A categorical $\mathfrak{sl}_2$ action on some moduli spaces of sheaves

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Outline

1) spherical twists + stratified Mukai flops

2) $\mathfrak{sl}_2$ actions

3) an example

4) some ingredients

5) future directions

S = K3 surface $\qquad \mathrm{Pic}(S) = \mathbb{Z}h$

spherical twist $T_{\mathcal{O}_S} : D(S) \xrightarrow{\cong} D(S)$

$$T_{\mathcal{O}_S}(F) \longmapsto \mathrm{cone}\left(\mathcal{O}_S \otimes H^*(F) \xrightarrow{\text{eval}} F \right)$$

$M = M(r, h, s)$ $\qquad$ compact hyperkähler variety

$r \hookrightarrow$ rank, $h \hookrightarrow c_1$, $s \hookrightarrow$ repackages c_2

$\chi = r + s$

Suppose $\chi \geq 0$. generic $F \in M$ has

Brill-Noether locus $\subset M$

$h^0 = \chi = r+s$ | may jump $\leadsto$ $\chi+1$ $\chi+2$ ...

$h^1 = 0$ 1 2 ...

$h^2 = 0$ $\leadsto$ always

strata

Markman, Yoshioka ~99.
jumping happens in expected codim.

what does $T_{\mathcal{O}}$ do?

if $0 \leq \chi \leq r$ $(s \leq 0)$

gen $F \in M$ $T_{\mathcal{O}}(F)$ = coker of :-

$$0 \to \mathcal{O}^{\chi} \xrightarrow{\text{eval}} F \to \underline{\underline{\text{coker}}} \to 0$$

$\hookrightarrow \in M(r-\chi, h, s-\chi)$
$= M(-s, h, -r)$

if $F \in$ BN locus $\subset M$
then $T_{\mathcal{O}}(F)$ is an unstable sheaf
or a 2-term complex

if $\chi > r$ $(s \geq 0)$

gen F: $0 \to \ker \to \mathcal{O}^{\chi} \to F \to \text{coker} \to 0$

$T_{\mathcal{O}}(F)$ has $\mathcal{H}^{-1} = \ker$ (v.b.)
$\mathcal{H}^0 = \text{coker}$ (fin. supp.)

not a sheaf, but

but $T_0(F)^\vee[1]$ is a stab. sheaf

$$\text{in } M(\chi-r, h, \chi-s) = M(s,h,r)$$

either way: T_0 gives a birational map

$$M = M(r,h,s) \dashleftarrow\dashrightarrow M(-s,h,-r) \text{ or } M(s,h,r)$$

if only one BN stratum $\rightsquigarrow$ this is a Mukai flop
more than one $\rightsquigarrow$ stratified Mukai flop

example soon.

local model of strat. Mukai flop:

$$T^*Gr(k,n) \dashleftarrow\dashrightarrow T^*Gr(n-k,n)$$

Cautis + Kamnitzer + Licata '09
introduced "geom. categorized sl_2 action"

their example: "weight spaces" are

$$\overset{-n}{D(T^*Gr(0,n))} \overset{E}{\underset{F}{\rightleftarrows}} \overset{-n+2}{D(T^*Gr(1,n))} \overset{E}{\underset{F}{\rightleftarrows}} \cdots \overset{-n+2k}{D(T^*Gr(k,n))} \cdots$$

$$\cdots \overset{n-2k}{D(T^*Gr(n-k,n))} \cdots \overset{n}{D(T^*Gr(n,n))}$$

Cor(CKL): equivalence $D(T^*Gr(k,n)) \cong D(T^*Gr(n-k,n))$

Nakajima: $T^* Gr(k,n)$ is quiver variety

$\rightsquigarrow$ (affine) sl_2 action on $\bigoplus_k H^*(Gr(k,n))$

'03: by analogy with quiver vars,

$sl_2 \circlearrowright \bigoplus H^*$(moduli sp. of sh. on K3)

Yoshioka '07: allows several spherical vector bundles in ADE config.

$\rightarrow$ fancier Lie alg. $\circlearrowright H^*$(Moduli spaces)

Thm (NA + RT) we construct a geom. cat. sl_2 action on

wt $= \chi - 2$	wt $= \chi = r+s$	wt $= \chi + 2$
$\cdots\ D(M(r-1,h,s-1))$	$D(M(r,h,s))$	$D(M(r+1,h,s+1))\ \cdots$

Cor: equivalence $D(M(r,h,s)) \cong D(M(-s,h,-r))$

or $D(M(s,h,r))$ if $s \leq 0$

Example:

$h^2 = 10$ $\quad S = Gr(2,5) \cap 3$ hyperplanes $\cap$ quadric

$S \subset \mathbb{P}^6$

$\boxed{} \leftarrow \{F,\ \mathbb{C}^2 \subset H^0(F)\}$ $\quad \{F \in M(1,h,0),\ \mathbb{C} \subset H^0(F)\}$

$Gr(3,5)$ $\quad$ gen. $\mathbb{P}^2$-bundle $\quad$ resolves BN

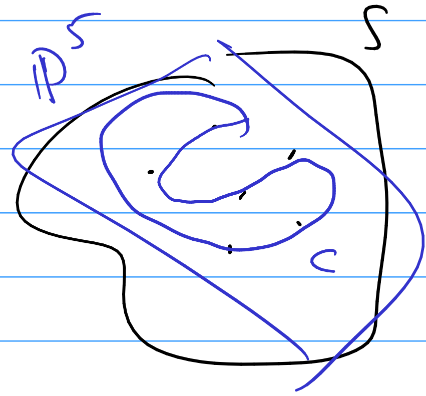
$\cancel{M(2,h,-3)}$ $\quad \cancel{M(-1,h,-2)}$ $\quad M(0,h,-1) \dashleftarrow\dashrightarrow M(1,h,0)$ $\quad M(2,h,1)$ ...

$M(2,h,3)$ $\quad M(1,h,2)$ $\quad \cap$ $\quad \| \quad \mathcal{I}_\xi(h)$

$=$ point $\quad = Hilb^4(S)$ $\quad$ Beauville-Mukai int. system $\quad Hilb^6(S) \ni \xi$ $\quad M(3,h,2)$

$\{T^*\}$ $\quad$ gen: deg-4 l.b. on curve $C = S \cap \mathbb{P}^5$ $\quad$ pt.

BN strat of $Hilb^6(S) = M(1,h,0)$:



gen: 6 pts $\in S$ span $\mathbb{P}^5$ $\quad h^0(\mathcal{I}_\xi(h)) = 1$

T_0: $\quad \mathcal{O} \to \mathcal{I}_\xi(h) \to \omega_C(-\xi)$

where $C = S \cap \mathbb{P}^5$

codim 2: 6 pts only span a $\mathbb{P}^4$ $\quad S \cap \mathbb{P}^4 = 10$ points

have 6

$\rightsquigarrow$ take other 4

$\mathcal{O}(-h) \to \mathcal{O}^2 \to \mathcal{I}_\xi(h) \to \mathcal{O}_{4 \text{ points}}$

$\operatorname{cone}(\mathcal{O}^2 \to \mathcal{I}_\xi(h))^\vee[1] = \mathcal{I}_{4\,pts}(h)$

codim 6: 6 pts span only a $\mathbb{P}^3$

$0 \to T \to \mathcal{O}^3 \to \mathcal{I}_\xi(h) \to 0$

$T =$ tant bundle on $Gr(2,5)|_{K3}$

Ingredients in construction:

spaces

$$\begin{tikzcd} & \text{corresp.} \arrow[dl] \arrow[dr] & \\ M(r,h,s) & & M(r+t,h,s+t) \end{tikzcd}$$

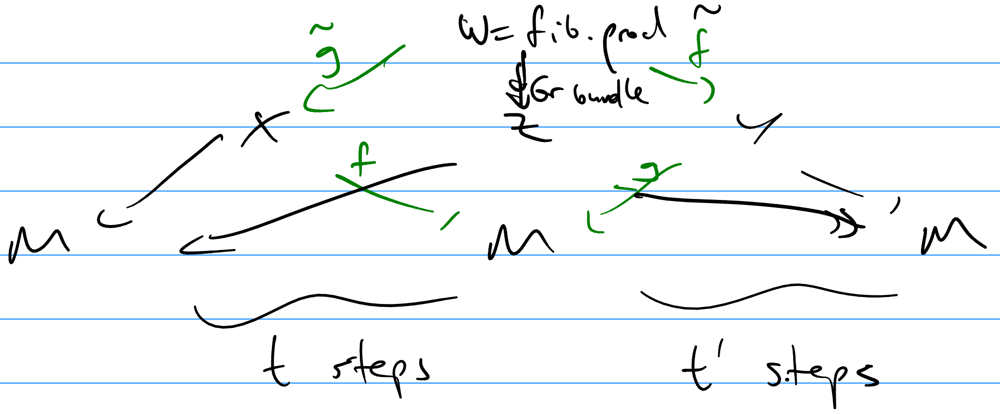
well-chosen line bundles on correspondences

check easy conditions – omitted

medium-hard conditions : $E^{(t)} \circ E^{(t')} = E^{(t+t')}$

$\otimes H^*(Gr(t,t+t'))$

①



$g^* f_* \simeq \tilde{f}_* \tilde{g}^* \otimes \bigoplus \Lambda^i$ excess normal bun'le [i]

$T_{W/Z}$ or $\Omega_{W/Z}$

② need defo. of each moduli sp.
such that sl_2 machinery doesn't deform

CKL's innovation : allows to show
kernels have $\cong \mathcal{H}^*$
rather than $\cong$

them: defo from
$T^* Gr \to \boxed{??} \to \mathcal{O}_{Gr}$ $\quad Ext = H^1(\Omega_{Gr}) = \mathbb{C}$

us: defo is a twistor fam from HK metric on M.

hard step: categorify $EF - FE = $ wt. id.

Open Directions (not me... you + Ryan?)

① replace $T_{\mathcal{O}_S}$ with T_{E_i}

several spherical v.b's in ADE config.

categorify Yoshioka '07

② had some freedom choosing line bundles on correspondences

$\longrightarrow$ affine sl_2 action?

(compare CKL 1112.6189)

③ $sl_2 \circlearrowright \bigoplus_t H^*(M(r+t, h, s+t))$ Nakajima

we cat'ified

Heisenberg $\circlearrowright \bigoplus_t H^*(Hilb^t)$ Nakajima + Groj.

$\circlearrowright \bigoplus_t H^*(M(r, h, s+t))$ Baranovsky

cat'ified by Negut, Yu Zhao...

ask how they interact...

double affine sl_2? ask Cautis.