Bogomolny Equations on $\mathbb{R}^3$ with a Knot Singularity

Weifeng Sun

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The Bogomolny Equations

- Suppose A is an $SU(2)$ connection on $\mathbb{R}^3$ whose curvature is F_A. Suppose Φ is a section of the adjoint $su(2)$ bundle, then the Bogomolny equations are:

$$F_A = *d_A\Phi.$$

- Assymototic conditions: $|F_A| + |d_A\Phi| = O(\frac{1}{r^2})$, $|\Phi| = 1 + O(\frac{1}{r})$.
- Connected components of the configuration space $\{(A, \Phi)\}$ can be indexed by the "monopole number" k. (Classified by the degree of the map $\frac{\Phi}{|\Phi|}: S^2 \to S^2 \subset su(2)$.)

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Donaldson's Description of the Moduli Space

Let M_k be the moduli space of solutions up to $SU(2)$ gauge transformation with monopole number k.

- **Theorem** (Donaldson, 1984) There is a circle bundle $\tilde{M}_k$ over M_k, such that $\tilde{M}_k$ can be identified with rational maps $f : \mathbb{CP}^1 \to \mathbb{CP}^1$ with degree k and $f(\infty) = 0$.
- In particular, dim $\tilde{M}_k = 4k$, dim $M_k = 4k - 1$.
- Donaldson's result was based on a study of the moduli space of "Nalm's equations" over $(-1, 1)$ with certain boundary conditions. The relationship between 1-d Nalm's equations and the 3-d Bogomolny equations was established earlier by Nalm and Hitchin.

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Hitchin's Algebraic Geometrical Approach

Hitchin's algebraic geometrical method:

- Consider the moduli space of straight oriented lines in $\mathbb{R}^3$ (identified with $T\mathbb{CP}^1$).
- If an oriented line (with direction v) has $\mathbb{L}^2$ solutions to the ODE equation $\nabla_v^A s + \Phi s = 0$ on it, then all such lines form an algebraic curve in $T\mathbb{CP}^1$, namely the "spectral curve".
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Knot Singularity?

- It is trending now to study the gauge theoretic PDEs with a knot singularity. For example
  - Kronheimer and Mrowka's studied the instanton floer theory with a knot singularity;
  - Witten proposed to study Kapustin-Witten equations with certain knot singularity.
- It is natural to ask (and proposed by Taubes), what can we say about the moduli space of the solutions on $\mathbb{R}^3$ with a knot singularity? Can Bogomolny equations with a knot singularity be studied by algebraic geometrical method? This may have the potential to bring knot into algebraic geometry in the future.
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Taubes' Analytical Approach

Taubes have also studied the moduli space of the Bogomolny equations on $\mathbb{R}^3$ using an analytical method in his Ph.D. thesis (Vortices and Monopoles: Structure of Static Gauge Theories).

- Suppose (A, Φ) is a configuration. Let L be the linearization of the Bogomolny equations with an extra gauge fixing condition at (A, Φ). Let Q be the quadradic term in the Bogomolny equations: $F_A - *D_A\Phi = 0$.
- Let $S = \Lambda^0(T^*M) \oplus \Lambda^1(T^*M)$ equipped with a "Clifford multiplication $\cdot$", g be the adjoint $su(2)$ bundle. Then $L : S \otimes g \to S \otimes g$ can be written as

$$L = \sum_{j=1}^{3} dx_j \cdot \nabla_{A_j} + [\Phi,\ ].$$

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Taubes' Analytical Approach

- Ignoring the boundary, integration by part shows that:

$$\int |L\psi|^2 = \int |\nabla_A \psi|^2 + |[\Phi, \psi]|^2 + < \psi, *[(*F_A + d_A\Phi) \wedge \psi] >,$$

$$\int |L^\dagger\psi|^2 = \int |\nabla_A \psi|^2 + |[\Phi, \psi]|^2 + < \psi, *[(*F_A - d_A\Phi) \wedge \psi] > .$$

- Define $\|\psi\|_{\mathbb{H}_{(A,\Phi)}} = (\int_{\mathbb{R}^3} |\nabla_A \psi|^2 + |[\Phi, \psi]|^2)^{\frac{1}{2}}$. Immediately,
  - L is Fredholm from $\mathbb{H}_{(A,\Phi)}$ to $\mathbb{L}^2$.
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Taubes' Analytical Approach

- The quadratic part Q is a bounded map from $\mathbb{H}_{(A,\Phi)} \times \mathbb{H}_{(A,\Phi)}$ to $\mathbb{L}^2$.
- A skechy proof:

  Write ψ as $\psi^{/\!/} + \psi^{\perp}$, where $\psi^{/\!/} /\!/ \Phi, \psi^{\perp} \perp \Phi$.
  Note that $Q(\psi_1^{/\!/}, \psi_2^{/\!/}) = 0$. So $Q(\psi_1, \psi_2) = Q(\psi_1, \psi_2^{\perp})$.

  By Sobolev embedding, $\psi_1 \in \mathbb{L}^6$, $\psi_2^{\perp} \in \mathbb{L}^6 \cap \mathbb{L}^2$.

  So $\left\| Q(\psi_1, \psi_2^{\perp}) \right\|_{\mathbb{L}^2}$ is bounded by $\|\psi_1\|_{\mathbb{H}_{(A,\Phi)}} \cdot \|\psi_2\|_{\mathbb{H}_{(A,\Phi)}}$.
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Knot Singularity

- If there is a knot singularity, then the usual integration by part does not go through: the boundary term near the knot doesn't vanish.
- Suppose ρ is the distance to the knot and N_ϵ is the ϵ-neighbourhood of the knot.
  By examining the scale near the knot, I make the following adaption:

$$\|\psi\|^2_{\mathbb{H}_{(A,\Phi),\epsilon}} = \epsilon(\int_{\mathbb{R}^3\setminus N_\epsilon} |\nabla_A\psi|^2+|[\Phi,\psi]|^2)+(\int_{N_\epsilon} \rho(|\nabla_A\psi|^2+|[\Phi,\psi]|^2)),$$

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- **Definition** If after a gauge transformation, $\Psi = (A, \Phi)$ is close enough to a model solution $\Psi_\gamma = (A_\gamma, \Phi_\gamma)$ near the knot $(\int_{N_\epsilon} \rho(|\nabla^{A_\gamma}(\Psi - \Psi_\gamma)|^2 + |[\Phi_\gamma, \Psi - \Psi_\gamma]|^2) < +\infty)$, then it has a knot singularity with monodromy γ.
  Here $A_\gamma = \gamma\sigma\omega$ is the flat connection with γ monodromy , $\Phi = \sigma$ is covariantly constant.
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- **Theorem** (Sun20) Suppose (A, Φ) satisfies the assymtotic conditions and both F_A and $D_A\Phi$ have bounded $\mathbb{L}^2_\epsilon$ norm. Then the only possible singularity is a knot singularity.
- The drawback is: the cokernel of L is not guaranteed to be 0.
- Luckily, the quadratic term Q is still a bounded map from $\mathbb{H}_{(A,\Phi),\epsilon} \times \mathbb{H}_{(A,\Phi),\epsilon}$ to $\mathbb{L}^2_\epsilon$.

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Fredholm Theory

- **Theorem** (Sun20) If $\gamma \in (0, \frac{1}{8}) \cup (\frac{3}{8}, \frac{1}{2})$, then L is Fredholm. In this situation, the moduli space has a local "analytical structure": Locally it can be identified with the pre-image $f^{-1}(0)$ of a real analytical map between f Euclidean spaces.
- A sketchy proof:

$$\int_{N_\epsilon} \rho |L_\Psi \psi|^2 = c\|\psi\|^2_{\mathbb{H}(N_\epsilon)} + \int_{\partial N_\epsilon} \text{boundary term } A + (\cdots),$$

$$\epsilon \int_{\mathbb{R}^3 \setminus N_\epsilon} |L_\Psi \psi|^2 \geq \epsilon \|\psi\|_{\mathbb{H}_{\mathbb{R}^3 \setminus N_\epsilon}} + \int_{\partial N_\epsilon} \text{boundary term } B + (\cdots).$$

Here $\int_{\partial N_\epsilon} (2A + B)$ is compact relative to $\|\psi\|_{\mathbb{H}_{(A,\Phi),\epsilon}}$, so

$$2\int_{N_\epsilon} \rho |L_\Psi \psi|^2 + \epsilon \int_{\mathbb{R}^3 \setminus N_\epsilon} |L_\Psi \psi|^2 \geq c\|\psi\|^2_{\mathbb{H}_{(A,\Phi),\epsilon}} - \text{compact terms.}$$

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Conjecture

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Examples from gluing

- One way to describe the Prasad-Sommerfield monopole (N $=$ 1) is (in a certain gauge):
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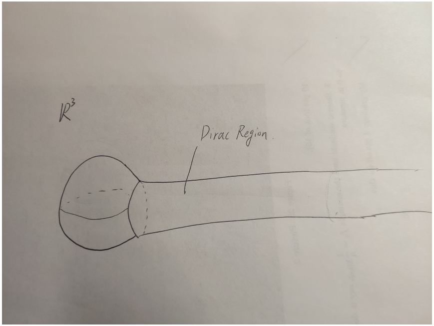

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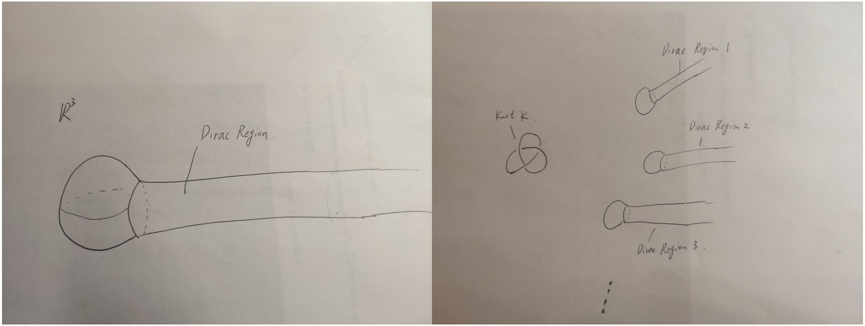

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- If the Dirac regions and the knot are far away, then it is possible to "glue" any number of Prasad-Sommerfield monopoles onto the model solution with knot singularity.
- (Sun20) If I shrink the knot to be small enough, then the relative compact term in

$$\|L_\Psi \psi\|^2_{\mathbb{L}^2_\epsilon} \geq c\|\psi\|^2_{\mathbb{H}_{(A,\Phi),\epsilon}} - \text{relatively compact term.}$$

  can be bounded $\frac{c}{2}\|\psi\|^2_{\mathbb{H}_{(A,\Phi),\epsilon}}$, which implies that the cokernel of L is 0. (So the moduli space has a manifold structure nearby in this situation.)

Thank you!