Folding Sevens: The Power of Origami

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Banff International Research Station for Mathematical Innovation and Discovery.

Folding Axioms

There are seven Huzita-Justin axioms that describe the possible geometric operations through paper folding:

1. Two points define a unique fold that passes through both of them (similar to "two points define a line" in standard Euclidean geometry)
2. Two points define a unique fold that places one point onto the other (the fold thus created being the perpendicular bisector of the line segment with the given two points as endpoints)
3. Given any two lines (folds), there exists a fold that places one line onto the other (if the lines intersect, this line is an angle bisector)
4. Given one point and one line, there exists a unique fold perpendicular to the given line that passes through the given point (accomplished by folding the line onto itself)
5. Given two points and one line, there exists a fold that places one point on the line that passes through the other point (equivalent to finding the intersection of a line with a circle)
6. Given two points and two lines, there exists a fold that places one point onto each line (equivalent to finding the mutual tangent line to two parabolas whose foci are the given points and directrices are the given lines)
7. Given one point and two lines, there exists a fold perpendicular to one line that places the point on the other line

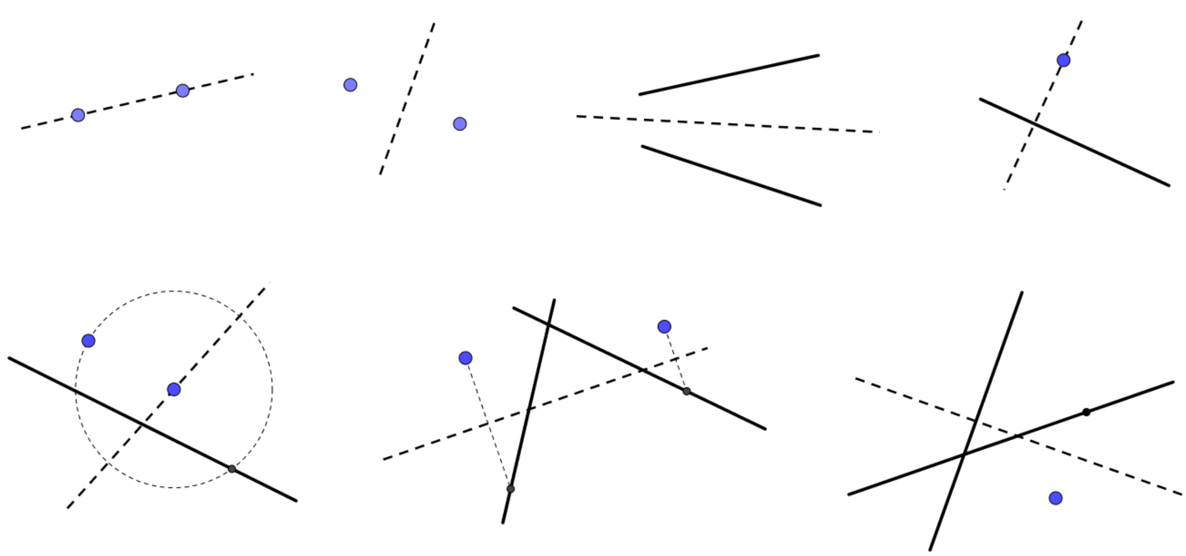

Figure 1 – The seven origami axioms. Dashed lines indicate the folds defined by each axiom.

Of particular interest to us is the 6th axiom and the Beloch fold named for Italian mathematician Margharita Beloch who used this technique to demonstrate the solution of cubic equations. This origami move allows for the construction of many geometric solutions that are not possible with a compass and straightedge, including the ancient problem of doubling the cube, trisecting angles, and constructing regular heptagons.

Mathematical background

Above, we mentioned that the Beloch move is "equivalent to finding the mutual tangent line to two parabolas whose foci are the given points and directrices are the given lines." As the reader may be unfamiliar with or have forgotten some of these terms, we review these concepts here.

One way to think of the graph of a quadratic function is as the set of points in the plane that are equidistant from a single point (the focus) and a line (the directrix). In Figure 2, point C is the focus and line AB is the directrix. Line segments g and j represent the shortest distances from point D on the parabola to each of the focus and directrix.

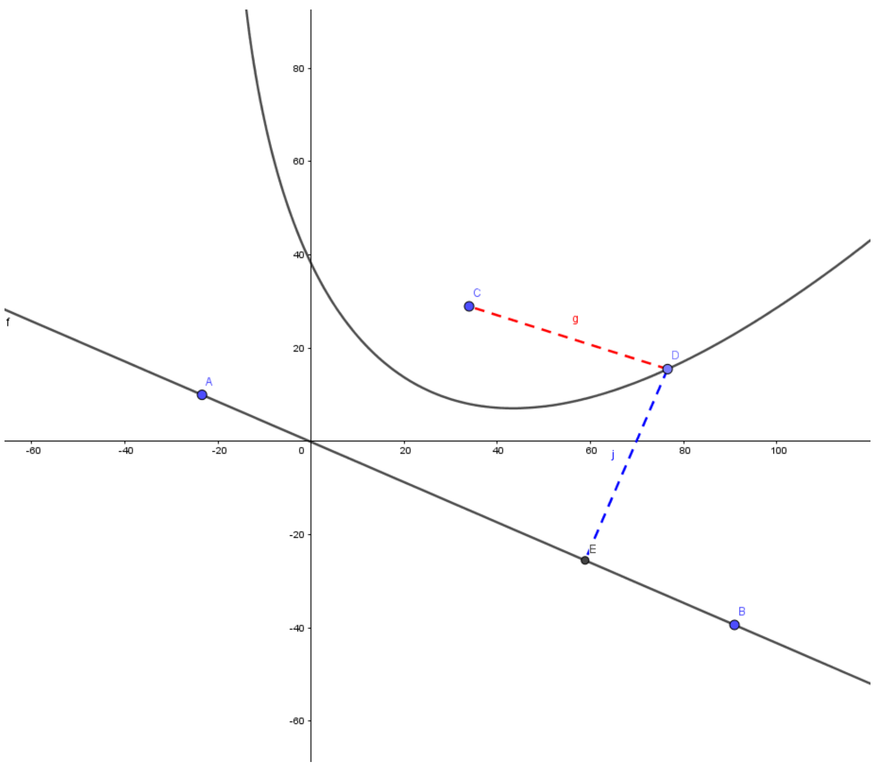

Figure 2 – A parabola is the set of points in the plane that are equidistant from a single point and a line.

Through origami, we can construct a so-called "envelope" of a parabola, or the approximation of the parabola's curve determined by a set of tangent lines, by repeatedly folding a given point onto a given line. Tom Hull has an excellent exploration of this exercise in [Hul06]. This is significant for many reasons, one of which is that determining the equation of a tangent line to a function in general is not trivial and usually requires calculus. The crease created when we fold the focus of a parabola to any point on the directrix is exactly a tangent line to the parabola defined by that focus and directrix. Repeating this process across the directrix results in a lovely set of lines that approximate the shape of that parabola. See Figure 3.

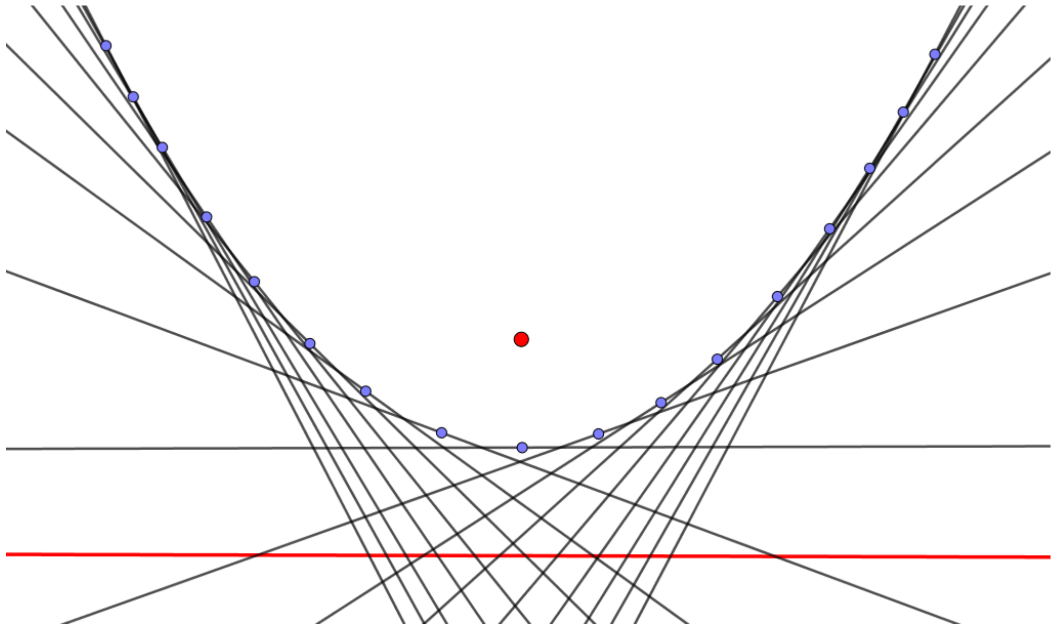

Figure 3 – It is possible to approximate the shape of a parabola by constructing a series of tangent lines.

Now, imagine that we have two parabolas, each defined by a given point and line representing their respective focus and directrix. The sixth origami axiom tells us that there is at least one fold we can make that places each point onto each line. The resulting crease is tangent to both parabolas. Figure 4 shows a mathematical model of three mutual tangent lines to two parabolas.

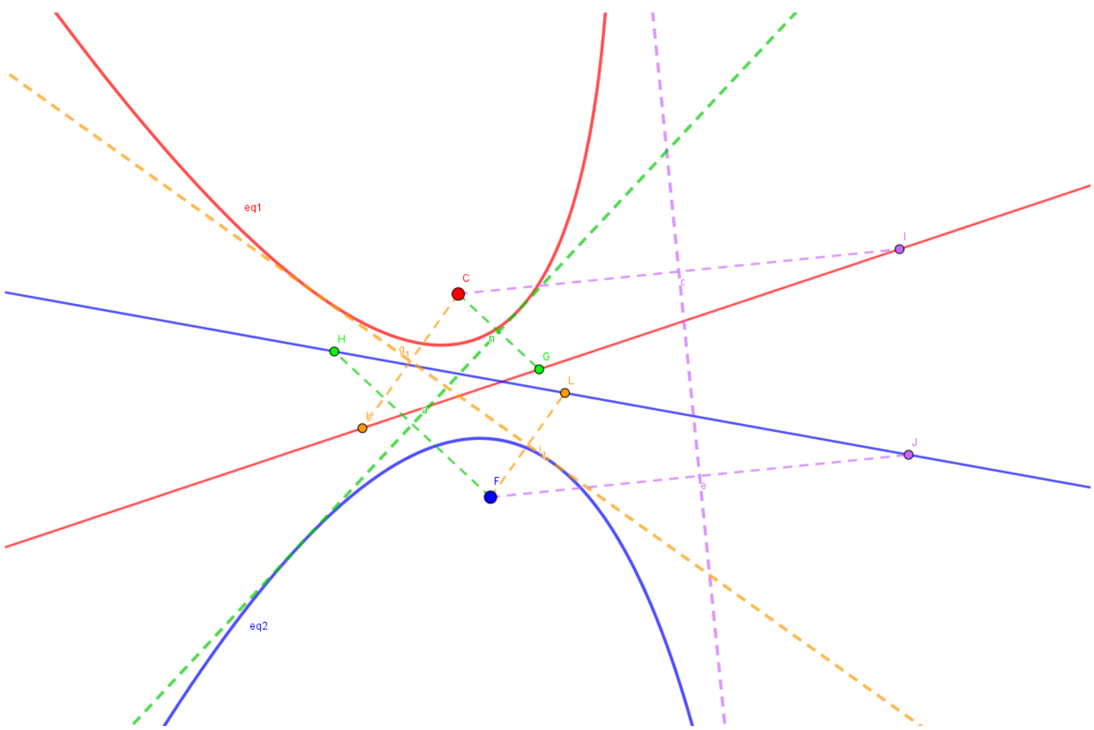

Figure 4 – Two parabolas can have up to three mutual tangent lines.

Folding Sevens

Of particular interest to the authors is the power of origami to construct regular polygons that are not constructable with a compass and straightedge. Because the compass and straightedge only allow for angle bisection, the only regular polygons that one can construct using those tools are those with sides that are either a power of two or a product of a power of two and any number of unique Fermat primes. A Fermat prime is a prime number of the form $2^{2^n} + 1$. The only known Fermat primes are 3, 5, 17, 257, and 65537. So, for example, we can construct through compass and straightedge a regular hexadecagon ($16 = 2^4$) or a dodecagon ($12 = 2^2 \cdot 3$), but not a regular enneagon ($9 = 3^2$) or a regular heptagon (7). If we allow for angle trisection, then polygons with a Pierpont prime ($2^u 3^v + 1$) number of sides are constructable, as well as powers of two times powers of three times any unique combination of Pierpont primes. Because origami allows us to trisect angles exactly, we can construct a regular triscadecagon ($13 = 2^2 \cdot 3 + 1$) and, as we are concerned with here, a regular heptagon ($7 = 2 \cdot 3 + 1$).

We like the heptagon construction that Tom Hull demonstrates and has clear step-by-step constructions of in [Hul09]. Figure 5 shows the two parabolas defined by having the vertical and horizontal axes or midlines of the square as their directrices and the bolded points in red and blue as their foci superimposed on the heptagon construction that folds these two points to these two lines in order to locate the key distance from the origin required in order to construct the heptagon: $2cos(\frac{2\pi}{7})$, represented by point H. Also visible in this diagram is the path between our two foci determined by the coefficients of the polynomial of which $2cos(\frac{2\pi}{7})$ is a solution, $z^3 + z^2 - 2z - 1 = 0$ and the congruent angles along that path, which Lill's method demonstrates as a way to solve such cubic polynomials. Finally, the crease resulting from the Beloch move determines one side of the so-called Beloch square, which Margharita Beloch used to demonstrate through folding the solution of the same cubic polynomials. We will not get into the details of why all of this works here, but see [Hul09], [Hul11], and [Alp09] for detailed explanations of why $2cos(\frac{2\pi}{7})$ is a solution of the equation $z^3 + z^2 - 2z - 1 = 0$ and how Lill's method allows us to solve such cubic equations.

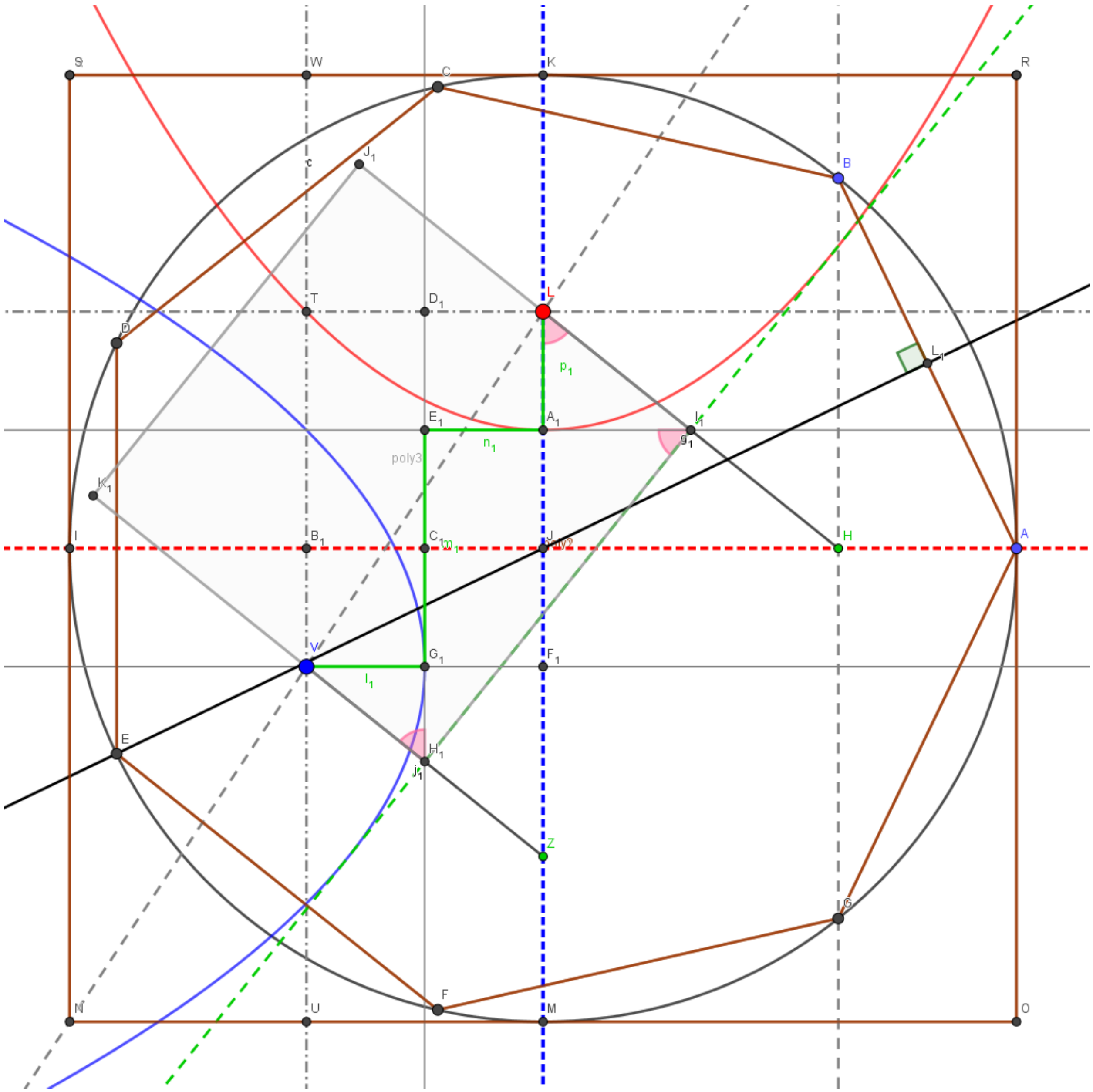

Figure 5 – Folding the mutual tangent to two parabolas allows us to construct a regular heptagon.

Sevens in Islamic Art

Seven-fold motifs are not common in Islamic Geometric designs, but they do appear occasionally. Often, they are irregular heptagons that are a byproduct of other parts of a pattern's construction, but sometimes the seven-foldness is central to a design. Because ruler and compass do not allow for the construction of regular heptagons and Islamic artisans were following the principles of Euclidean geometry, they generally used approximations. Both Heron of Alexandria (d. 70 AD) in his *Metrica* and Abu al-Wafa' Buzjani (d. 998 AD) in his treatise for craftsmen used $\sqrt{3}/2$ as an approximation for the side length of a heptagon with circumscribed circle radius 1. This and several other historical heptagon constructions can be found in [Sut09]. Trigonometry was well-formed and had spread to Islamic scholars from India by the 8th century, and we have documentation that by 1000 AD, Persian scholars could solve cubic equations. Documents like the *Anonymous Persian Compendium*, of which a facsimile appears in [Nec17], reveal the process by which patterns with regular heptagons (as well as the many compass-constructable polygons) were constructed. The instructions for the pattern on Folio 192 (See Figure 6) of that compendium begin with the construction of angle BAG that is "three-sevenths of the right angle" with no further instruction on how to determine that angle. Other diagrams in the compendium show the use of conic sections to solve cubic equations, but this particular angle was probably achieved using a set square, as there are no marks in the diagram indicating any preliminary construction prior to this step.

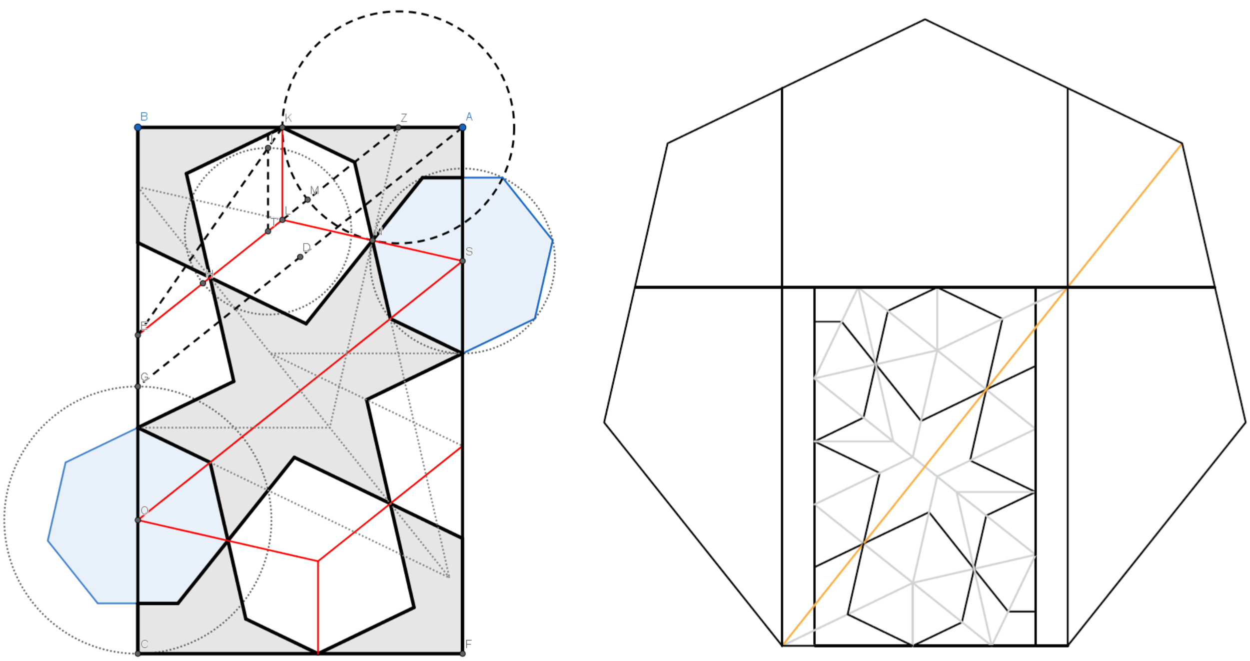

Figure 6 – Pattern from Folio 192 of the *Anonymous Persian Compendium*. Red and dark dashed lines on the left are the only direction given in the compendium and the rest "should be easy, God…willing." The construction placed inside of a regular heptagon on the right (drawing by @kamikyodai 2021) reveals some of the elusiveness and unique symmetry that this pattern possesses.

Star rosette patterns in Islamic art are characterized by petals whose outer four sides are of equal length and (usually irregular) five-pointed stars inscribed in circles. Often these petals have parallel edges (as seen in Figure 7), but styles vary across regions and to fit the needs of the particular larger pattern that such a star rosette motif may be situated within.

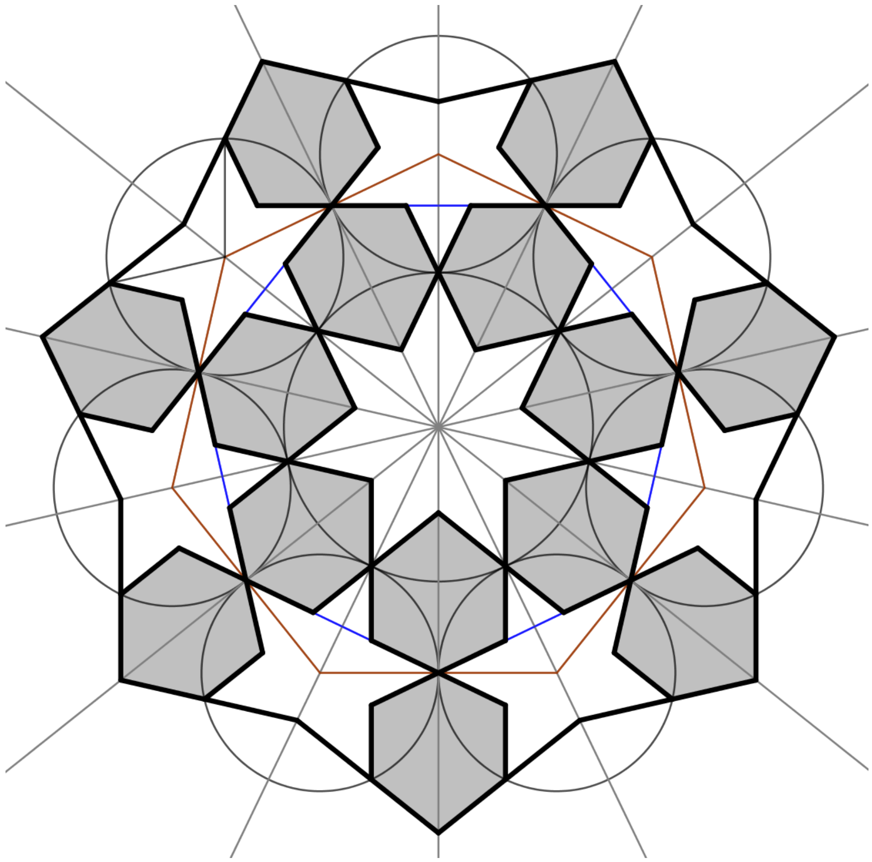

Figure 7 – A seven-fold Islamic star rosette pattern with canonically-proportioned petals.

The authors feel that origami is especially well-suited to the construction of Islamic geometric patterns, not only for its ability to form constructions impossible with ruler and compass alone, but for the ease with which mirror symmetry can be carried out through folds. There is also potential for Islamic geometric patterns to be used as crease patterns for three-dimensional origami models in addition to lending themselves quite nicely to corrugations and tessellations. We demonstrate how to construct, through origami folds alone, both a seven-fold rosette pattern and the hexagonal motifs derived from heptagons that make up the pattern from Folio 192 of the *Anonymous Persian Compendium* in the accompanying folding instructions.

Sources

All figures except 6b constructed by Sarah Brewer with Geogebra Classic 5.
Interactive Geogebra files for select figures can be accessed through the following links:

Figure 1 – Origami axioms, https://www.geogebra.org/m/grdxewgw

Figure 2 – Parabola from focus and directrix, https://www.geogebra.org/m/qnxw6p27

Figure 4 – Mutual tangents to two parabolas, https://www.geogebra.org/m/kqckkez7

For further reading:

[Alp09] R. Alperin and R. Lang, "One-, two-, and multi-fold origami axioms," *Origami4: Proceedings of the 4th International Meeting of Origami Science, Mathematics and Education*, R. Lang, ed., 2009.

[Buz] Abu al-Wafa' Buzjani, *A Book on Those Geometric Constructions Which Are Necessary for a Craftsman*, available from https://gallica.bnf.fr/ark:/12148/btv1b52503120k/

[EM94] J. Emert, K. Meets, and R. Nelson, "Reflections on a Mira," *The American Mathematical Monthly*, Vol 101, No 6, Jun-Jul 1994.

[Gle88] A. Gleason, "Angle trisection, the heptagon, and the triskadecagon," *The American Mathematical Monthly*, Vol 95, No 3, Mar 1988.

[Hul06] T. Hull, *Project Origami: Activities for Mathematics Classes*, AK Peters, 2006.

[Hul09] T. Hull, "Folding regular heptagons," in *Homage to a Pied Puzzler*, E. Pegg Jr., et al., eds., A.K. Peters, 2009.

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[Jus89] J. Justin, "Résolution par la pliage de l'équation du troisième degré et applications géométriques," *Proceedings of the First International Meeting of Origami Science and Technology*, H. Huzita, ed. 1989.

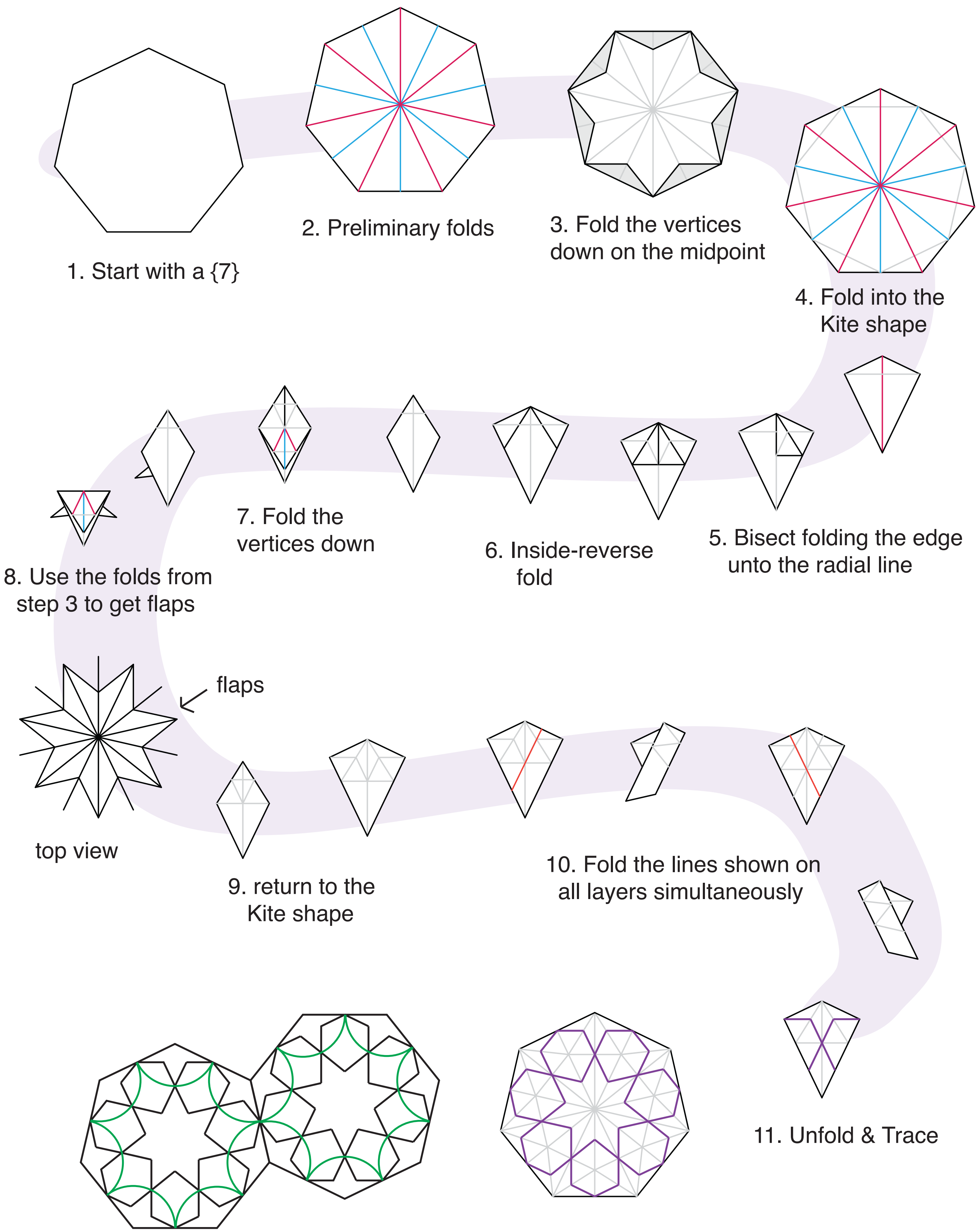
[Kat18] V. Katz, *A History of Mathematics: An Introduction, 3rd ed.*, Pearson, 2018.

[Lee75] A. Lee, "Islamic star patterns ~ notes: summaries of main results of researches into the geometry of Islamic star patterns during Nov 1964-," available from http://www.tilingsearch.org/tony/

[Nec17] G. Necipoğlu, ed., *The Arts of Ornamental Geometry, A Persian Compendium on Similar and Complementary Interlocking Figures. A Volume Commemorating Alpay Özdural*, Brill, 2017.

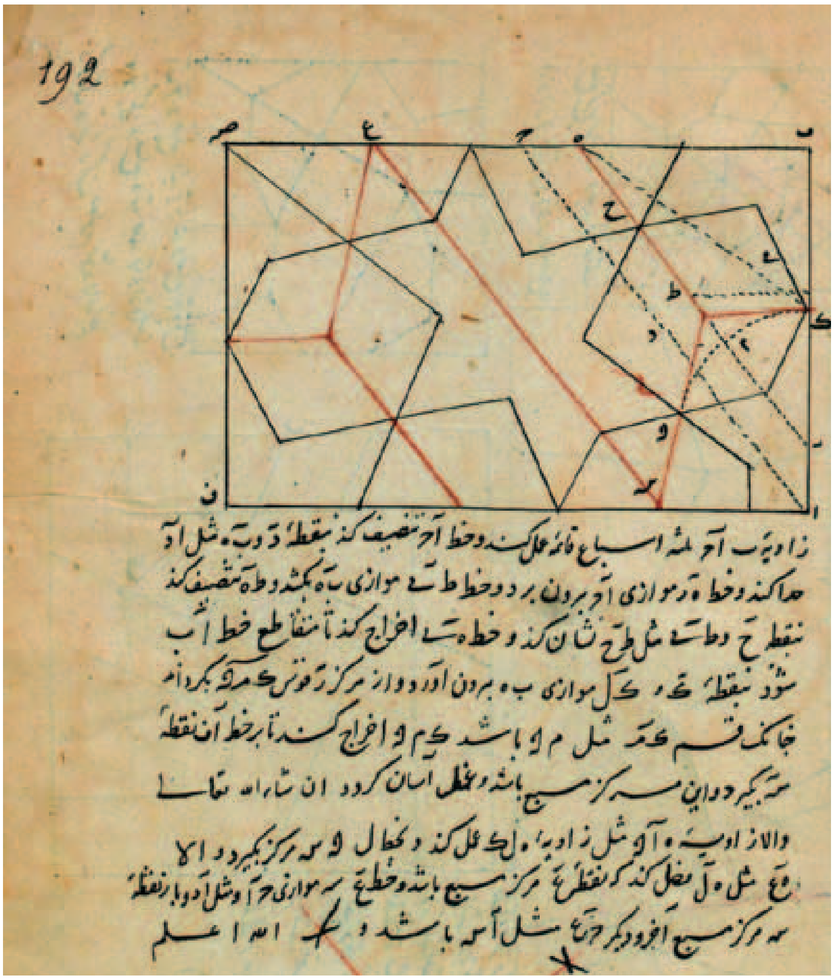
[Sut09] A. Sutton, *Ruler & Compass: Practical Geometric Constructions*, Wooden Books, Walker & Company, 2009.

7 Fold Rosette
GEAR 2021



7-Fold Islamic Pattern from the Anonymous Persian Compendium
Folio 192r - GEAR 2021

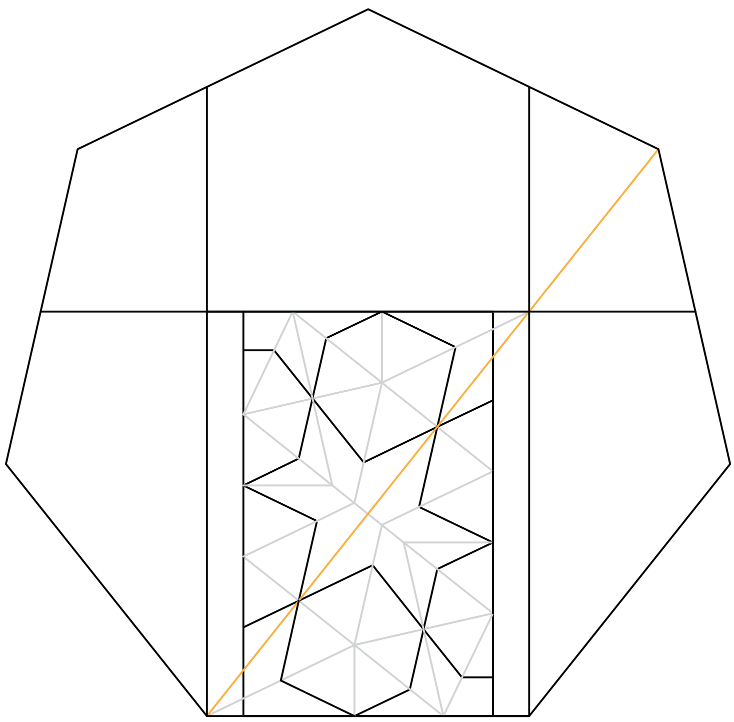
Folding Sevens: The power of origami
A workshop by Sarah “Mathemartiste” Brewer and Ricardo “Kamikyodai” Hinojosa



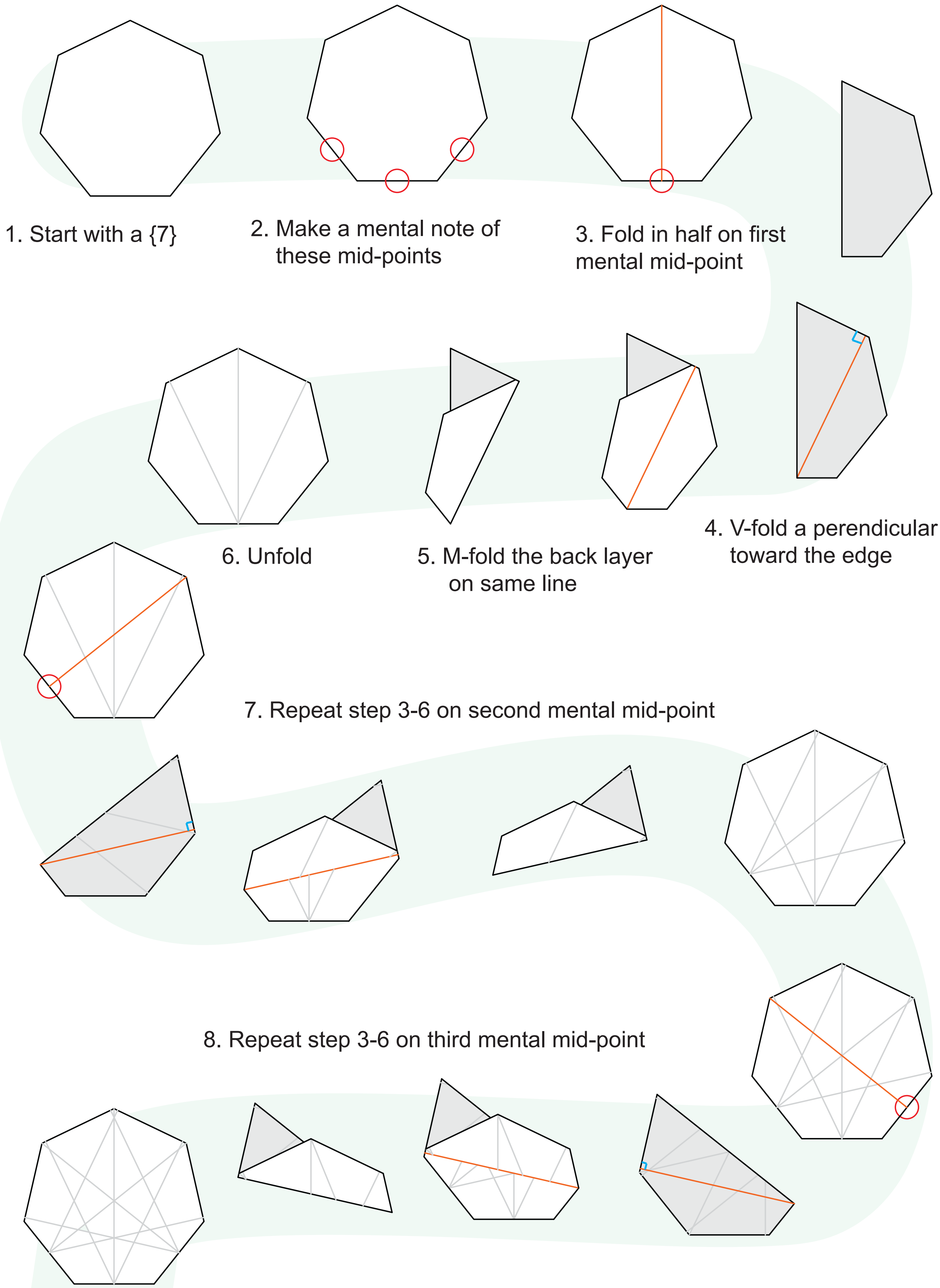
Rectangular repeat-unit of a geometric construction with regular heptagons, irregular hexagons and two different kind of six-pointed stellate forms.

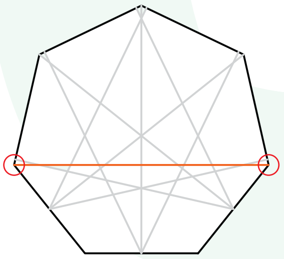
From the Anonymous Compendium. Paris, Bibliotheque nationale de France, Ms. Persan 169, fol. 192r

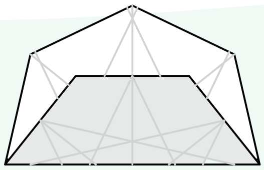
The construction placed inside of a regular heptagon reveals some of the elusiveness and unique symmetry this pattern posseses.



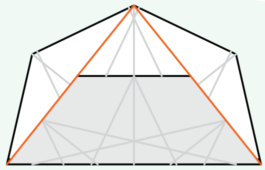
7-Fold Islamic Pattern from the Anonymous Persian Compendium
Folio 192r - Hexagon Tile A - Folding design by Kamikyodai - GEAR 2021



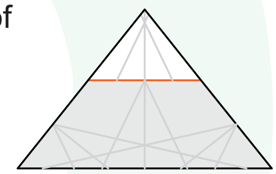


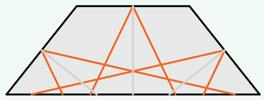


9. V-fold a line that connects these two vertices

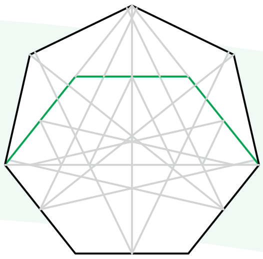


10. M-fold the outline of the top folded layer

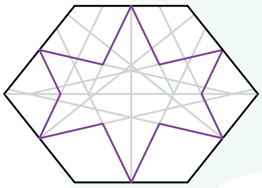




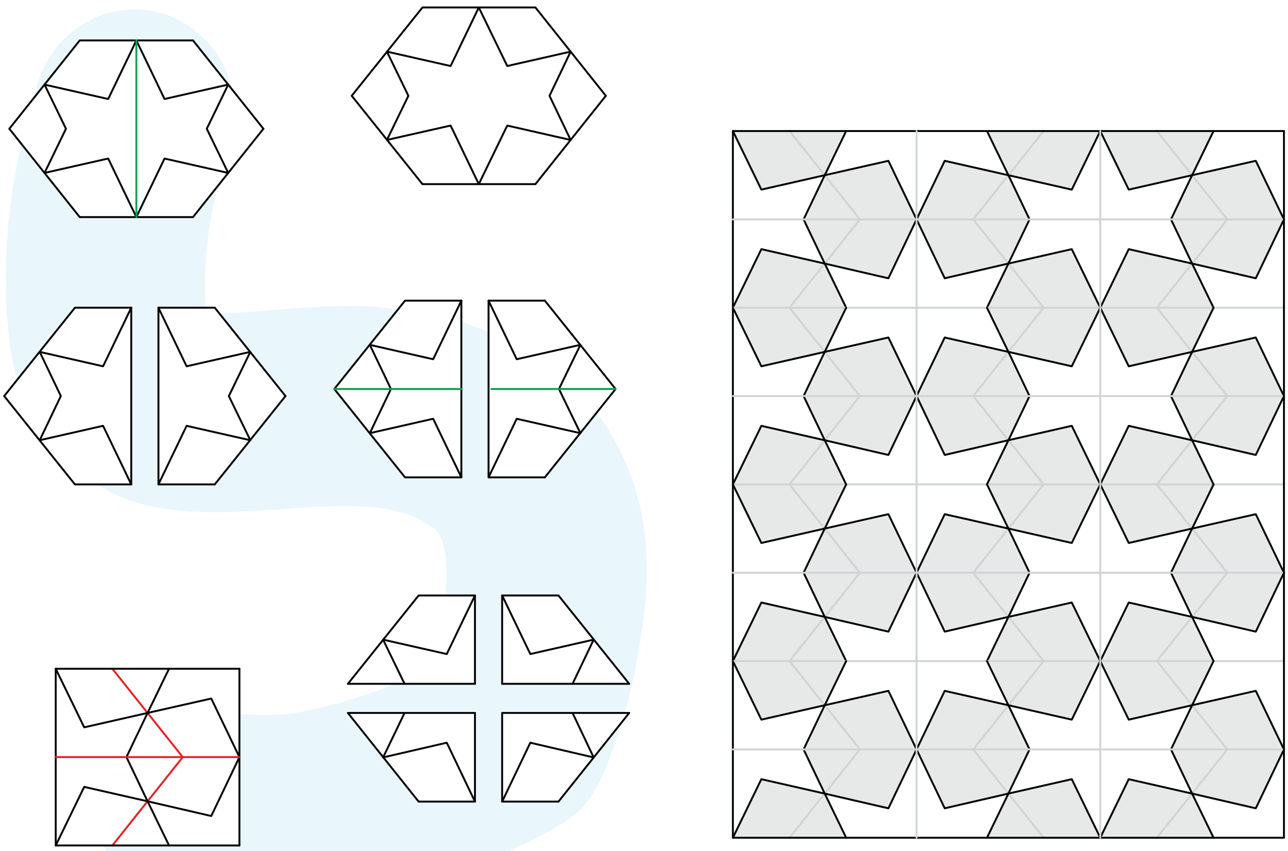
11. Fold the highlighted lines on both layers



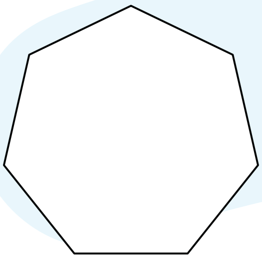
12. Cut the hexagon



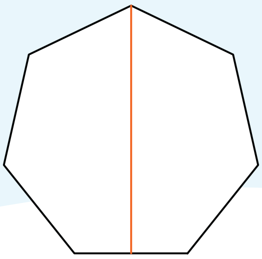
13. Trace



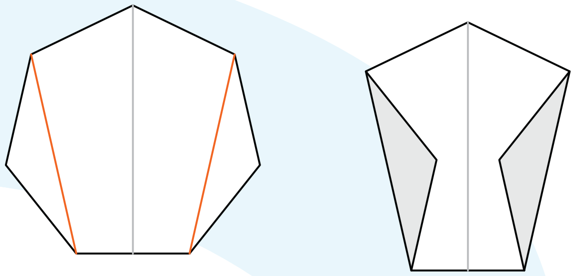
7-Fold Islamic Pattern from the Anonymous Persian Compendium
Folio 192r - Hexagon Tile B - Folding design by Kamikyodai - GEAR 2021



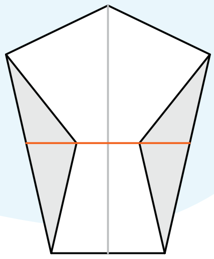
1. Start with a {7}



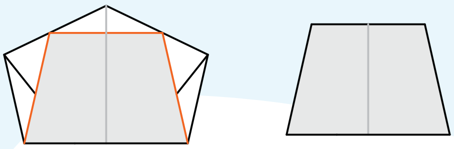
2. Fold in half



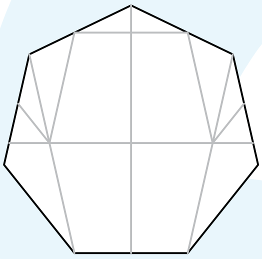
3. V-fold lines from the bottom vertices to the top vertices.



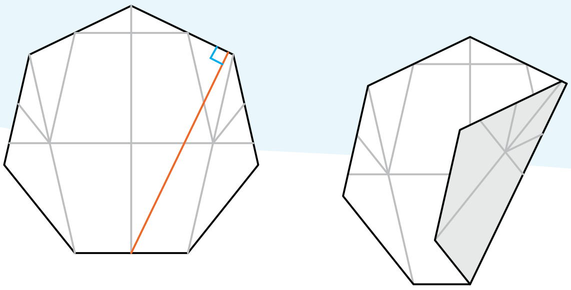
4. where the vertices line up, V-fold so the vertices touch the edge



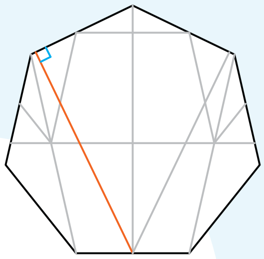
5. M-fold to only see the trapezoid



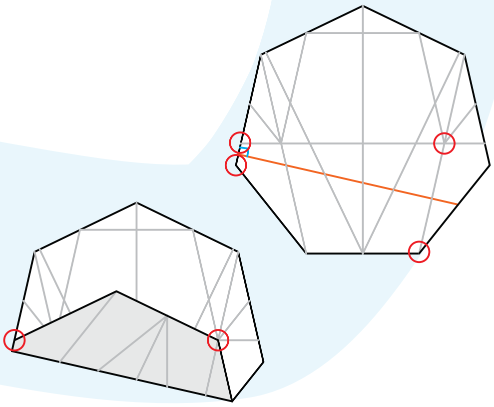
6. Unfold



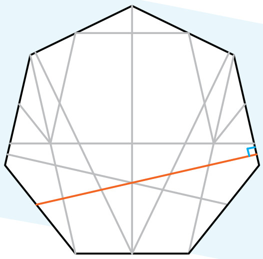
7. From the mid point of the bottom edge, V-fold a perpendicular toward the top edge



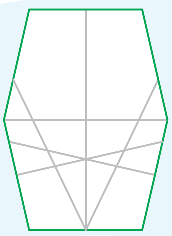
8. Repeat on other side



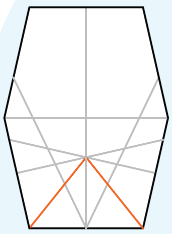
9. V-Fold the bottom vertices toward the intersection on the line from step 4



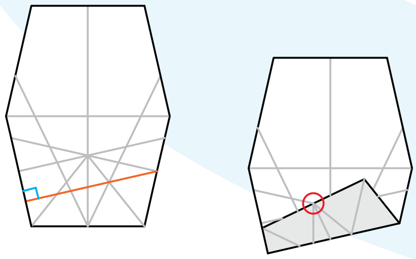
10. Repeat on other side



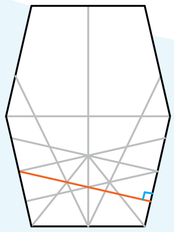
11. Cut out the hexagon



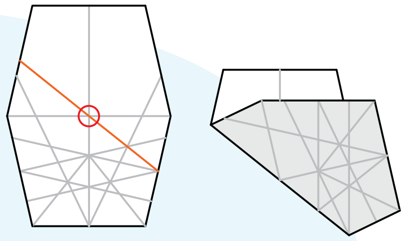
12. Fold the vertices toward the intersection



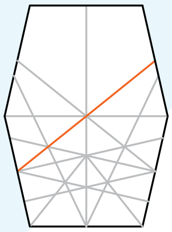
13. Fold a perpendicular on the line from step 9 & 10



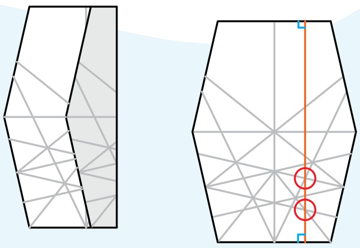
14. Repeat on other side



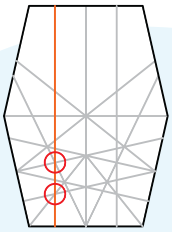
15. From the same point fold a line to the center



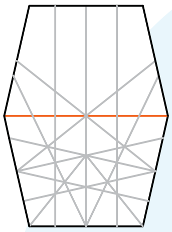
16. Repeat on other side



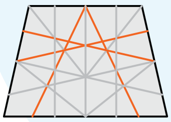
17. Fold a perpendicular that goes through intersections



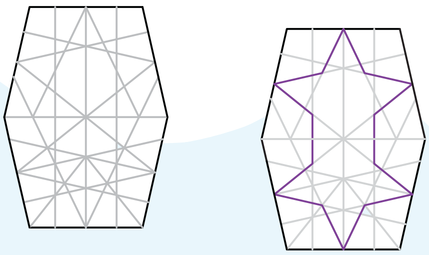
18. Repeat on other side



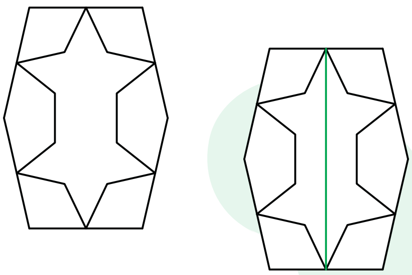
19. Fold in half

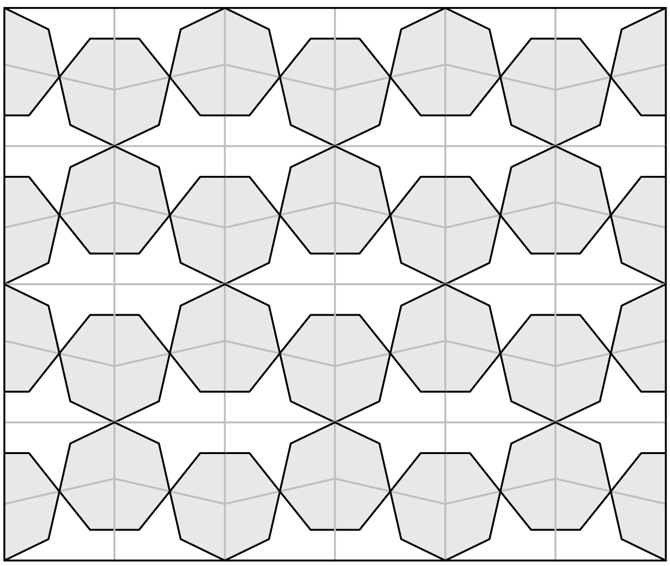


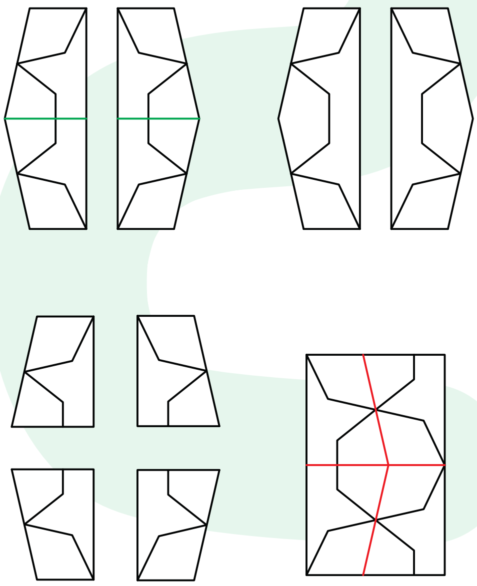
20. Fold lines shown on back layer



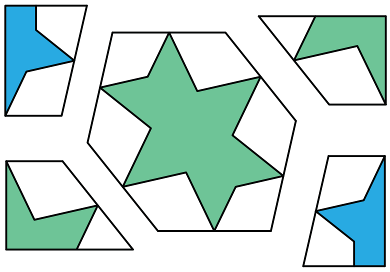
21. Unfold and Trace



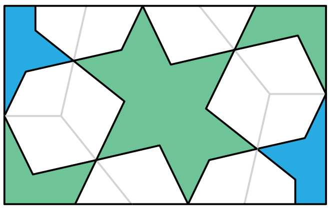


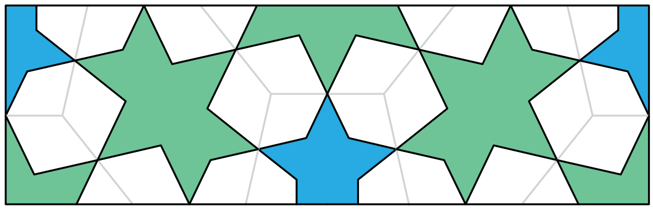


7-Fold Islamic Pattern from the Anonymous Persian Compendium
Folio 192r - Tile Fusion - Folding design by Kamikyodai - GEAR 2021

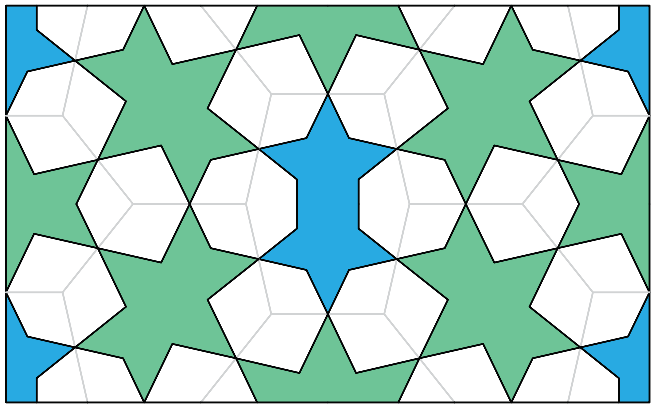


To get the rectangle from the Compendium, you will need:
-- One full Tile A
-- Two 1/4 of Tile A
-- Two 1/4 of Tile B





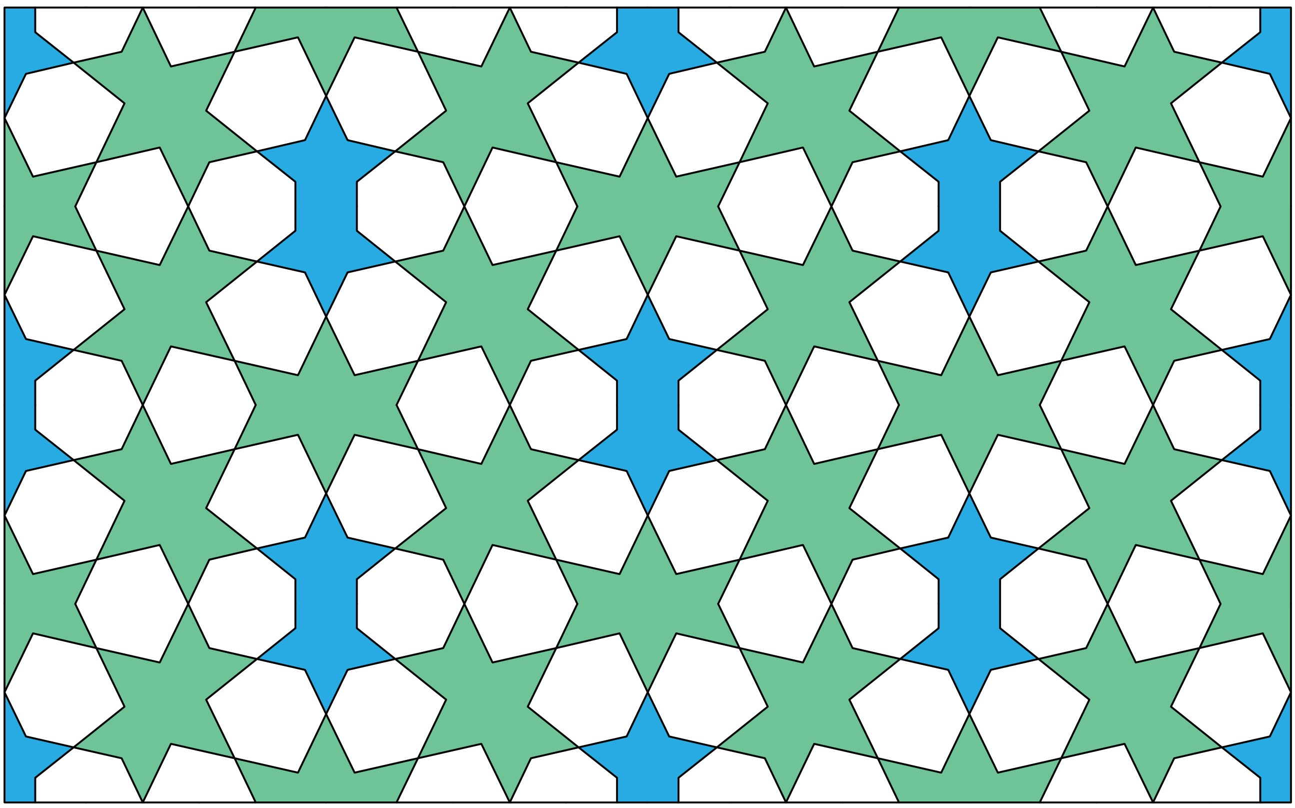
You will notice that the tile needs to be mirrored not once, but twice, in order to tessellate.



For a translational rectangular tile you will need:
-- Four full Tile A
-- Two 1/2 of Tile A cut horizonally
-- Two 1/2 of Tile A cut vertically
-- One full Tile B
-- Four 1/4 of Tile B

A total of:

-- Six Tile A
-- Two Tile B



Happy folding!

Contact

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