Dynamical Algebraic Combinatorics (Online)

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1 Overview of the Field

Rowmotion and Homomesy. *Rowmotion* was introduced by Duchet in [Duc74]; studied for the Boolean lattice (and the product of two chains) by Brouwer and Schrijver [BS74, Bro75]; and (still for the Boolean lattice) related to matroid theory by Deza and Fukuda [DF90]. Cameron and Fon-der-Flaass considered rowmotion on the product of two and then three chains [Fon93, CF95]. Because the orbit structure of rowmotion on Boolean lattices is so wild, much of the effort in the references above is dedicated to understanding which orbit sizes are realizable.

Its study then apparently lay dormant for over a decade until Panyushev resurrected it in the form of a series of conjectures on the orbit structure of rowmotion on the root posets of Lie algebras [Pan09]. The focus then shifted to finding equivariant bijections to natural combinatorial objects, and Stanley (and, independently, Thomas) completely characterized the orbit structure of rowmotion on the product of two chains combinatorially (using the Stanley-Thomas word) [Sta09]. Armstrong, Stump, and Thomas [AST13] resolved Panyushev's conjectures using an equivariant bijection to noncrossing partitions under the Kreweras complement, while Striker and Williams unified and extended various results by relating rowmotion to *jeu-de-taquin* and made terminological innovations to the theory [SW12]. Influentially, Propp and Roby returned to the product of two chains and introduced the notion of *homomesy* [PR15]. This popularization of rowmotion led to a swell of related work falling under Propp's heading of *dynamical algebraic combinatorics*.

K-Theoretic Promotion and Resonance. Work of Dilks, Pechenik, Striker, and later Vorland connected rowmotion to Thomas and Yong's K-theoretic *jeu-de-taquin*, developed to compute structure coefficients in K-theoretic Schubert calculus [DPS17, DSV19]. The quasi-periodicity (under the name *resonance*) of K-theoretic promotion of rectangular tableaux was studied by Dilks, Pechenik, and Stiker using the relationship to rowmotion on plane partitions [DPS17]; this relationship was exploited in the other direction by Patrias and Pechenik to resolve a long-standing conjecture of Cameron and Fon-Der-Flaass [PP20]. The relationship to K-theoretic slides was later picked up by Dao, Wellman, Yost–Wolff, and Zhang [DWYWZ20] via a bijection of Hamaker, Patrias, Pechenik, and Williams between plane partitions of trapezoidal and rectangular posets [HPPW20].

Piecewise-Linear Rowmotion and Promotion of Semistandard Tableaux. Motivated by Berenstein and Kirillov's piecewise-linear Bender-Knuth involutions on Gelfand–Tsetlin patterns [KB95], Einstein and Propp considered a piecewise-linear lifting of rowmotion to the order polytope of a poset [EP13+, EP14]. Einstein and Propp [EP14] (and Hopkins [Hop20, Appendix A], as well as Propp [1]) elucidated the connection between piecewise-linear rowmotion on rectangular plane partitions and promotion of rectangular semistandard Young tableaux, further solidifying the representation-theoretic connections. With Bernstein and Vorland, Striker generalized semistandard tableaux and promotion to P-strict labelings under promotion, establishing a more general correspondence between PL-rowmotion on plane partitions and promotion on (flagged and symplectic) tableaux [BSV20]. Thus, rowmotion on plane partitions recovers K-theoretic promotion on increasing tableaux, while PL-rowmotion on plane partitions recovers promotion of semistandard tableaux.

Birational Rowmotion Einstein and Propp further lifted rowmotion to the birational setting (from which tropicalization recovers the piecewise-linear definition) [EP13+, EP14]. Birational rowmotion on rectangles and shifted staircases was studied by Grinberg and Roby [GR15, GR16]; Glick and Grinberg later showed in unpublished work that one of the Grinberg–Roby results was equivalent to an instance of Zamolodchikov periodicity. Building on the Grinberg–Roby result, Musiker and Roby gave a precise description using families of non-intersecting lattice paths for powers of birational rowmotion on rectangles, giving an independent proof of periodicity and homomesy [MR19].

Joseph and Roby have continued in this birational direction with a generalization of the Stanley–Thomas word [JR20b], as well as a definition of birational toggles on antichains and a generalization to the noncommutative (skew field) setting [JR20a].

Minuscule Posets and Representation Theory. Rush and Shi placed rowmotion in a natural representation-theoretic setting, which gives a partial explanation for the reappearance in the theory of certain posets with preferred properties [RS13]. Using the Striker–Williams conjugacy result, they obtained a uniform and conceptual explanation for the periodicity of rowmotion on combinatorial models of bases for minuscule representations V_λ (notably, all weight-spaces have dimension one), by connecting rowmotion to the action of a Coxeter element of W. Rush later built on this machinery in [RW15+, Rus16+] to establish homomesy results. Through the connection with Bender–Knuth involutions in type A, piecewise-linear rowmotion corresponds to the action of the cactus group on $V_{m\lambda}$, although this connection remains mysterious in general. Certain homomesies were established by Bloom, Pechenik, and Saracino in [BPS16].

At the piecewise-linear level, using the reflection functors of quiver representation theory (and embedding minuscule posets inside the Auslander–Reiten quiver of the root category), Garver, Patrias, and Thomas gave a uniform proof of periodicity of piecewise-linear rowmotion on minuscule posets [GPT18]. They further show that the RSK and Hillman–Grassl correspondence are recovered as special cases of their constructions via certain compositions of PL-toggles.

At the birational level, Okada checked the remaining E_7 case by computer to complete the proof of periodicity of birational rowmotion for minuscule posets (the other cases were essentially dealt with above by Grinberg and Roby) [Oka20]. Okada also established homomesies via a case-by-case check.

Rowmotion in other Settings. Striker extended the definition of rowmotion by concentrating on generalizing the concept of toggle [Str15, Str16]. Joseph completed one branch of this program by establishing the relationship between toggles on antichains and toggles on order ideals [Jos19], and other variants of toggles on more varied combinatorial objects appeared in [STWW17, CHHM17, EFG16].

Joseph and Roby studied the structure of rowmotion on order ideals in a zigzag poset, with the interpretation as independent sets of a path graph [JR18]. Motivated by this, lattice theory, Catalan combinatorics, and quiver representation theory, Thomas and Williams worked on independent sets of directed acyclic graphs, giving their independent sets a partial order from which rowmotion could be computed in several different ways [TW19b, TW19a]. Galashin and Pylyavskyy's R-systems give a different generalization (of birational rowmotion) to strongly connected directed graphs [GP19].

2 Recent Developments and Open Problems

- It is still open to show that PL-rowmotion on general trapezoidal posets and Hopkins' V poset has the predicted order (a good survey is given in [Hop20b]).

- Cyclic sieving phenomena (CSP) for plane partitions in minuscule and coincidental root posets under PL-rowmotion remain unproven. There is no equivariant correspondence between minuscule and their coincidental doppelgängers (beyond height one).

- There is no representation-theoretic interpretation of birational toggles. Such an interpretation could lead directly to a proof of the CSPs above using the representation-theoretic paradigm for proving the CSPs above [RSW04, Lemma 2.4].
- It would be good to generalize the relationship in type A between the cactus group and Bender–Knuth involutions to other types. This is being currently studied by Dranowski, Elek, Kamnitzer, Libman, and Morton-Ferguson.
- Williams and his student Kraushal are developing piecewise-linear and birational analogues of independence posets.

Further open problems were presented in the problem session on the last day of the conference, and are available on Zulip.

3 Presentation Highlights

All talks were given remotely, using Zoom.

Monday, October 19. Striker gave the first talk of the workshop, giving a historical overview of dynamical algebraic combinatoics (DAC) through her work extending notions of rowmotion and promotion to increasingly broad classes of objects [SW12, Str16, DPS17, DSV19]. Vorland gave the second talk, introducing the notion of homomesy and illustrating several examples of the phenomenon. The day concluded with Garver speaker on his joint work with Patrias and Thomas on PL-rowmotion on minuscule posets, interpreting the problem using the language of quiver representation theory [GPT18].

Wednesday, October 21. Pechenik spoke of his recent work with Patrias [PP20] resolving a conjecture of Cameron and Fon-der-Flaass regarding the orbit lengths of rowmotion on the product of three chains by using his previous work connecting this problem to K-theoretic promotion [DPS17, BPS16, Pec17]. Patrias spoke of her work in generalizing promotion on webs (invariant tensors in $V^{\otimes n}$ for V the first fundamental representation of $\mathfrak{sl}_2$) to webs with two colors (now invariant tensors using factors V and V^*). Gunawan gave an expository talk on classical friezes, building to cluster algebras.

Friday, October 23. Roby's talk "Let's birational" explained his continuing work extending "classical" homomesy and periodicity results from the combinatorial to piecewise-linear and birational levels. Okada followed with his recent proof of homomesy results for all minuscule posets at the birational level of generality. The day concluded with 14 posters, given via breakout rooms; participants were free to roam the breakout rooms, or stay in the main room to chat. A moderator stayed in the room to help direct participants with an older version of Zoom.

1. Carlos Alejandro Alfaro - The sandpile groups of outerplanar graphs (joint work with Ralihe Raúl Villagrán)
2. Joseph Bernstein - P-strict promotion and piecewise-linear rowmotion (joint work with Jessica Striker and Corey Vorland)
3. Colin Defant - Promotion Sorting (joint work with Noah Kravitz)
4. Ben Drucker, Eli Garcia, and Rose Silver - RSK algorithm and the box-ball system (joint work with Aubrey Rumbolt)
5. Noah Kravitz - Friends and strangers walking on graphs (joint work with Noga Alon and Colin Defant)
6. Matthew Macauley - Abstract Algebra through Cayley diagrams, actions, and lattices
7. Rene Marczinzik - Distributive lattices and Auslander regular algebras (joint work with Osamu Iyama)
8. Jaeseong Oh - Cyclic sieving and orbit harmonics (joint work with Brendon Rhoades)
9. GaYee Park - Naruse Hook length formula for linear extensions of mobile posets
10. Matthew Plante - Periodicity and Homomesy for the $V \times [n]$ poset and center-seeking snakes

11. Samu Potka - Refined Catalan and Narayana Cyclic Sieving (joint work with Per Alexandersson, Svante Linusson, and Joakim Uhlin)
12. James Propp - A Spectral Theory for Combinatorial Dynamics
13. Bruce Sagan - Fences, unimodality, and rowmotion (includes joint work with Thomas McConville and Clifford Smyth)
14. Hugh Thomas - Independence posets (joint work with Nathan Williams).

Monday, October 26. Williams spoke on his joint work "Independence Posets" with Thomas (also presented as an interactive poster); independence posets are a generalization of distributive lattices, eliminating the lattice requirement and allowing for several definitions of rowmotion. Barnard then followed with her related representation-theoretic work on generalizing the Kreweras complement (or rowmotion) to semidistributive lattices. Yıldırım discussed her thesis work on the Coxeter transformation on cominuscule posets, discussing the similarities of this transformation to rowmotion.

Wednesday, October 28. Grinberg spoke about Littlewood-Richardson coefficients and birational combinatorics. Following this, Joseph discussed recent work on a birational lifting of the Lalanne–Kreweras involution on Dyck paths and Hopkins spoke on rowvacuation of root posets. This topic relates to work of Panyushev, whose work initiated a lot of the recent activity in DAC. Hopkins noted the open problem he discussed could serve to "bring DAC full circle."

Friday, October 30. The last day was dedicated to a moderated open problem session, with participants then joining breakout rooms to work on their preferred problem, also using the Zulip platform. Problems were presented by Sagan and Gunawan, Joseph, Pechenik, Hopkins, and Propp.

- **Sagan and Gunawan.** A fence is a poset induced by an orientation of a path graph — its distributive lattice of order ideals has many interpretations, including as certain perfect matchings and certain weak-order intervals in the symmetric group. What can be said about rowmotion orbits on fences with more than two segments?
- **Pechenik.**
  1. Give a direct proof that the order of K-promotion on $\mathrm{Inc}^{a+b}(a \times b)$ is $a+b$, and similarly for $\mathrm{Inc}^{a+b+1}(a \times b)$ and $a+b+1$.
  2. Is there a Cameron/Fon-Der-Flaass theorem for products of 4 chains/other "nice" posets?
- **Joseph.** The trapezoid poset $T(a,b) = \{(i,j) : 1 \leq i \leq a \text{ and } i \leq j \leq a+b-i\}$, defined for $a < b$, is a "doppelganger" of the poset $[a] \times [b]$, and the two posets have some similar behavior under rowmotion.
  1. For $T(2,b) \times [c]$, antichain cardinality appears to be homomesic for rowmotion.
  2. Orbit structure on $T(2,b) \times [c]$ appears to be the same as that of $[2] \times [b] \times [c]$.
  3. For $T(a,b) \times [2]$ with $3 \leq a \leq b$, the order of rowmotion appears to be $2(a+b+1)$, which is twice that of $[a] \times [b] \times [2]$.
  4. Find a birational lifting of the OY (Oksana Yakimova) invariant of [Pan04].
  5. There are several interesting problems regarding toggling independent sets of a cycle graph.
- **Hopkins.** Prove Narayana symmetry for W-nonnesting partitions under the antichain cardinality (slides at `http://www.birs.ca/workshops/2020/20w5164/files/Hopkins_talk.pdf`).
- **Propp.** Prove that the antichain cardinality statistic is homomesic under Propp's q-generalization of rowmotion, described in `http://jamespropp.org/q-rowmotion.pdf`.

4 Scientific Progress Made

A team of (mostly young) researchers who attended the workshop has solved a problem raised by organizer Jim Propp during the open problem session. Colin Defant, Sam Hopkins, Alex Lazar, Svante Linusson, Svetlana Poznanovik, and Jim Propp, participating in a shared Zulip thread hosted by BIRS, have shown that organizer Jessica Striker's toggleability statistic and its generalizations, and the "rooks" technique developed by Hopkins and his collaborators, are the key to obtaining both proofs of new results in the field (specifically, the desired homomesy result for q-rowmotion) and streamlined proofs of already-known results. One or more articles are expected to result from this burst of effort.

Hopkins and Joseph have already posted a preprint to the arXiv (`https://arxiv.org/abs/2101.02329`) studying a PL and birational generalization of the Lalanne–Kreweras involution on Dyck paths. In particular, they prove that the number of valleys and major index symmetry properties of the Lalanne–Kreweras involution extend to these lifts.

Defant and Hopkins used the work of Armstrong, Stump, and Thomas [AST13] to find an explicit map on nonnesting partitions of classical type that proves Narayana symmetry. This work has been posted to the arXiv (`https://arxiv.org/abs/2012.15795`)

The group considering Sagan and Gunawan's problem on fence posets (Elizalde, Gunawan, Plante, Roby, Sagan, Serhiyenko, and Yıldırım) has proven a number of results characterizing orbit structures and proving homomesic statistics for rowmotion acting on fence posets with specific parameters. These could be turned into a preprint at any time, but the group is currently trying to explore the wide range of possibilities further.

After engaging in some computer exploration, a team consisting of Karen Collins, Oliver Pechenik, Anne Schilling, and Jessica Striker concluded that the Cameron/Fon-Der-Flaass theorem does not extend to products of 4 chains (counterexample in $[2]\times[2]\times[2]\times[3]$) or the simplest "bird" poset from Proctor's d-complete posets.

5 Outcome of the Meeting

Although the community was disappointed to have to postpone our face-to-face meeting due to the coronavirus pandemic, we felt that the online meeting went very well. We made a point of emphasizing early-career mathematicians among our invited speakers. The overall quality of the presentations was excellent.

A number of open problems were presented at the end of the meeting, and continue to be actively pursued by subgroups of the participants (some on the Zulip thread, others via email and Overleaf).

The video lectures make a valuable permanent record of the state of dynamical algebraic combinatorics in Fall 2021, and will be useful to other young researchers who wish to become involved.

We think the meeting generated a lot of enthusiasm for the subfield, and created interpersonal connections between researchers at various levels of their professional careers, which will stand us in good stead for having a successful in-person meeting at Banff when the pandemic permits.

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