Fences, unimodality, and rowmotion

Thomas McConville
Kennesaw State University

Bruce Sagan
Michigan State University
www.math.msu.edu/~sagan

Clifford Smyth
University of North Carolina, Greensboro

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Fences

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Outline

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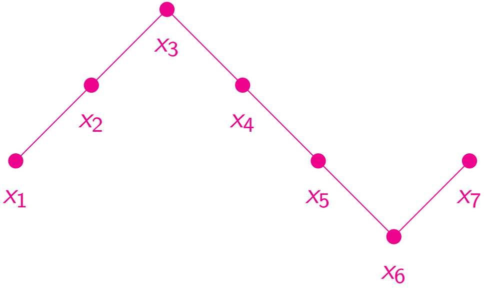
Let $\alpha = (a, b, \ldots)$ be a composition.
A *fence* is a poset $F = F(\alpha)$ with elements $x_1, \ldots, x_n$ and covers

$$x_1 \lhd x_2 \lhd \ldots \lhd x_{a+1} \rhd x_{a+2} \rhd \ldots \rhd x_{a+b+1} \lhd x_{a+b+2} \lhd \ldots \ .$$

Ex.

$F(2,3,1) =$



The maximal chains of F are called *segments*.
Note that if $\alpha = (\alpha_1, \alpha_2, \ldots)$ then

$$n = \#F(\alpha) = 1 + \sum_i \alpha_i.$$

Let $L = L(\alpha)$ be the distributive lattice of order ideals of $F(\alpha)$. These lattices can be used to compute mutations in a cluster algebra on a surface with marked points.

Who	When	What
Propp	2005	perfect mathings on snake graphs
Yurikusa	2019	perfect matchings of angles
Schiffler	2008, 2010	T-paths
Schiffler and Thomas	2009	T-paths
Propp	2005	lattice paths on snake graphs
Claussen	2020	lattice paths of angles
Claussen	2020	S-paths

Outline

Lattice $L(\alpha)$ is ranked with rank function $\operatorname{rk} I = \# I$.
We let

$$R_k(\alpha) = \{I \in L(\alpha) \mid \operatorname{rk} I = k\} \quad \text{and} \quad r_k(\alpha) = \# R_k(\alpha).$$

We will also use the rank generating function

$$r(q;\alpha) = \sum_k r_k(\alpha) q^k.$$

This generating function was used by Morier-Genoud and Ovsienko to define q-analogues of rational numbers.
Call a sequence $a_0, a_1, \ldots$ or its generating function *unimodal* if there is an index m with

$$a_0 \le a_1 \le \ldots \le a_m \ge a_{m+1} \ge \ldots \ .$$

Conjecture (Morier-Genoud and Ovsienko, 2020)

For any α we have that $r(q;\alpha)$ is unimodal.

Previous work: Gansner (1982), Munarini and Salvi (2002), Claussen (2020).

Call sequence $a_0, a_1, \ldots, a_n$ *symmetric* if, for all $k \le n/2$,

$$a_k = a_{n-k}.$$

Call the sequence *top heavy* or *bottom heavy* if, for all $k \le n/2$,

$$a_k \le a_{n-k} \quad \text{or} \quad a_k \ge a_{n-k}, \qquad \text{respectively.}$$

Call the sequence *top interlacing (TI)* if

$$a_0 \le a_n \le a_1 \le a_{n-1} \le a_2 \le \ldots \le a_{\lceil n/2 \rceil}$$

or *bottom interlacing (BI)* if

$$a_n \le a_0 \le a_{n-1} \le a_1 \le a_{n-2} \le \ldots \le a_{\lfloor n/2 \rfloor}.$$

Note that interlacing implies unimodality and heaviness.

Conjecture (MSS)

Suppose $\alpha = (\alpha_1, \ldots, \alpha_s)$.

(a) *If s is even, then $r(q; \alpha)$ is BI.*

(b) *Suppose $s \ge 3$ is odd and let $\alpha' = (\alpha_2, \ldots, \alpha_{s-1})$.*

   (i) *If $\alpha_1 > \alpha_s$ or $\alpha_1 < \alpha_s$ then $r(q; \alpha)$ is BI or TI, respectively.*

   (iii) *If $\alpha_1 = \alpha_s$ then $r(q; \alpha)$ is symmetric, BI, or TI depending on whether $r(q; \alpha')$ is symmetric, TI, or BI, respectively.*

A *chain decomposition (CD)* of a poset P is a partition of P into disjoint saturated chains.

If P is ranked then the *center* of a chain C is

$$\text{cen } C = \frac{\mathrm{rk}(\min C) + \mathrm{rk}(\max C)}{2}.$$

If $\mathrm{rk}\, P = n$ then a CD is *symmetric (SCD)* if for all chains C in the CD

$$\text{cen } C = \frac{n}{2}.$$

A CD is *top centered (TCD)* if for all chains C in the CD

$$\text{cen } C = \frac{n}{2} \quad \text{or} \quad \frac{n+1}{2}.$$

A *bottom centered CD (BCD)* has $\text{cen } C = n/2$ or $(n-1)/2$ for all chains C.

If P has an SCD, TCD, or BCD then its rank sequence is symmetric, top, or bottom interlacing, respectively.

Conjecture (MSS)

For any α, the lattice $L(\alpha)$ admits an SCD, TCD, or BCD consistent with the previous conjecture.

Theorem (MSS)

Let $\alpha = (\alpha_1, \ldots, \alpha_s)$ and suppose that for some t we have

$$\alpha_t > \sum_{i \neq t} \alpha_i.$$

Then $r(q;\alpha)$ is unimodal.

Theorem (MSS)

Let $\alpha = (\alpha_1, \ldots, \alpha_s)$ where for some t

$$\alpha_t = 1 + \sum_{i \neq t} \alpha_i.$$

If $L(\alpha)$ has an SCD, TCD, or BCD then so does $L(\beta)$ where

$$\beta = (\alpha_1, \ldots, \alpha_{t-1}, \alpha_t + a, \alpha_{t+1}, \ldots, \alpha_s)$$

for any $a \geq 1$.

Theorem (MMS)

If α has at most three parts then $L(\alpha)$ has an SCD, TCD, or BCD.

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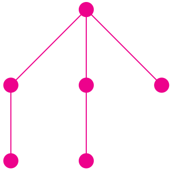
Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be an integer partition with all parts at least two. The *extended star*, $S = S(\lambda)$, consists of k chains with λ_i elements in chain i where the maximum elements of the chains have been identified.

Ex.

$S(3,3,2) =$ 

Note that $S(a, b) = F(a-1, b-1)$.

Theorem (S)

Rowmotion on the antichains of $S(\lambda)$ has the following properties.

1. *All orbits have size $\ell = \mathrm{lcm}(\lambda_1, \lambda_2, \ldots, \lambda_k)$ except for one which has size $\ell + 1$.*
2. *The number of orbits is $\prod_i \lambda_i / \ell$.*
3. *The number of antichain elements in the orbit of size $\ell + 1$ is m where m is a multiple of $\#S(\lambda) = \lambda_1 + \cdots + \lambda_k - k + 1$. The number of antichain elements in the other orbits is $m - 1$.*

For fences with more that two segments, the picture is less clear. Let (1^s) denote the composition consisting of s ones.

Theorem (S)

Consider $F = F(1^s)$.

1. *F always has an orbit of length 3.*
2. *F has an orbit of size $3(s-2)+2$ for $s \geq 2$.*
3. *F has an orbit of size $3(s-3)+1$ for $s \geq 5$.* □