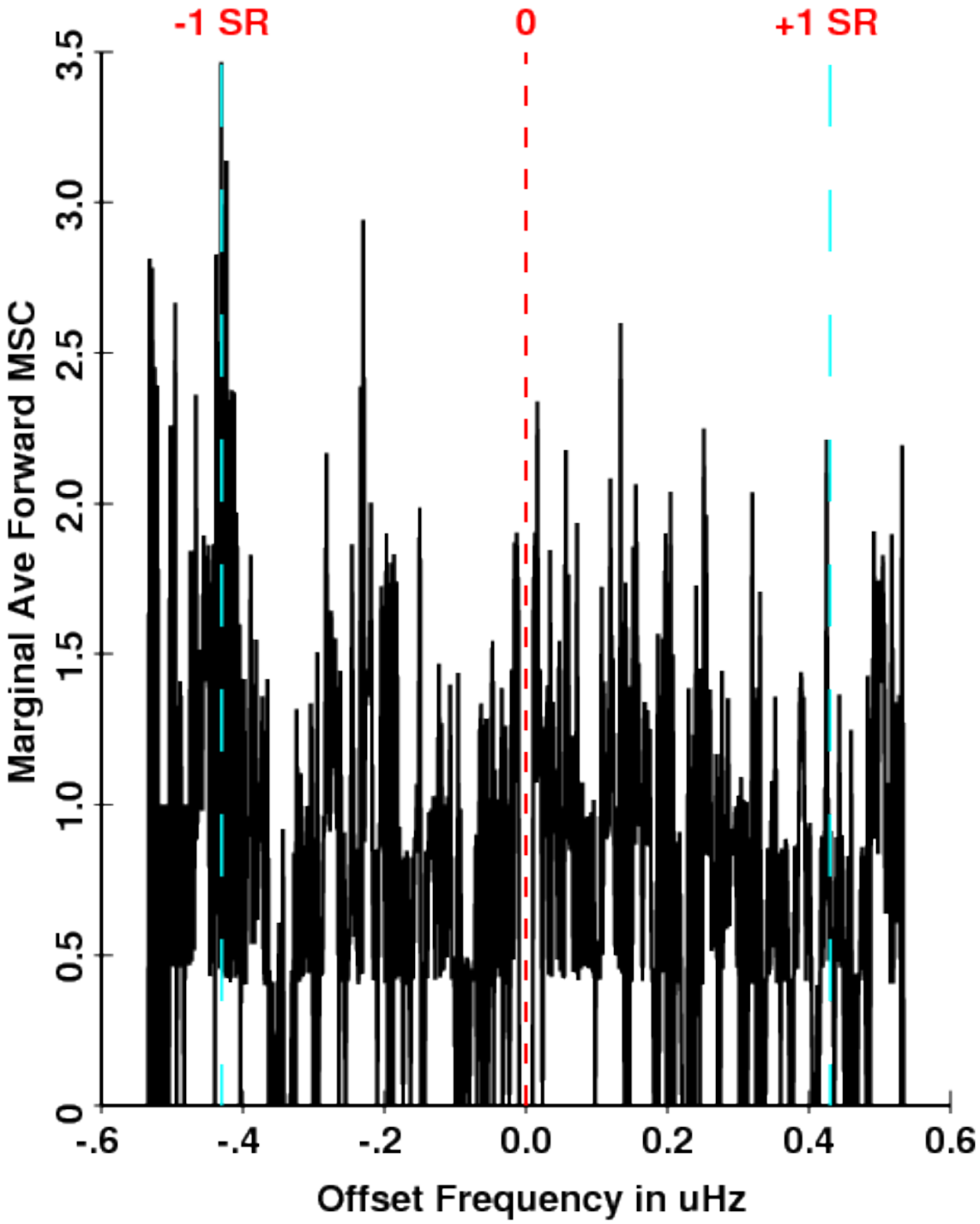
Temperature, Marseille, France,1920 to 1936



Data:
Marseille, France
daily, 1920 to 1936
(avoids problems)
$NW = 5,\ K = 9$

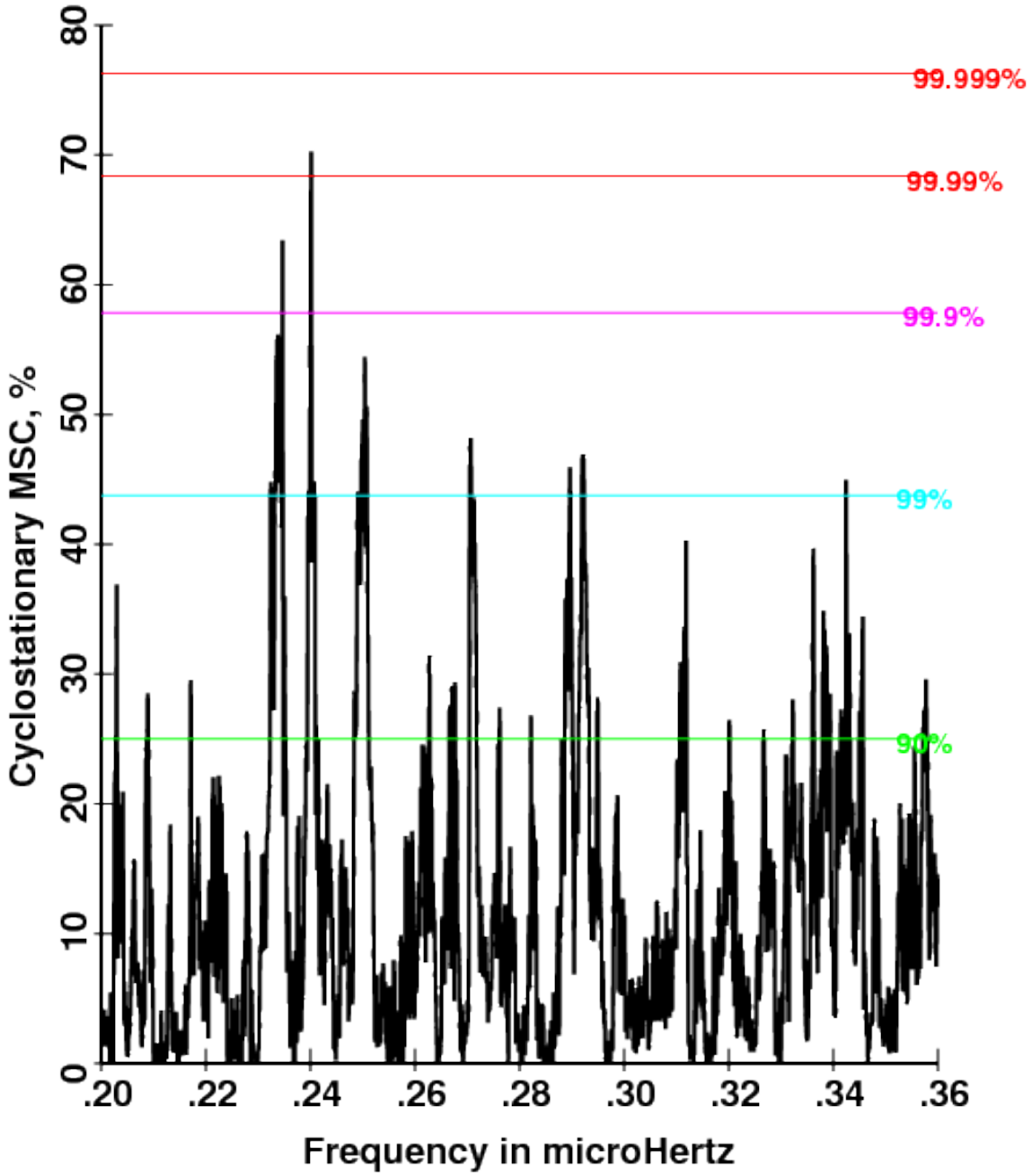
Rotated $-45°$
So f and δf
Not obvious!

Temperature, Marseille, France,1920 to 1936



Data:
Marseille, France
daily, 1920 to 1936
Horizontal Sums
(of 2D MSC)
98% Sig Low Clip
Markers: (SR)
Solar Rotation
27 day = 428 nHz
NOT symmetric!

Temperature, Marseille, France, 1897 to 2003



Offset MSC

Specified Offset

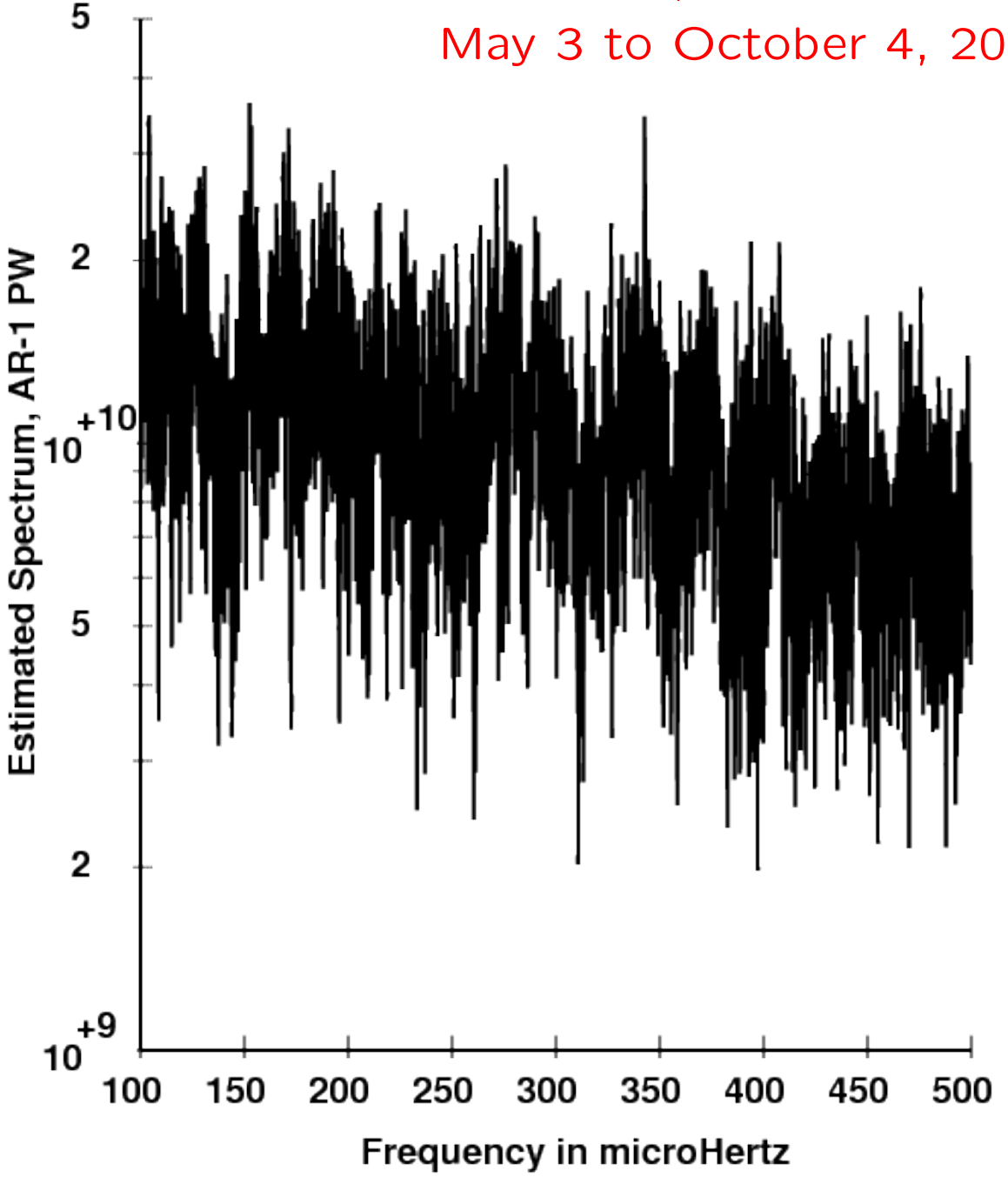
$f_2 = f_1 + f_R$

$P_R = 27$ days

Peak $\approx 2 \times 24.8$ day

$\approx$ Solar Rotation

Barometric Pressure, Black Forest Observatory
May 3 to October 4, 2000



Pre-whitened

AR–1, $r_1 = 0.99959$

$N = 19,320$

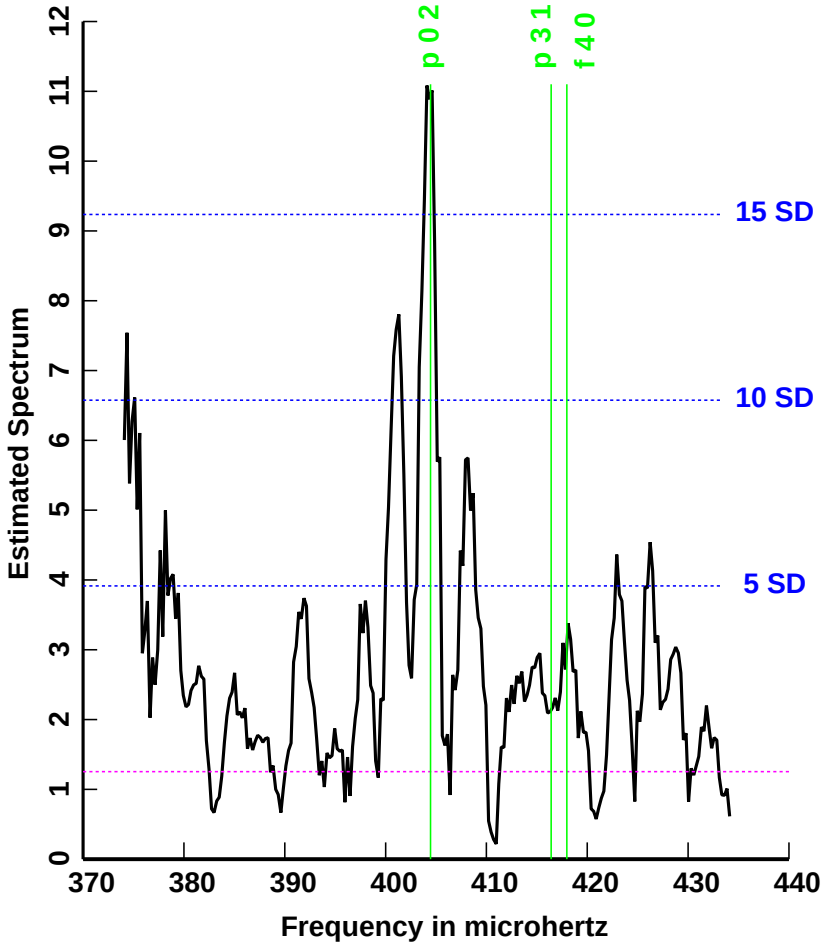
$C_{\mathcal{R}} = 6$

$K = 10$

Too variable

Plot too dense!

Solar Modes in Spectra of Barometric Pressure Data



Black Forest Obs.

$\sim 20\sigma$ peak

– Center: 401.1 μHz

– Width $\lesssim 2.8\mu$Hz

$\approx$ Solar $p_{0,2}$ Mode

Doppler splitting

Proc. IEEE (2007)

In Solar Wind

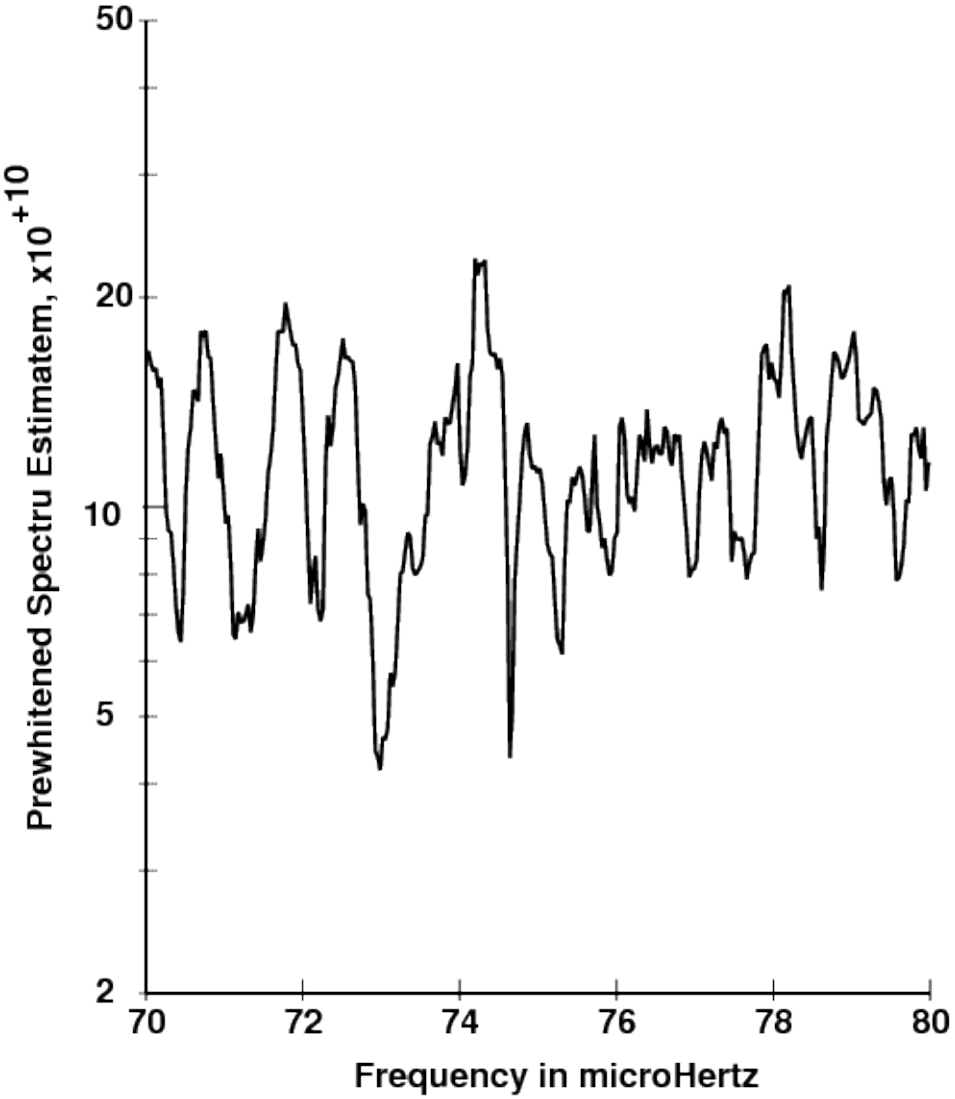
Ghosh, djt, *et al.*

JGR (2009)

Can do science

with this.

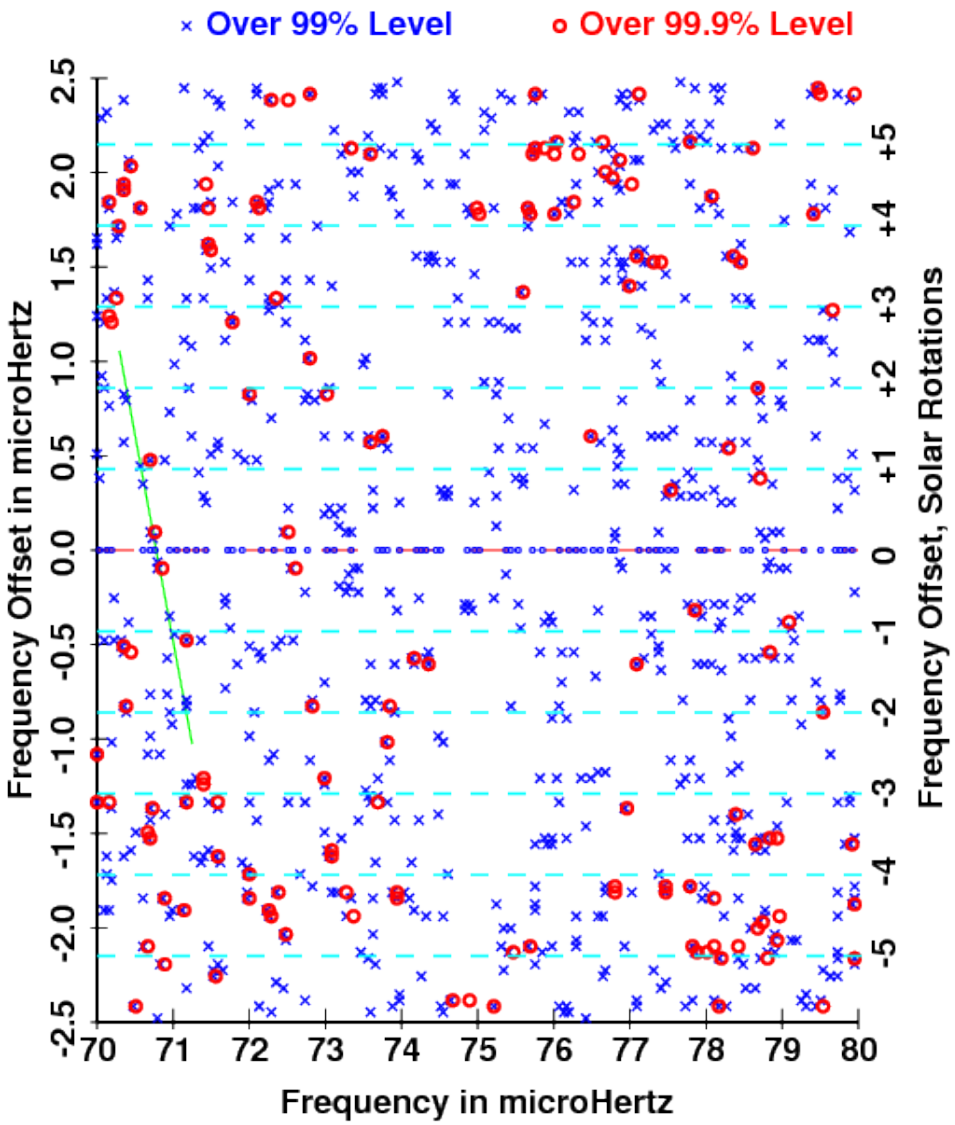
Solar Modes in Spectra of Barometric Pressure Data ?



Black Forest Obs.
Same data
g–mode band
10 tapers

99.9% point 2.266 x mean
Ave = 1.18, Max = 2.27
SD = 0.36 all $\times 10^{10}$
Max = Ave + 2.6σ
No big peaks!

Barometric Pressure Data: BFO, Mar 3 to Oct 4, 2000



Loève spectrum

99% and 99.9% levels

$C_{\mathcal{R}} = 6,\ K = 10$

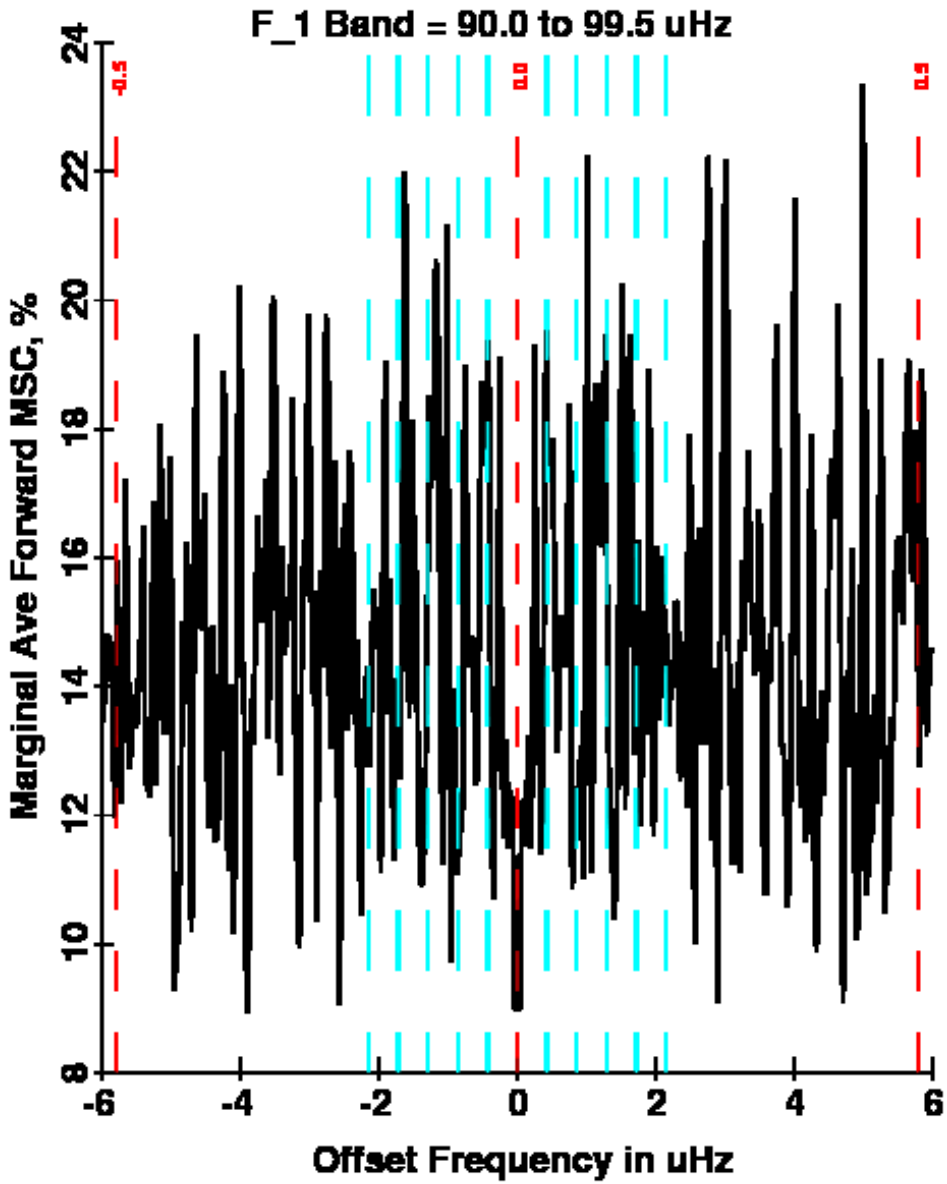
Lowest Earth mode

${}_0S_2$ at 309.49μHz

Many more than expected!

Systematic "features" ?

Barometric Pressure Data: BFO, Mar 3 to Oct 4, 2000



"Horizontal Averages", MSC

$C_{\mathcal{R}} = 6,\ K = 10$

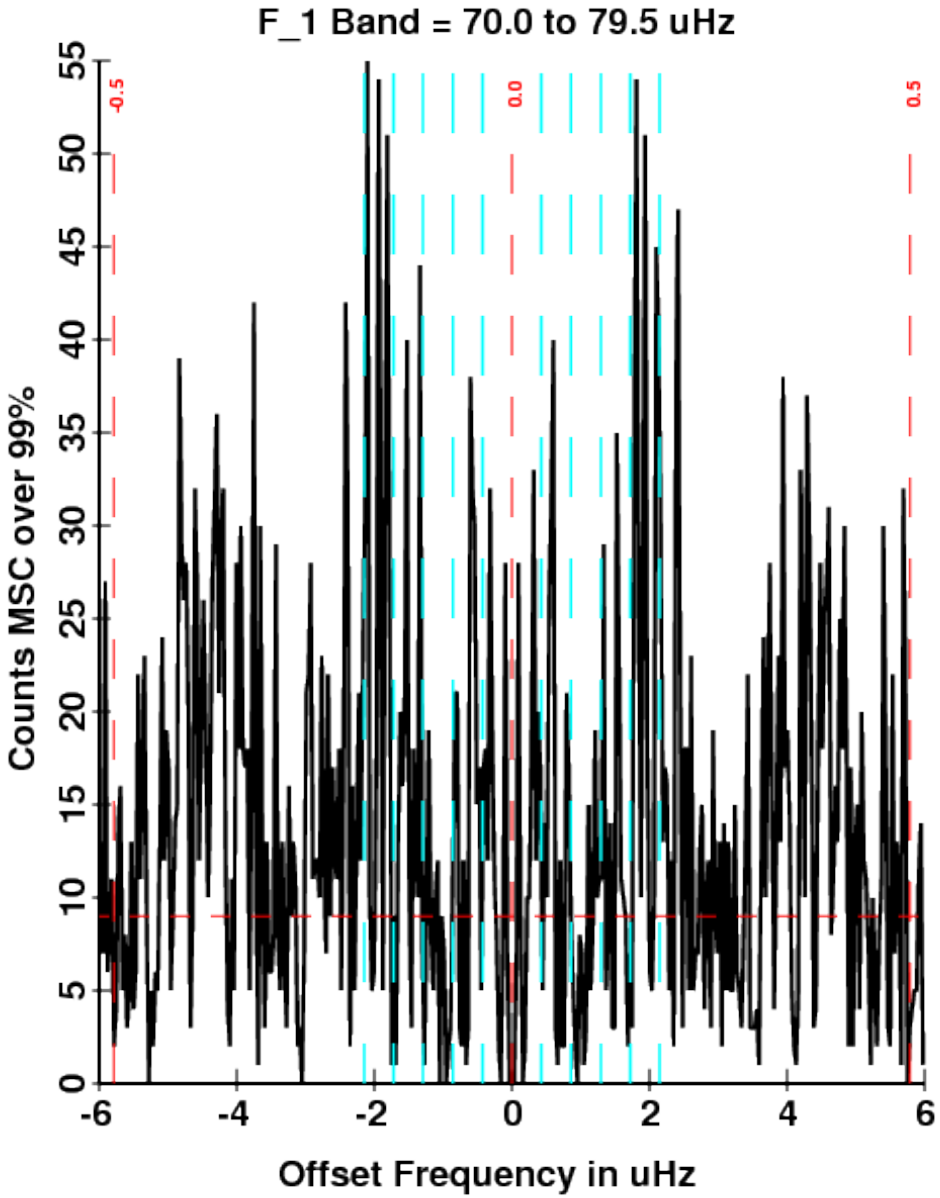
Asymmetric

PDF complicated

Quasi two day mode

Which peaks important?

Barometric Pressure Data: BFO, Mar 3 to Oct 4, 2000



"Horirizontal Peak Count"

Peaks above 99%

Expected ≈ 9

$C_{\mathcal{R}} = 6,\ K = 10$

"zero" supressed

NOT symmetric

Same pattern

Solar rotation

Only Ordinary MSC

Summary - What I'm trying to do

Convert Time–Series from a "Black Art" to a Science

Still underway!

Work in progress! (since *c.* 1965)

40 years since the 1982 paper!

MT spectrogram: works well

Ordinary "single taper" spectrogram: obsolete

Loève spectrum:

- Still being developed
- High false detection rate

But: Finds things others mostly miss!

Needs "auxillary" procedures

Horizontal, vertical, diagonal statistics.

(Expect about 200 plots/Simple series!)

Stationary processes: *Extremely* rate !!!

Some Lessons more–or–less Learned

Pay attention to maxims. Learn from other people's mistakes,
— you won't live long enough to make them all yourself.

Don't ever use periodograms. Ditto, Bartlett autocorrelations.

Avoid unverifiable assumptions. Test the ones you make.

You are *always* working with small samples.

Keep Frequencies in Hertz. Cycles/year OK for paleoclimate.

Use degrees and avoid Radians. *Really* avoid radian frequency!

Specify frequencies precisely. $\sim$ C–R bound.

Chaoplexology $\Rightarrow$ Run away!

Avoid dogma — it always gets you in trouble!

Analyze data, don't assume dogma is correct

Pay attention to theory, but not too much.

Most *great* scientists have one idea in their working life, except for Lord Rayleigh, who had two. — Fermi.

— Corollary: Expect < 0.01 idea/paper.

Useful wisdom from the past

"It's better to be approximately right than exactly wrong" - Tukey

"If your experiment needs statistics, you ought to have done a better experiment" - Rutherford

"All models are wrong but some are useful" - Box

— Corollary: *Some aren't*

"One can't be too paranoid about spectra" - Dewan

"As simple as possible, but not simpler" - Einstein

KISS: Keep It Simple, Stupid!

"The hallmark of good science is that it uses models and 'theory' but never believes them." - Wilk

All things are made of atoms. - Feynman

"A statistician is someone who is good with numbers, but who lacks the personality to be an accountant" — RSS News

Thank you!