The cycle structure of Euclidean random permutations

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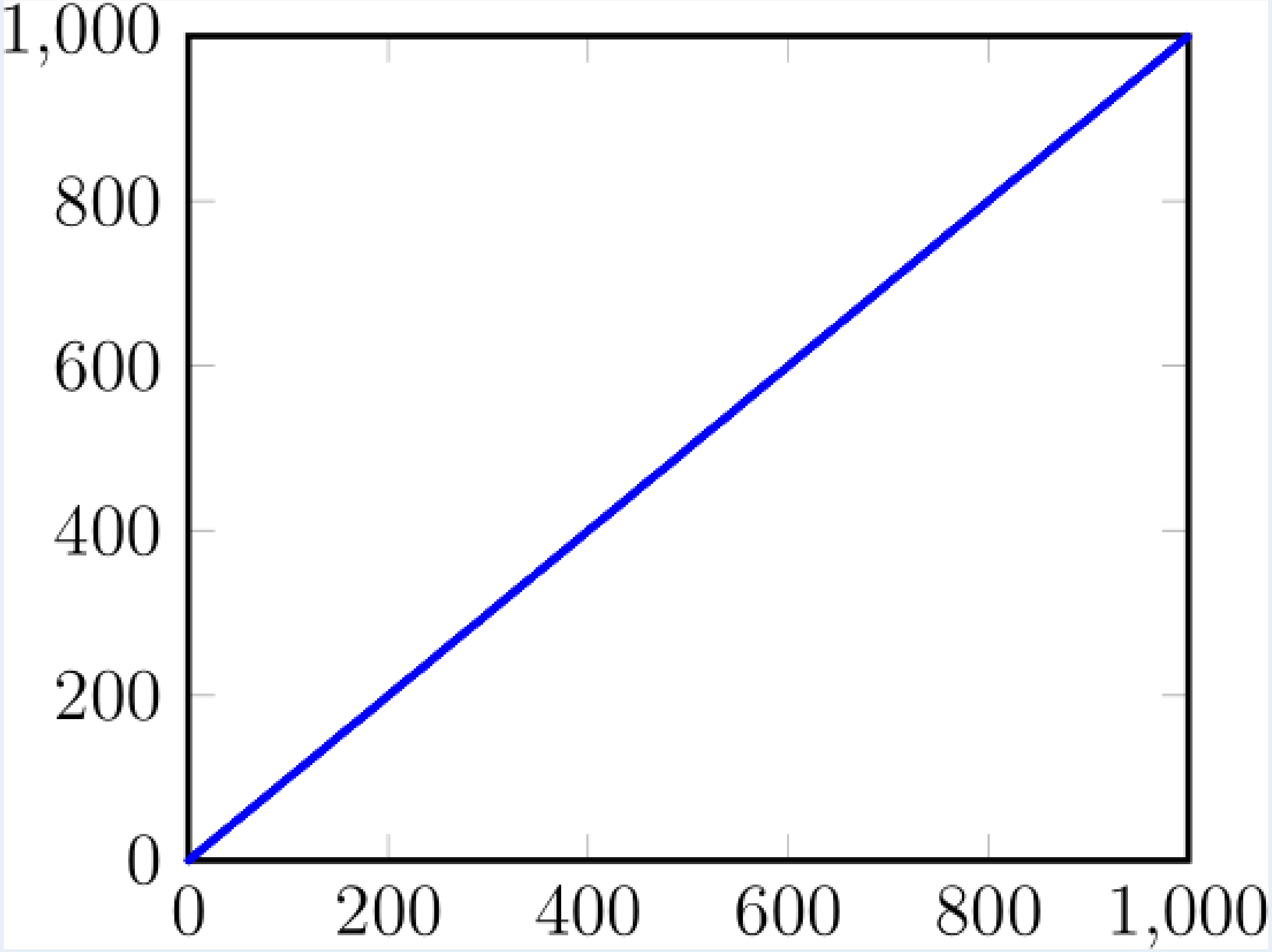
Spatial random permutations

- A spatial random permutation is a random permutation which is biased towards the identity in some underlying geometry.
- Example: (X, d) finite metric space. Sample a permutation π of the points of X with probability proportional to

$$\exp\left(-\sum_{x\in X} d(x, \pi(x))\right)$$

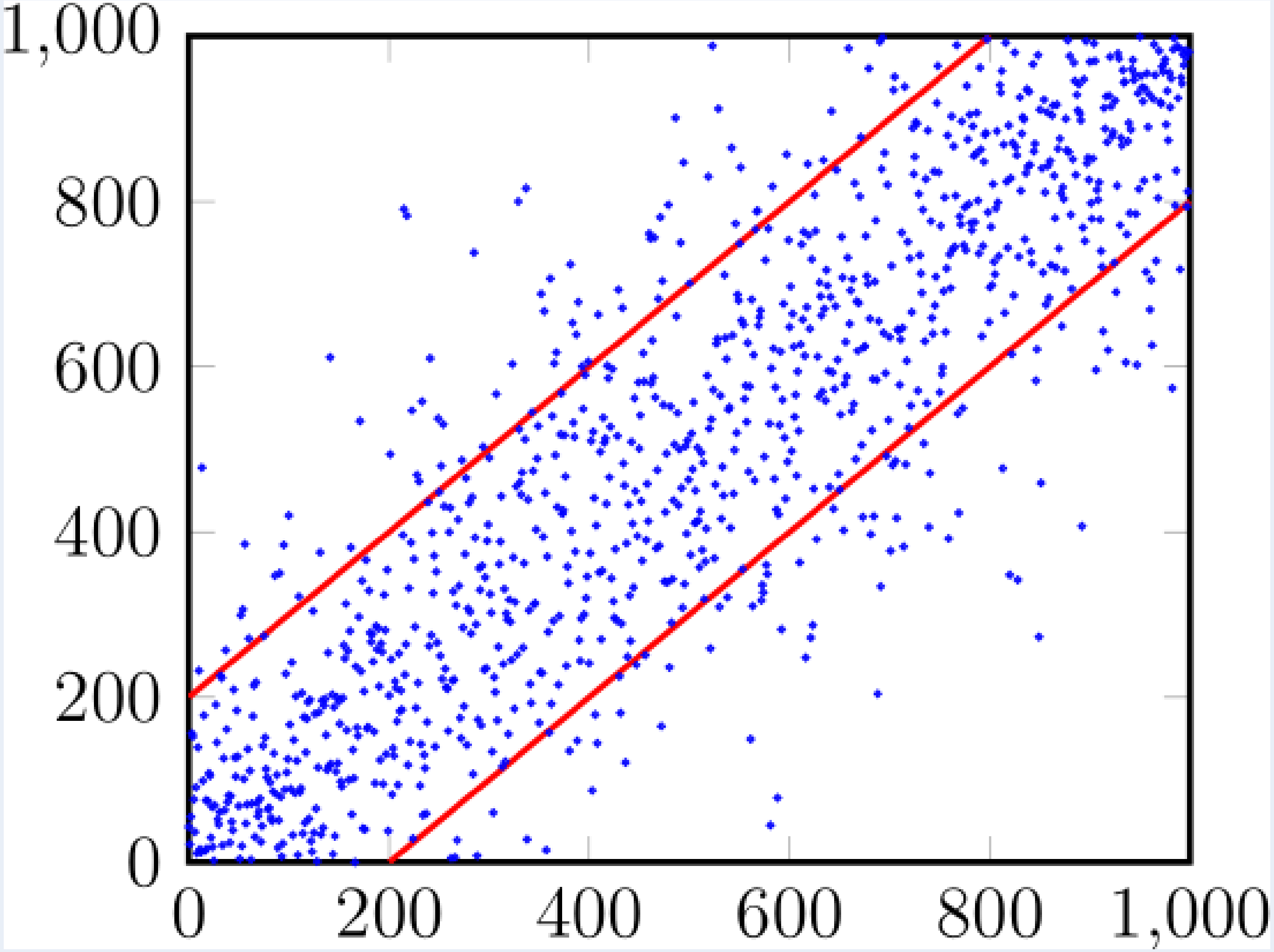
- Motivation:
  - Random permutations of this type usually have a band structure:
    the typical distance $\mathrm{d}(\mathrm{x}, \pi(x))$ is smaller than in a uniform permutation.
    How does the geometric structure affect the permutation statistics?
    Our focus is on the cycle structure of π.
  - Related to phase transitions in other models (see later slides).
- We focus on the case of Euclidean geometry – that of $\mathbb{R}^d$ or $\mathbb{Z}^d$.
- There are several rigorous results on models of the above type (Armendariz, Betz, Biskup, Ferrari, Fyodorov, Gandolfo, Groisman, Leonardi, Muirhead, Richthammer, Ruiz, Taggi, Ueltschi) but our focus will be on other models which are similar in spirit but possess an integrable structure which facilitates finer analysis.

One-dimensional band permutations



Graph of the identity permutation on 1000 points.

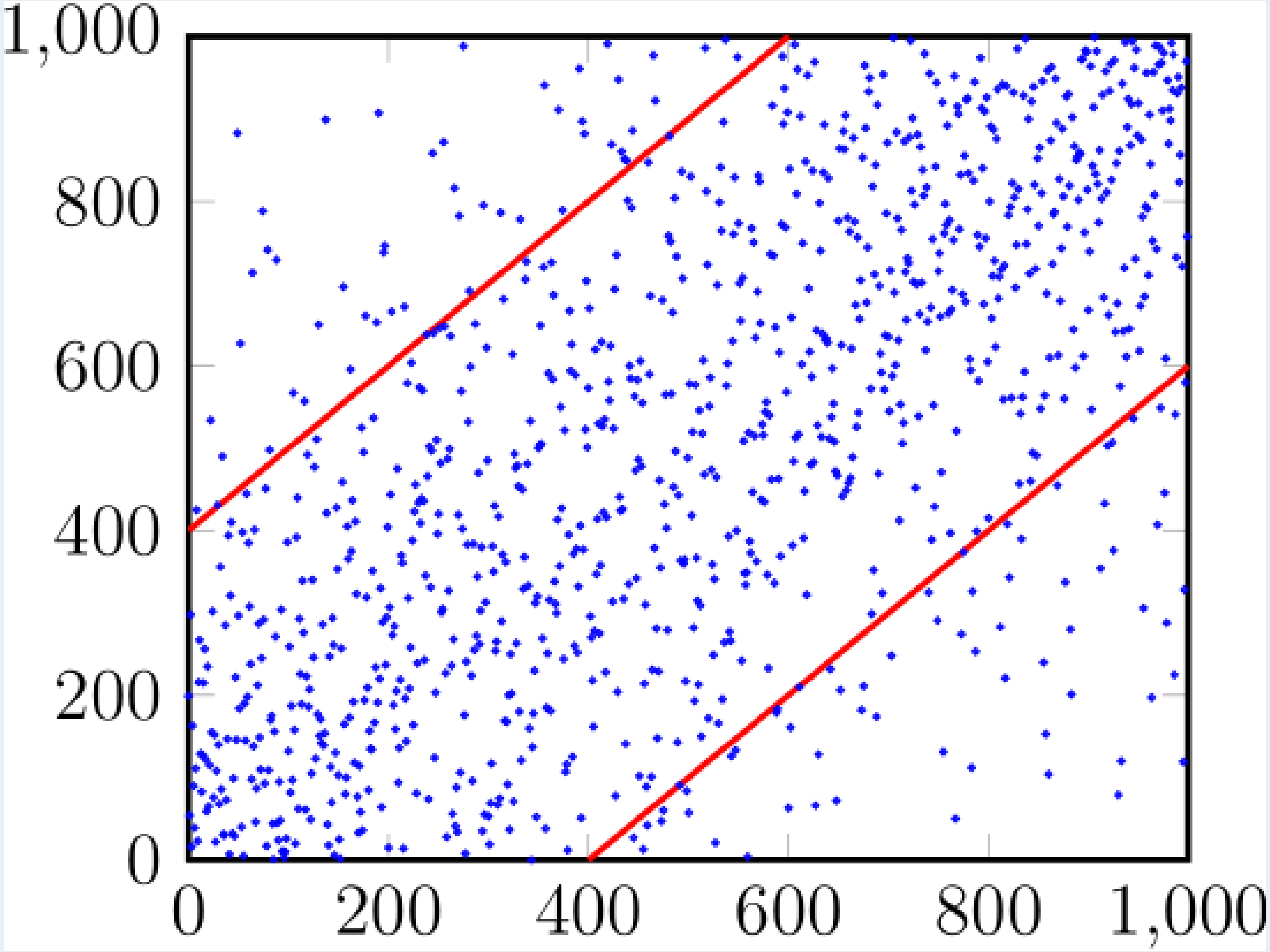
One-dimensional band permutations



Graph of a band permutation on 1000 points (Mallows distribution, q=0.99).

Red lines at distance 200 from the diagonal delimit most of the points.

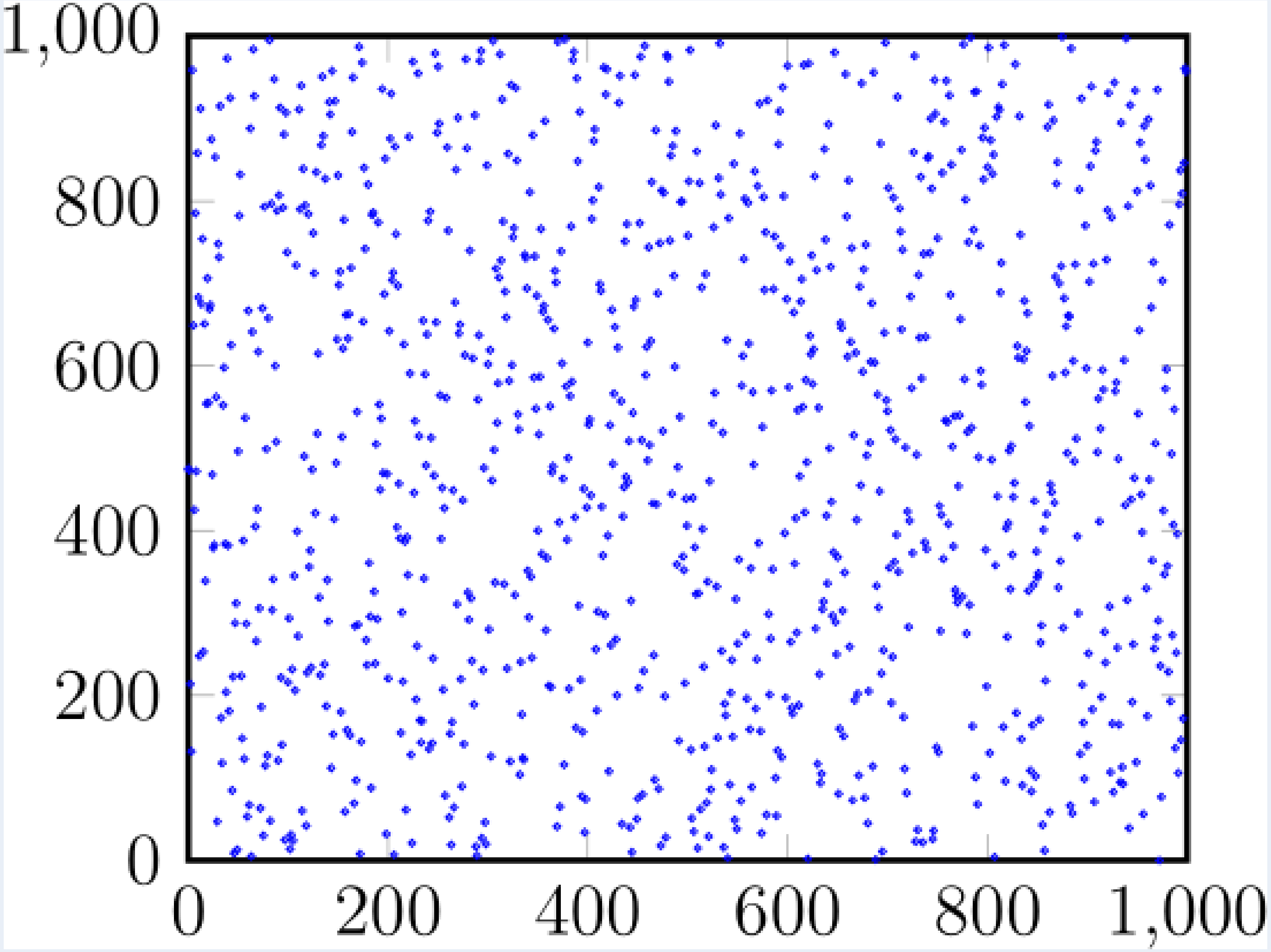
One-dimensional band permutations



Graph of a band permutation on 1000 points (Mallows distribution, q=0.995).

Red lines at distance 400 from the diagonal delimit most of the points.

One-dimensional band permutations



Graph of a uniform permutation on 1000 points.

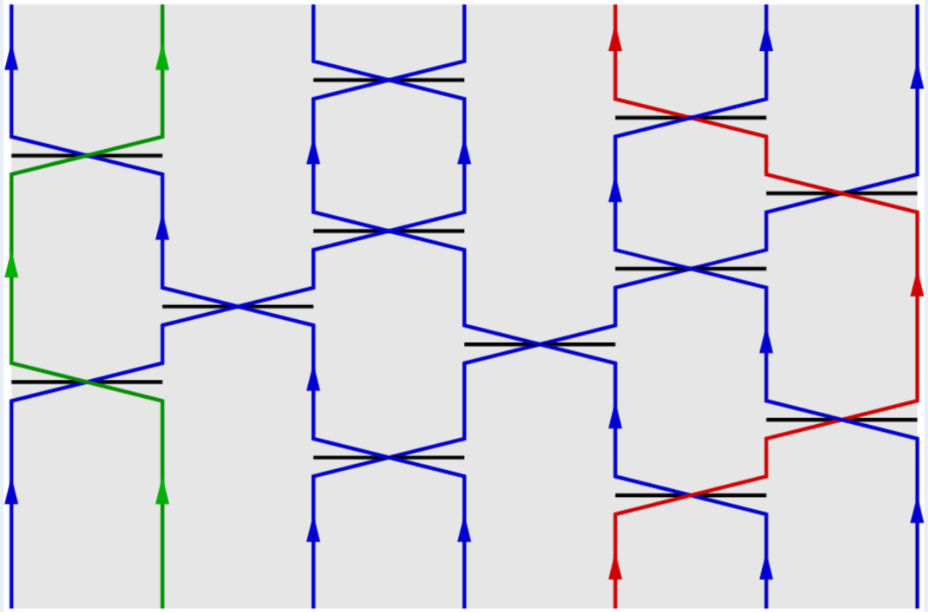
The cycle structure of the uniform and Ewens permutations

- Let π be a uniformly random permutation on $\{1, \dots, n\}$.
- The cycle structure of π is described by a "stick breaking construction":
  - The length of the cycle containing 1 is uniform on $\{1, \dots, n\}$.
  - Proceeding inductively, suppose k points are outside the previously determined cycles. The length of the cycle containing the first of these points is uniform on $\{1, \dots, k\}$.

- Let $C(\pi)$ be the number of cycles in π.
- Let $\ell_1 \geq \ell_2 \geq \cdots \geq \ell_{C(\pi)}$ be the sorted lengths of the cycles of π.
- Then $\frac{1}{\mathrm{n}}\left(\ell_1, \ell_2, \dots, \ell_{C(\pi)}\right)$ converges in distribution as $n \to \infty$ to the Poisson-Dirichlet distribution with parameter $\theta = 1$ (denoted PD(1)).
- Similarly, a random permutation π on $\{1, \dots, n\}$ has the Ewens distribution with parameter $\theta > 0$ when the probability of π is proportional to $\theta^{C(\pi)}$.
- This is a spatial random permutation on the complete graph (Mallows model).
- Its cycle lengths are given by a stick breaking construction with a beta distribution replacing the uniform distribution.
- The sorted cycle lengths $\frac{1}{\mathrm{n}}\left(\ell_1, \dots, \ell_{C(\pi)}\right)$ converge to $PD(\theta)$.

Spatial random permutations I: The interchange (stirring) process

- Let G be a (finite or infinite) graph. The interchange (or stirring) process is a dynamical process $(\pi_t)_{t\geq 0}$ of permutations on the vertex set of G.
- Set π_0 to be the identity permutation: $\pi_0(v) = v$ for each $v \in V(G)$.
- Place a Poisson process of intensity 1 on each edge, independently.
- When edge e "rings", swap the values of π on the endpoints of e.
- The process is well defined on bounded-degree (and more general) graphs.



An illustration of the interchange process when the graph is a path of length 7.

Picture taken from Goldschmidt-Ueltschi-Windridge (2011).

Interchange process: First results

- We are interested in the cycle structure of π_t for fixed t, when $G = \mathbb{Z}^d$.
- Observe that for each v, the process $\pi_t(v)$ is exactly a simple random walk on G. Thus, on $G = \mathbb{Z}^d$, the distance of $\pi_t(v)$ from v is of order $\sqrt{t}$.
- However, these random walks are coupled, making the analysis of the cycle structure of π_t a difficult task.
- It is not hard to prove that the cycles are short when t is sufficiently small.
- Indeed, the probability that a given edge does not "ring" by time t is $\exp(-t)$.
- Define a percolation process by declaring an edge open if it rings (at least once) by time t and closed otherwise.
- Then the cycles of π_t are contained within the connected components of the percolation process. They are thus short when the percolation is sub-critical, i.e., when $t < \log\left(\frac{1}{1-p\_c(G)}\right)$.

Interchange process: Tóth conjecture

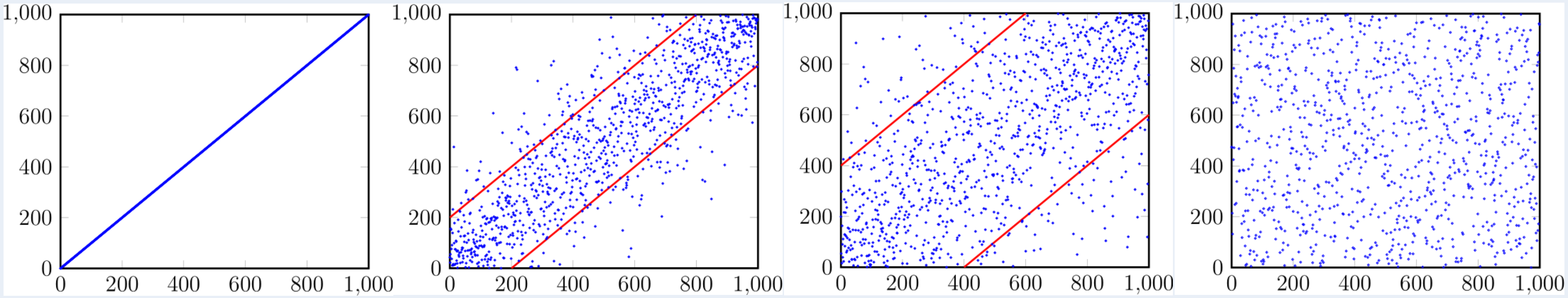
- Let $G = \mathbb{Z}^d$. Write $L_\mathbf{0}(\pi)$ for the length of the orbit containing the origin in π (if the orbit is finite then it forms a cycle).
- Conjecture (Tóth 1993):
  - There are no infinite orbits in dimension $d = 2$.
    Precisely, $\mathbb{P}(L_\mathbf{0}(\pi_t) < \infty) = 1$ for all t.
  - There is a phase transition in dimensions d≥ 3.
    Precisely, there exists $t_\mathrm{c}(d) < \infty$ such that
    $\mathbb{P}(L_\mathbf{0}(\pi_t) < \infty) = 1$ for $t < t_\mathrm{c}(d)$,
    $\mathbb{P}(L_\mathbf{0}(\pi_t) < \infty) = 0$ for $t > t_\mathrm{c}(d)$.
- Conjecture is wide open!
  The $d \geq 3$ case is closely related to the existence of a phase transition for the quantum ferromagnetic Heisenberg model – a fundamental open problem.
- Results exist on complete graph (Alon, Schramm, Berestycki, Kozma), trees (Angel, Betz, Ehlert, Hammond, Lees, Roth), hypercube (Miłoś, Kotecký, Ueltschi) and hamming graph (Adamczak, Kotowski, Miłoś, Şengül).

Spatial random permutations II: The Mallows model

- Let G be a finite graph. Let $0 < q \leq 1$.
- For a permutation π of the vertices of G set $d^{\text{adj}}(\pi, Id)$ to be the minimal number of adjacent transpositions (transpositions along edges of the graph G) required to move from π to the identity permutation.
- The Mallows model with parameter q on permutations of the vertex set of G is the model in which the probability of a permutation π is proportional to $q^{d^{\text{adj}}(\pi, Id)}$.
  - In the limit $q \to 0$ the model concentrates on the identity permutation.
  - When $q = 1$ the model becomes the uniform distribution.
- Integrability: The model has additional structure when G is an interval in $\mathbb{Z}$.
  In this case $d^{\text{adj}}(\pi, Id)$ equals the number of inversions in π. That is,
  $$d^{\text{adj}}(\pi, Id) = \{(i,j) : i < j \text{ and } \pi(i) > \pi(j)\}$$
  and the equality implies a relatively simple sampling algorithm for the model.
- Mallows (1957) introduced the model on an interval as a statistical model for rankings.
- Gnedin and Olshanski found an extension of the model (for $0 < q < 1$) to permutations on the infinite graphs $\mathbb{N}$ and $\mathbb{Z}$ (q-exchangeability).

Mallows model: Displacement results

- Let $0 < q < 1$ and $n \geq 1$ integer.
- Sample π from the Mallows model with parameter q on the interval $\{1, \ldots, n\}$ in $\mathbb{Z}$.
- Displacements / Band structure (Braverman-Mossel 2009, Gnedin-Olshanskii 2012, Bhatnagar-Peled 2015): For each $1 \leq s \leq n$,
$$c \cdot \min\left\{\frac{1}{1-q}, n-1\right\} \leq \mathbb{E}|\pi(s) - s| \leq \min\left\{\frac{2q}{1-q}, n-1\right\}$$
- The graph of a Mallows permutation becomes closer to that of a uniform permutation as $q \uparrow 1$. Their displacements are similar when $1 - q \leq O\left(\frac{1}{n}\right)$.
Starr 2009 found the permuton limit of the Mallows graph when $1 - q = \Theta\left(\frac{1}{n}\right)$.
- For the total variation distance between the Mallows model and a uniform permutation to be small one needs a still smaller order of magnitude for $1 - q$.



Mallows model: Cycle structure

- Write $x \approx y$ if $cy \leq x \leq Cy$ for absolute constants $C, c > 0$.
- Theorem (Gladkich-Peled 2018): Let $0 < q < 1$ and $n \geq 1$ integer. Sample π from the Mallows model with parameter q on the interval $\{1, \dots, n\}$ in $\mathbb{Z}$.
  - Write L_s for the length of the cycle containing $1 \leq s \leq n$. Then
  $$\mathbb{E}|L_s| \approx \min\left\{\frac{1}{(1-q)^2}, n\right\}$$
  - Write $\ell_1 \geq \ell_2 \geq \cdots$ for the sorted lengths of cycles in π.
  If $1 - q = o\left(\frac{1}{\sqrt{n}}\right)$ then
  $$\frac{1}{n}(\ell_1, \ell_2, \dots) \text{ converges weakly to the Poisson}-\text{Dirichlet law with } \theta = 1$$
- Transition from mesoscopic to macroscopic cycles as $1 - q$ goes below $\frac{1}{\sqrt{n}}$.
- At the onset of the transition, displacements are only of order $\sqrt{n}$.
Permutation is far from uniform but its long cycles are already similar to it.
- Mukherjee found Poisson limits for number of short cycles when $1 - q \leq O\left(\frac{1}{n}\right)$.
- Other types of results on the one-dimensional Mallows model include: descents form a determinantal process (Borodin-Diaconis-Fulman), longest monotone subsequence (Bhatnagar-Peled, Basu-Bhatnagar), relations with stable marriage and finitely-dependent proper colorings (Angel, Holroyd, Hutchcroft, Levy).

Spatial random permutations III: Permutation representation of Non-interacting Bose gas

- This model originates from a permutation representation of the non-interacting Bose gas (Matsubara 1951, Feynman 1953), devised in order to better understand the phenomenon of Bose-Einstein condensation.
- Configurations consist of N points in a box $[0, L]^d$ and a permutation of them. Both the point locations and the permutation are random, sampled jointly.
- Formally, a configuration is a pair (x, π) with $x = (x_1, \dots, x_N) \subset [0, L]^d$. It is sampled from a density (wrt. Lebesgue times counting measure) proportional to

$$\theta^{C(\pi)} \exp\left(-\sum_{i=1}^{n} U\left(x_i - x_{\pi_i}\right)\right)$$

for a parameter $\theta > 0$, with $C(\pi)$ the number of cycles of π.

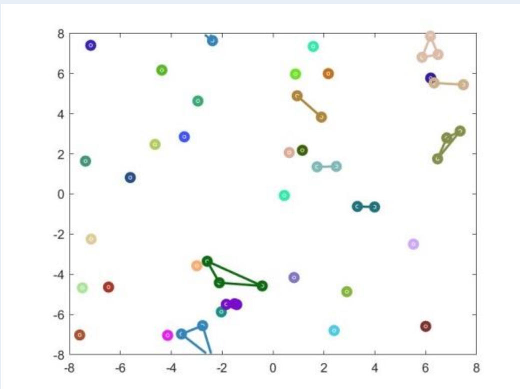
- The function U governs the distance that points typically travel in the square.
- The analysis requires that the differences $x_i - x_{\pi_i}$ be regarded on the torus (i.e., $[0, L]^d$ is endowed with periodic boundary conditions).
- To define U let X be a random variable in $\mathbb{R}^d$ with density φ (e.g., Gaussian). Assume that $\mathbb{E}(X) = 0$ and that φ is a Schwartz function (i.e., smooth with fast decaying derivatives). Then set

$$\exp\left(-U(y)\right) = \sum_{k \in \mathbb{Z}^d} \varphi(y + L \cdot k).$$

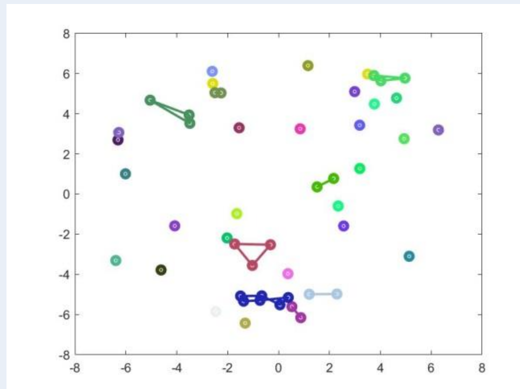
- Density of particles $\rho \coloneqq \frac{N}{L^d}$. For fixed U, increasing ρ widens the band structure of π.

Model III: Simulations

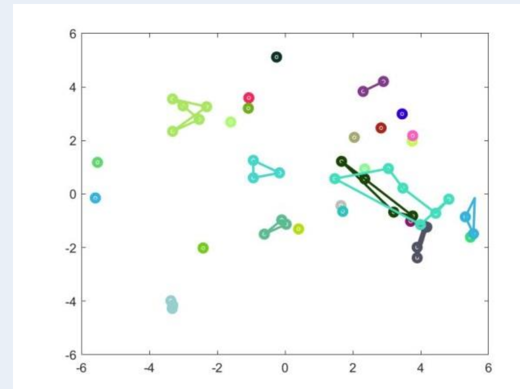
- The random variable X is a standard 2-dimensional Gaussian. $N = 50, \theta = 1$.
- Results show a phase transition asymptotically at $\rho = \alpha_c \log(N) \approx 0.623$.



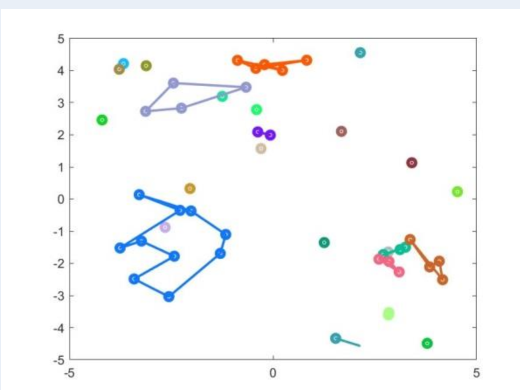
$\rho = 0.2$



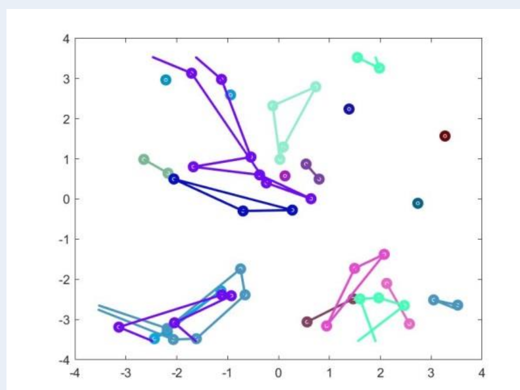
$\rho = 0.3$



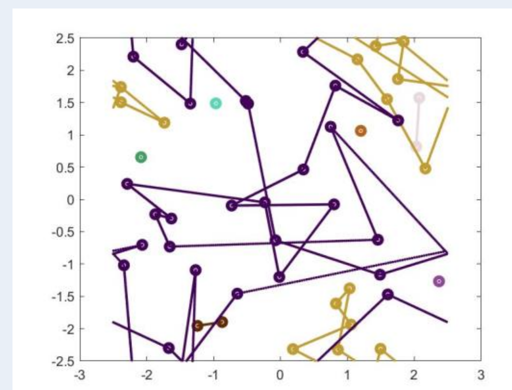
$\rho = 0.4$



$\rho = 0.6$



$\rho = 1$



$\rho = 2$

Model III: Results ($d = 1$)

- Theorem (Elboim-Peled 2019): Dimension $d = 1$. Suppose $\rho, N \to \infty$. Let L_1 be the length of the cycle containing the first point.
  - Sub-critical: If $\rho = o\left(\sqrt{N}\right)$ then

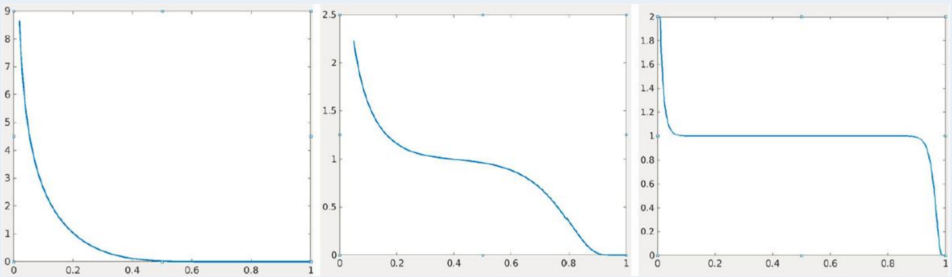
$$\frac{\theta^2 L_1}{2Var(X)\rho^2} \to \mathrm{gamma}\left(\frac{1}{2}, 1\right) \qquad \left(\mathrm{density}\, \frac{c}{\sqrt{x}} \exp(-x) \ \text{ on } [0, \infty)\right)$$

  - Critical: If $\frac{\rho}{\sqrt{N}} \to \alpha \in (0, \infty)$ then $\frac{L_1}{N} \to \mu_\alpha$ where μ_α has the density on (0,1),

$$\frac{1}{Z}\left(\sum_{m \in \mathbb{Z}} e^{-2\pi^2\sigma^2\alpha^2 m^2 x}\right)(1-x)^{-\frac{3}{2}} \sum_{n=0}^{\infty} (-1)^n \binom{-2\theta}{n} (\theta + n) e^{-\frac{(\theta+n)^2}{2\alpha^2\sigma^2(1-x)}}$$

$$\text{where } Z = \frac{1}{\theta} \sum_{n=0}^{\infty} (-1)^n \binom{-2\theta}{n} (\theta + n) e^{-\frac{(\theta+n)^2}{2\alpha^2\sigma^2}} \ \text{ and } \sigma^2 := Var(X)$$

  - Super-critical: If $\rho = \omega\left(\sqrt{N}\right)$ and $\rho \le N$ then $\frac{1}{N}(\ell_1, \ell_2, \dots) \to PD(\theta)$



Critical density for $\alpha = 0.3, 0.8, 2$. Interpolates between a delta mass at 0 and a uniform density as α goes from 0 to ∞.

Model III: Results ($d = 2$)

- Theorem (Elboim-Peled 2019): Dimension $d = 2$. Suppose $\rho, N \to \infty$. Let L_1 be the length of the cycle containing the first point. Set $\alpha_c \coloneqq \frac{\theta}{2\pi\sqrt{\det(\mathrm{Cov}(X))}}$.
  - Sub-critical and critical: If $\frac{\rho}{\log(N)} \to \alpha \in [0, \alpha_c]$ then
  $$\frac{\alpha_c \log(L_1)}{\rho} \to \mathrm{Uniform}[0,1]$$
  - Super-critical: Suppose $\frac{\rho}{\log(N)} \to \alpha \in (\alpha_c, \infty]$and $\rho \leq N$.
  Then asymptotically $1 - \frac{\alpha_c}{\alpha}$ fraction of the points lie in macroscopic cycles (cycles of length proportional to N) and
  $$\frac{1}{\left(1 - \frac{\alpha_c}{\alpha}\right) N} (\ell_1, \ell_2, \dots) \to PD(\theta)$$
  In addition, $\frac{\log(L_1)}{\log(N)} \to \mu$ where μ is a probability measure on $[0,1]$ with an atom of mass $1 - \frac{\alpha_c}{\alpha}$ at the point 1 and constant density on $(0,1)$.

Model III: Results ($d \geq 3$)

- Theorem (Elboim-Peled 2019): Dimension $d \geq 3$. Suppose ρ is fixed as $N \to \infty$. Let L_1 be the length of the cycle containing the first point. Set $\rho_c := \theta \sum_{j=1}^{\infty} \varphi^{\star j}(0)$ (where $\varphi^{\star j}$ denotes the convolution of φ with itself j times).
  - Sub-critical: If $\rho < \rho_c$ then L_1 converges in distribution to an explicit integer-valued random variable with exponential tail decay.
  - Critical: If $\rho = \rho_c$ then L_1 converges in distribution to an explicit integer-valued random variable Y with tail decay

$$\mathbb{P}(Y = j) \sim \frac{\theta}{\rho_c (2\pi)^{\frac{d}{2}} \sqrt{\det\left(\operatorname{cov}(X)\right)}} \cdot j^{-\frac{d}{2}} \quad as \quad j \to \infty$$

  - Super-critical: If $\rho > \rho_c$ then asymptotically $1 - \frac{\rho_c}{\rho}$ fraction of the points lie in macroscopic cycles (cycles of length proportional to N) and

$$\frac{1}{\left(1 - \frac{\rho_c}{\rho}\right) N} (\ell_1, \ell_2, \ldots) \to PD(\theta)$$

  In addition, $\mathbb{P}(L_1 = j) \to \frac{\theta}{\rho} \varphi^{\star j}(0)$ for $j \geq 1$ so that asymptotically a fraction one of the points are in cycles of length of order $\Theta(1)$ or $\Theta(N)$.

Model III: Previous results and discussion

- Mathematical investigations of the model start with Sütő 1993, 2002 and Betz-Ueltschi 2009, 2011. They focused on dimensions $d \geq 3$ (for a slightly different class of distributions of X) and identified the formula for the critical density, the fraction of points in macroscopic cycles in each regime and the convergence to the Poisson-Dirichlet distribution in the super-critical regime.
  Betz-Ueltschi further showed that in some cases when X has heavy tails, the behavior in dimensions $d = 1{,}2$ can be similar to that of $d \geq 3$.
- Our results for dimensions $d = 1{,}2$ are new and our analysis adds to the existing results in dimensions $d \geq 3$ by providing explicit limit laws in all regimes (beyond the already-established Poisson-Dirichlet convergence).
- Our analysis, as well as the previous ones, relies on an integrability property of the model and analytic techniques. Our analysis was inspired by and borrowed from the methods used by Bogachev-Zeindler 2015 in analyzing a related "surrogate-spatial model". We introduce additional innovations for the analysis in dimensions $d = 1{,}2$ and in analyzing the critical case in dimensions $1 \leq d \leq 4$.
- The typical displacement of the points is $\rho^{\frac{1}{d}}$. The onset of the phase transition is thus at displacement of order $\sqrt{N}$ in $d = 1$, of order $\sqrt{\log(N)}$ in $d = 2$ and constant order for $d \geq 3$. For $d = 1$ this is analogous to the Mallows model.

Open questions and conjectures

- Cycle structure in other models (e.g., the interchange process).
  The Poisson-Dirichlet limiting distribution is expected to be universal.
  Are there other universal features?
  For instance, in terms of the typical displacement of a point D, we have seen that the transition to cycles of macroscopic length (proportional to the size of the permutation) occurs when D is of order $\sqrt{n}$ in $d = 1$, of order $\sqrt{\log(n)}$ in $d = 2$ and of constant order for $d \geq 3$.
  Moreover, the length of the cycles in the sub-critical regime is of order D^2 when $d = 1$ and of order $\exp(cD^2)$ when $d = 2$.
  Kozma-Sidoravicius (still unpublished) prove such results for the one-dimensional interchange process.
- This behavior is analogous to the predicted localization/delocalization transition for the eigenvectors of random band matrices (and indeed, the cycles of a permutation can be read simply from the eigenvectors of the permutation matrix).
- Berezinskii-Kosterlitz-Thouless transition in two dimensions for certain spatial random permutation models? (Betz 2014 has numerical simulations).