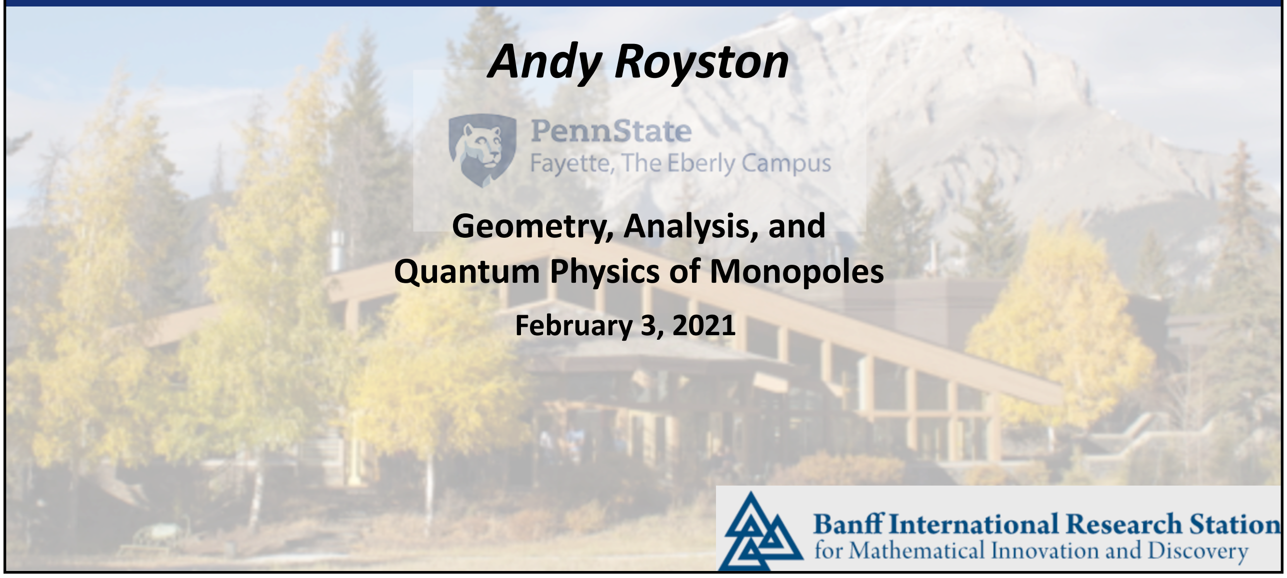
Monopoles and Quantum Field Theory



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Geometry, Analysis, and Quantum Physics of Monopoles

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1

The Magnetic Monopole turns 90

Quantised Singularities in the Electromagnetic Field.

By P. A. M. Dirac, F.R.S., St. John's College, Cambridge.

(Received May 29, 1931.)

§ 1. *Introduction.*

The steady progress of physics requires for its theoretical formulation a mathematics that gets continually more advanced. This is only natural and to be expected. What, however, was not expected by the scientific workers of the last century was the particular form that the line of advancement of the mathematics would take, namely, it was expected that the mathematics would get more and more complicated, but would rest on a permanent basis of axioms and definitions, while actually the modern physical developments have required a mathematics that continually shifts its foundations and gets more abstract. Non-euclidean geometry and non-commutative algebra, which were at one time considered to be purely fictions of the mind and pastimes for logical thinkers, have now been found to be very necessary for the description of general facts of the physical world. It seems likely that this process of increasing abstraction will continue in the future and that advance in physics is to be associated with a continual modification and generalisation of the axioms at the base of the mathematics rather than with a logical development of any one mathematical scheme on a fixed foundation.

2

The Quantization of Charge

- Wave functions in QM may carry a nonintegrable phase – a consequence of the presence of an electromagnetic field: $\psi = \psi_0 e^{i\beta}, \qquad \vec{\nabla}\beta = \frac{e}{\hbar c}\vec{A}, \qquad \vec{\nabla}\times\vec{A} = \vec{B}$
- The change in this phase around a closed loop need not be the same for all wave functions, but can differ for two different wave functions by an integer multiple of 2π

> "…This new development requires *no change whatever* in the formalism when expressed in terms of abstract symbols denoting states and observables, but is merely a generalization of the possibilities of representation of these abstract symbols by wave functions and matrices. Under these circumstances one would be surprised if Nature had made no use of it."

- This w … nction vanish …
- Dirac t … um of the phase differences around all nodal lines ending on singularities within a closed surface is an invariant given by the magnetic flux through that surface:

$$2\pi\sum n_i + \frac{e}{\hbar c}\oint \vec{B}\cdot d\vec{S} = 0 \qquad \Rightarrow em = \frac{n\hbar c}{2}$$

- In so many words, Dirac showed it is consistent with all physical requirements of QM to regard the wave function as a section of a U(1) line bundle determined by the magnetic charge

3

The first 50 years

- Mathematicians steadily built and explored those "more abstract" foundations, including the theory of fiber bundles
- Physicists were preoccupied with another concern: the development of Quantum Field Theory
- QFT and the ideas that evolved from it are fundamental to the modern results I want to touch on

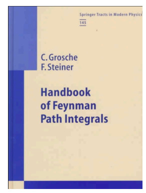
4

Feynman and Wilson's QFT Tutorial

- Feynman's Path Integral for Quantum Mechanics:
  - Given $\widehat{H}(\hat{p},\hat{q})$ and corresponding time evolution operator $\widehat{U}(t_f,t_i)$,

$$\langle\psi_f|\widehat{U}(t_f,t_i)|\psi_i\rangle = \int [DqDp]\psi_f(q(t_f))^*\psi_i(q(t_i))\exp\{i\int_{t_i}^{t_f}dt'(p\dot{q}-H)\}$$

- As good as any other formalism for QM:
  - any solvable model can also be handled with the PI:



- And better:
  - Analytic continuation to Euclidean time turns WKB for tunneling into a saddle-point analysis around a classical solution
  - See Sidney Coleman's "The Uses of Instantons"
- Note: if H is quadratic in p, we can integrate out p and recover "e^{iS}" form

5

The Feynman PI for QFT

- For a field theory of ϕ: $\mathbb{R}^{1,d} \to \mathbb{R}$, with Hamiltonian $H[\pi,\phi]$, let $q(t) \to \phi_x(t) = \phi(t,x)$ and $p(t) \to \pi_x(t) = \pi(t,x)$:

$$\langle\Psi_f|\widehat{\mathcal{O}}(t)|\Psi_i\rangle = \int [D\phi D\pi]\Psi_f[\phi(x,t_f)]^*\Psi_i[\phi(x,t_i)]\exp\{i\int_{t_i}^{t_f}dt'(\int dx\pi\dot{\phi}-H)\}\mathcal{O}$$

- Student:
  - I did the Klein-Gordon PI vacuum-to-vacuum amplitude, $\langle\Omega|\Omega\rangle$, and got something undefined
- Professor Feynman:
  - Of course it is. This computes $e^{-iT\sum\omega_i/2}$ and there are infinitely many normal modes.
  - In fact, we should divide out by this amplitude so as to define this as the zero of energy.
  - And you should not complain, since it is only changes in potential energy that matter

6

Wilson's Renormalization

- Student:
  - I tried to compute the first perturbative correction in g to the scalar propagator in ϕ^4 theory and got infinity
- Professor Wilson:
  - You're right, what you tried to compute is not defined. But you are thinking about QFT the wrong way. The parameters in the Hamiltonian – m, g – they are not constants. Rather, they change with the energy scale.
  - Let's truncate the number of modes in your computation by throwing away those with energies above a cut-off scale μ. (We should probably put the theory in a box of sides L too, so that the spectrum is discrete. We can send $L \to \infty$ later.)
  - Now, you give me values g_0, m_0 and a scale μ_0, and I will determine $g(\mu), m(\mu)$ order by order in perturbation theory such that
    - $g(\mu_0) = g_0$, $m(\mu) = m_0$, and
    - the limit of your computation as $\frac{\mu}{\mu_0} \to \infty$ exists provided we replace the "bare" g, m you had with my running $g(\mu), m(\mu)$
  - Furthermore, provided the theory is *renormalizable*, this procedure will cure the divergences in *any* computed observable, and thus we have not lost predictive power

7

Wilsonian low energy effective theories

- Using renormalization, we can define an effective Hamiltonian $H^{eff}_{\mu_0,\{g_0\}}$ (or action $S^{eff}_{\mu_0,\{g_0\}}$) by (path-) integrating out all modes with energies $\mu > \mu_0$

$$\int [D\phi] e^{iS} = \int [D\phi]_{\mu_0} e^{iS^{eff}_{\mu_0,\{g_0\}}}$$

  - Often, some fields are massive and some are massless – by choosing μ_0 to be the mass gap, we eliminate the massive fields from the action.
  - In an interacting theory, heavy and light modes couple, so integrating out heavy modes will result in new couplings in $S^{eff}_{\mu_0,\{g_0\}}$ for the light modes – indeed an infinite set of terms that can be organized in a double expansion:
    - In the coupling g_0 (since we are carrying out the PI perturbatively in this parameter)
    - And in the energy scale μ_0 (which takes the form of a derivative expansion)

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and so concludes the tutorial

8

The Return of the Monopole

- How broad is the class of renormalizable field theories? Well, it includes Yang-Mills-Higgs in ≤ 4 dimensions
  - (Which, not incidentally, is relevant to our current best description of nature!)

- And this is where the monopole re-enters our story
  - 1974: 't Hooft and Polyakov discover YMH contains finite-energy magnetic monopoles
  - 1974-1976: Formalism for defining soliton states in QFT was developed and fits nicely in the Feynman/Wilson framework:

$$(\pi(t,x);\phi(t,x)) \mapsto (\{p(t)\},\varpi(t,x);\{q(t)\},\chi(t,x))$$

$$\left.\begin{array}{l}\phi(t,x) = \phi_{mono}\big(t,x-q(t)\big)+\chi\big(t,x-q(t)\big)\\ \pi(t,x) = \ \ldots\end{array}\right\}\text{ such that } \int [D\phi D\pi] = \int [DqDp][D\chi D\varpi]$$

$$\Psi_{mono}[q,\chi] = \psi(q)\otimes\Psi_0[\chi]$$

9

The monopole at 50

- And so the state of affairs in physics after the first 50 years was nicely summarized by Coleman:

THE MAGNETIC MONOPOLE FIFTY YEARS LATER

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1. INTRODUCTION

This is a jubilee year. In 1931, P. A. M. Dirac[1] founded the theory of magnetic monopoles. In the fifty years since, no one has observed a monopole; nevertheless, interest in the subject has never been higher than it is now.

There is good reason for this. For more than forty years, the magnetic monopole was an optional accessory; Dirac had shown how to build theories with monopoles, but you didn't have to use them if you didn't want to. Seven years ago, things changed; 't Hooft and Polyakov[2] showed that magnetic monopoles inevitably occur in certain gauge field theories. In particular, all grand unified theories necessarily contain monopoles. (In this context, a grand unified theory is one in which a semi-simple internal symmetry group spontaneously breaks down to electromagnetic U(1).) Many of us believe that grand unified theories describe nature, at least down to the Planck length. So where are the monopoles?

10

But was QFT ready to be useful for Mathematics?

- Some pros:
  - The class of renormalizable field theories is pretty general, including YM, YMH
  - Don't have to be in Minkowski signature; can analytically continue to Euclidean time as in QM, and study vacuum to vacuum matrix elements on Riemannian manifolds
- Too many cons:
  - It's messy; tricky limits $T, L, \mu \to \infty$ involved
  - Do we even know if renormalization is well-defined nonperturbatively?
  - QFT observable should be something of interest mathematically, and probably want something better than an asymptotic series in a small parameter
  - What if we want to study YMH in > 4 dimensions?
- So while mathematicians forged ahead with many remarkable results, physicists needed two more tools before QFT could be brought to bear

11

The last 40 years: SUSY and Strings

- SUperSYmmetry: an odd extension of the Poincare algebra
  - In a supersymmetric theory, the Hilbert space of states forms a representation of the supersymmetry algebra
  - At the level of actions, one adds fermions (Grassmann-valued sections of appropriate spinor bundles) with certain special couplings such that the action is invariant under field transformations mixing fermions and bosons.
- With enough SUSY, some of the cons on the previous slide are bypassed because path integrals
  - can be computed exactly via localization (c.f. Witten's Topological QFT '88)
  - Or inferred by indirect arguments utilizing symmetry (c.f. SW '94)

12

String Theory

- A quantum "field" theory in 10 or 11 dimensions (depending on the formulation) with
  - an infinite number of fields
  - A class of solitonic objects of varying co-dimension capable of hosting localized modes and interacting/joining with each other to form networks
    - Dp-branes and NS5-branes of Type II string theories
    - M2- and M5-branes of M-theory
  - Many different low energy effective descriptions depending on which fields are integrated out and which aren't, including
    - Supergravity
    - Supersymmetric Yang-Mills-Higgs in various dimensions including $d > 4$
    - Strongly coupled superconformal theories without Lagrangian descriptions
  - Often providing multiple descriptions of the same underlying state in the string theory Hilbert space

13

Three Examples

- BPS states in 4d $\mathcal{N} = 2$ YMH on $\mathbb{R}^{1,3}$
  - Wall-crossing for the L^2-kernel of certain Dirac operators on monopole moduli spaces
  - A generalization of the Sen conjecture
- Vacuum structure in 3d $\mathcal{N} = 4$ YMH on $\mathbb{R}^{1,2}$
  - Monopole moduli spaces
- Vacuum structure of 4d $\mathcal{N} = 2$ YMH on $\mathbb{R}^{1,2} \times S^1$
  - Connections to periodic monopoles and Hitchin moduli spaces

14

BPS states in 4d $\mathcal{N} = 2$

- Classical theory:
  - Fields $(A, \varphi, \psi^{1,2})$; simple compact $\mathcal{G}$; couplings g_0, θ_0
  - potential energy $\supset \|[\varphi, \bar{\varphi}]\|^2$
  - Family of vacua parameterized by $u_i = \mathrm{tr}(\varphi^{i+1})$
  - ∃ ½ BPS finite-energy solutions representing monopoles and dyons:

$$d_A X = \star_3 F_A \ , \quad \Delta_A Y + [X,[X,Y]] = 0 \ , \quad E = d_A Y$$
$$\text{where} \quad \varphi = \zeta_{cl}^{-1}(Y + iX) \ , \quad \zeta_{cl} = \zeta_{cl}(\gamma_m, X_\infty, \gamma^e, Y_\infty)$$

- Semiclassically, (and in the Manton/adiabatic regime: $\dot{q} = O(g_0)$):
  - $\int [D\phi D\pi] e^{iS} = \int [DqDp][D\chi D\varpi] e^{iS} = \int [DqDp]\, e^{iS_{c.c.}}(1 + O(g_0))$, where
  - $H_{c.c.} = \frac{1}{2} p_m g^{mn} p_n + \frac{1}{2}\|\mathrm{G}(Y_\infty)\|^2$ + fermions

$\mathrm{G}: \mathfrak{t} \to \mathfrak{isom}_{\mathbb{H}}(\mathcal{M})$

15

Semiclassically

- The space of BPS states of electromagnetic charge $\gamma = \gamma_m \oplus \gamma^e$ is

$$\ker^{(\gamma^e)}_{L^2}\left(\not{D}^{\mathcal{Y}}_{\mathcal{M}_0(\gamma_m, \mathcal{X})}\right)$$

  - $\mathcal{M}_0$ is the (strongly) centered moduli space
  - $\not{D}^{\mathcal{Y}} = \not{D} - ie \cdot \mathrm{G}(\mathcal{Y})$ is the Dirac operator twisted by Clifford contraction with the one-form dual to $\mathrm{G}(\mathcal{Y})$ restricted to the centered moduli space
  - $\mathcal{X} = X_\infty + O(g_0^2)$, $\mathcal{Y} = Y_\infty + O(g_0^2)$
  - γ^e refers to the γ^e-subspace of the kernel with respect to the electric charge operator (which is given by Lie derivative and commutes with the Dirac operator)
  - Note: this kernel can only be nontrivial when $\mathrm{rnk}(\mathcal{G}) > 1$

- Semiclassical analysis ⇒ a family of Dirac operators on monopole moduli space parameterized by $(\mathcal{X}, \mathcal{Y}) \in \mathfrak{t} \times \mathfrak{t}$

16

Wilsonian Low Energy Effective Description

- $A = A^I H_I + A^\alpha H_\alpha$, $\varphi = a^I H_I + \varphi^\alpha H_\alpha$ corresponding to $\mathfrak{g} = \mathfrak{t} \oplus (\oplus_\alpha E_\alpha)$

- $\int [D\phi D\pi] e^{iS} = \int [D\phi^I D\pi_I] e^{iS^{eff}_{\mu_0,g_0}}$ with
  - $S^{eff}_{\mu_0,g_0} = S^{two-der} + O((\partial_x/\mu_0)^4)$
  - SUSY fixes the form of the two-derivative L.E.E.A. in terms of a single function – the prepotential: $\mathcal{F}(a^I,\Lambda)$, where $\Lambda = \mu_0 e^{2\pi i h^\vee \tau_0}$ with $\tau_0 = i\frac{4\pi^2}{g_0^2} + \frac{\theta_0}{2\pi}$
  - Perturbative computations, discrete symmetries, and SUSY dictate
  $$\mathcal{F} = \mathcal{F}^{cl} + \mathcal{F}^{1loop} + \sum \mathcal{F}_n^{inst}$$
  - The fields a^I take asymptotic values $a^I(u)$, depending on the vacuum $\{u\}$
  - Λ is the strong-coupling scale. When $u_i \sim \Lambda$, the running coupling $g(u_i)$ blows up, and the perturbative reasoning that identified A^I, a^I above as the "light" fields is invalid

17

Seiberg-Witten Solution

- SW found $\mathcal{F}$ using SUSY and consistency in known limits in conjunction with the action of an $SL(2r,\mathbb{Z})$ (electromagnetic duality) symmetry (SW '94)
  - The solution is in terms of an elliptic curve $\Sigma(u,\Lambda)$ and a differential λ.
  - $a^I(u), a_{D,I}(u)$ are the period integrals of the curve, and $a_{D,I} = \frac{\partial\mathcal{F}}{\partial a^I}$
- Later, Nekrasov obtained $\mathcal{F}$ directly from a Euclidean path integral computation through localization (Nekrasov '02)

- BPS states of e.m. charge $\gamma = \gamma_m \oplus \gamma^e$ in the SW description:
  - Have exact mass: $M_\gamma(u) = |Z_\gamma(u)|$, with $Z_\gamma(u) = \gamma_m^I a_{D,I}(u) + \gamma_I^e a^I(u)$
  - Can decay across real-codim. 1 walls $W_{\gamma_1,\gamma_2} \subset \mathcal{B}$
  - The degeneracy of states gained/lost upon crossing a wall is determined by the Kontsevich-Soibelman wall-crossing formula (KSWCF) (DM '07, KS '08, GMN '08)

18

Comparing the two Descriptions

- All quantum (g_0) corrections to collective coordinate QM at the two-derivative level are captured by identifying (MRV, '16)

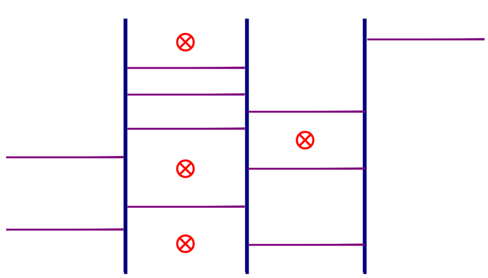
$$\mathcal{X} = Im(\zeta^{-1} a(u)) \qquad \text{where } \zeta = \frac{Z_\gamma(u)}{|Z_\gamma(u)|}$$
$$\mathcal{Y} = Im(\zeta^{-1} a_D(u))$$

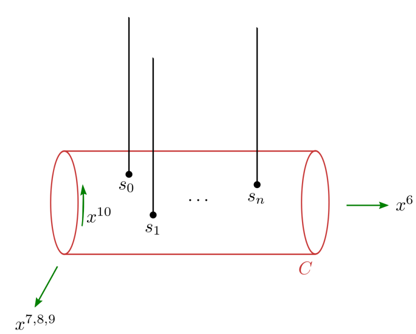
- Wall-crossing therefore dictates that the L^2-kernel of the Dirac operator must jump according to the KSWCF
  - Verified in cases where the WCF reduces to the primitive one (BMR, '18)
  - Intricacies of WCF in non-primitive cases must be due to intricacies of monopole moduli space asymptotics (see M. Singer and C. Kottke's talks)
- Absence of certain "exotic" BPS states implies all ($\mathrm{G}(\mathcal{Y})$-twisted) L^2-Dolbeault cohomology concentrated in middle (antiholomorphic) degree
  - When semiclassical construction generalized to include matter representations, this is a generalization of the Sen conjecture (BM, '16)

19

Generalizations

- The class of 4d $\mathcal{N} = 2$ theories for which the SW curve is known is vast. It includes:
  - Theories like the ones just discussed but with matter representations
  - More general quiver gauge theories (coupled to strongly interacting SCFT's)
- Almost all examples are of "class $\mathcal{S}$", meaning they descend from M5-branes wrapping a Riemann surface $\mathcal{C}$ (Witten '96, Gaiotto '09, GMN '09)
- Witten's IIA/M-theory brane construction:





20

3d $\mathcal{N} = 4$ YMH

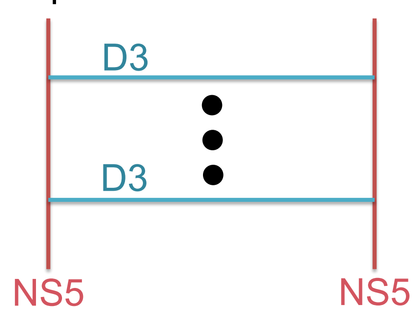
- Classically: (gauge group SU(k) for simplicity)
  - Fields: vector multiplets $(A, \vec{X}, \psi^{1,2})$ and hypermultiplets $(b, \vec{Y}, \tilde{\psi}^{1,2})$;
  - Potential energy $\supset \sum \|[X_i, X_j]\|^2$
  - Vacua: (coulomb branches and Higgs branches)

    coulomb branches → $(\mathbb{R}^3 \times S^1)^k / S_k$
- Low energy effective theory for $(\vec{x}^I, \theta_{m,I}),\ I = 1, \dots, k$
  - A 3d σ-model $\mathbb{R}^{1,2} \to \mathcal{M}$; supersymmetry dictates $g_{\mathcal{M}}$ must be hyperkahler
  - Integrating out massive fields at one loop corrects the flat metric on $(\mathbb{R}^3 \times S^1)^k / S_k$ to the Gibbons-Manton metric on the asymptotic charge k centered monopole moduli space
  - Claim: Instanton corrections (where the “instantons” are actually monopoles on the Euclidean spacetime $\mathbb{R}^3$!) will complete the metric to the true monopole moduli space metric (SW ‘96, CH ‘96, DKMTV ‘97)

21

3d $\mathcal{N} = 4$ YMH

- A string theory brane set-up makes this “obvious”: (HW ’96)



- Many generalizations and connections…

22

4d $\mathcal{N} = 2$ YMH on $\mathbb{R}^{1,2} \times S^1$

- Classical theory is one from before, but now on $\mathbb{R}^{1,2} \times S^1$ instead of $\mathbb{R}^{1,3}$
- Wilsonian low energy effective action?
  - Start with SW theory in terms of $(A^I, a^I, \psi^I_{1,2})$ and restrict to x^3-independent configuration
  - $2r$ new vacuum parameters:
    - The vevs of the periodic scalars obtained from dualizing the 3d gauge fields: $\theta_{m,I}$
    - The holonomy of A^I around the circle: θ^I_e
  - Space of vacua is an elliptic fibration over the Coulomb branch.
  - L.E.E.A is a 3d $\mathcal{N} = 4$ sigma model with this space as target – SUSY demands the metric on the target be hyperkahler
    - Simply changing variables $(A^I, a^I) \to (a^I, \theta^I_m, \theta^I_e)$ gives a hk metric – the "semi-flat metric", but it is singular
    - Are there new contributions to the Euclidean path integral over massive fields that we might have missed?

23

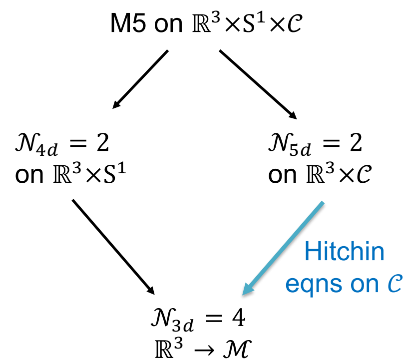
4d $\mathcal{N} = 2$ YMH on $\mathbb{R}^{1,2} \times S^1$

- Yes – the BPS states of 4d $\mathcal{N} = 2$ on $\mathbb{R}^{1,3}$ can be taken to wrap the circle in the Euclideanized theory on $\mathbb{R}^3 \times S^1$, where they become instantons!
  - In principle, if we know the spectrum of BPS states, then we know the form of these instanton corrections and we can write down a formula for the corrected metric
  - But the BPS spectrum undergoes wall-crossing – there must be an interplay between one-instanton and multi-instanton sector contributions, to the metric such that the metric is smooth
    - Initially, smoothness of the metric on the total space of this fibration was used as a criterion to infer the KSWCF for BPS states (GMN '08)
    - Now, this machinery is being used to extract explicit corrections to the semi-flat metric on these hyperkahler ALG spaces
- Do these ALG spaces have other descriptions?

24

String Theory Perspective

- The M5-brane picture (GMN '09):
  - ⇒ $\mathcal{M}$'s are Hitchin moduli spaces
  - Recovering metric aymptotics from the Hitchin picture a major area of active research (see Fredrickson's talk)



- The IIA perspective:
  - Make x^3 a circle and T-dualize
  - ⇒ CHW brane picture with D3's localized on S^1
  - ⇒ periodic monopoles
- Cherkis and Kapustin showed these two objects are Nahm transforms of each other – at least for $G = \mathrm{SU}(k)$ (CK '00, '01)
  - How far into "class $\mathcal{S}$" does this correspondence extend?

25

Other areas I should have touched on

- What happens if we compactify another circle?
  - Monowalls and ALH
  - What does String Theory have to say?
- (singular) monopoles
  - Supersymmetric defects in YMH
  - Geometric Langlands
- Higher d monopoles and their dimensional reductions
  - Brane constructions
  - BVP's and holography
- String compactifications on special holonomy manifolds

The monopole is alive and well. It has facilitated a new era of connectedness between math and physics.

26