The Kreweras Complement on the Lattice of Torsion Classes

Emily Barnard

Joint with Gordana Todorov and Shijie Zhu

DePaul University

October 26, 2020

Goal and Outline

The goal of this project is to study a certain purely combinatorial map (which I call the "kappa" map) in the context of the representation theory of quivers.

- Define κ and review an important example
- We'll make the connection to the Kreweras complement
- Focus the lattice of torsion classes

Take home...

The "kappa" map that I will define is an analog of the Kreweras complement.

Lattice-theoretic background

Definition

A lattice L is a poset such that for each pair of elements u and w

- the smallest upper bound or **join** $u \vee w$ exists and
- the greatest lower bound or **meet** $u \wedge w$ exists.

Definition

- An element $j \in L$ is **join-irreducible** if $j = \bigvee A$ implies $j \in A$, where A is finite.
- An element is **completely join-irreducible** if j is covers a unique element, which we write as j_*.
- For the purposes of this talk, all lattices will be finite—so these notions coincide.

Lattice-theoretic background

Definition

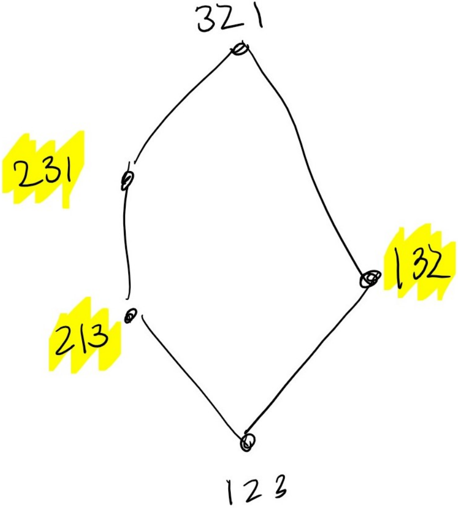
A lattice L is a poset such that for each pair of elements u and w

- the smallest upper bound or **join** $u \vee w$ exists and
- the greatest lower bound or **meet** $u \wedge w$ exists.

Definition

- An element $m \in L$ is **meet-irreducible** if $m = \bigwedge A$ implies $m \in A$, where A is finite.
- An element is **completely meet-irreducible** if m is covered by a unique element, which we write as m_*.

Running Example: The Tamari Lattice



The kappa map

The "kappa" map is a map which takes completely join-irreducible elements to completely meet-irreducible elements.

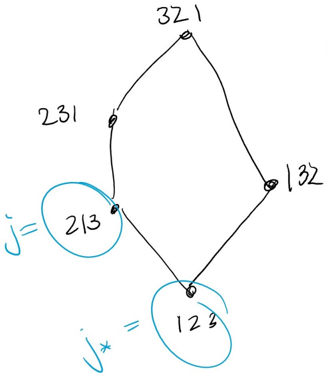
Main Definition

Let j be a (completely) join-irreducible element of a lattice L, and let j_* be the unique element covered by j. Define $\kappa(j)$ to be:

$$\kappa(j) := \text{unique max}\{x \in L : j_* \leqslant x \text{ and } j \nleqslant x\},$$

when such an element exists.

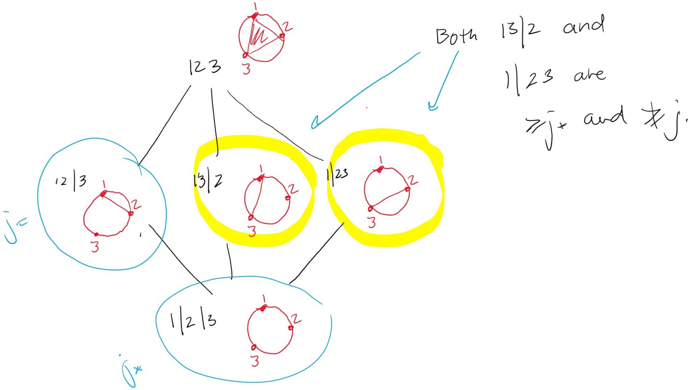
Running Example



Want the largest element x:

- $x \geq 123$
- $x \not\geq 213$

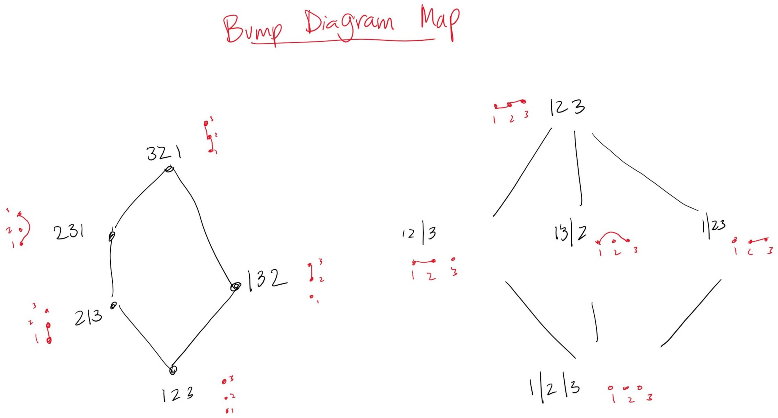
The Noncrossing Partition Lattice



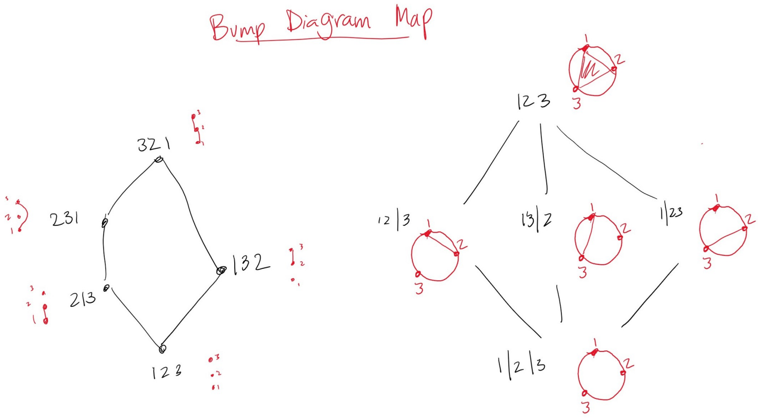
Takeaways

- If L is finite, then κ is defined if and only if L is **semidistributive**.
- Our noncrossing partition lattice is a minimal non-example of a lattice which fails to be semidistributive.
- There is an important connection to the Kreweras Complement.

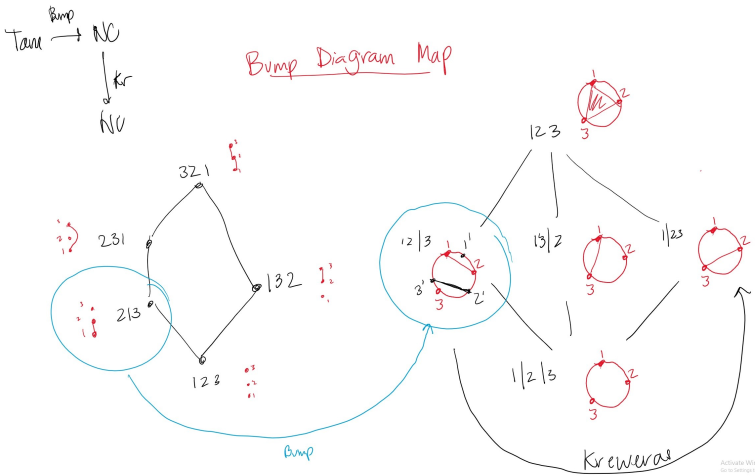
Running Example



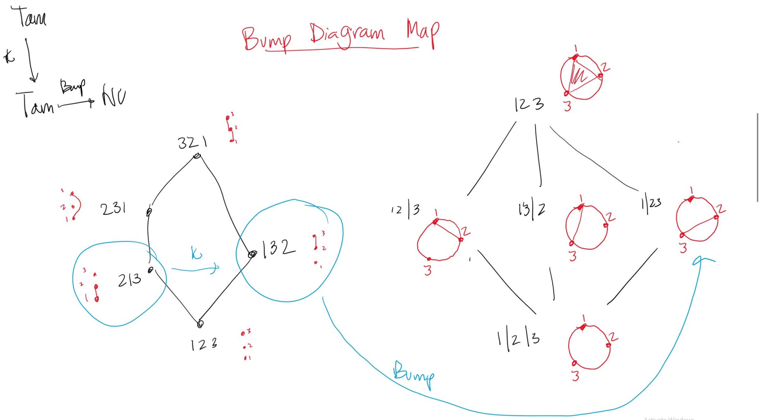
Running Example



Running Example



Running Example



Takeaways

- If L is finite, then κ is defined if and only if L is *semidistributive*.
- Our noncrossing partition lattice is a minimal non-example of a lattice which fails to be semidistributive.
- There is an important connection to the Kreweras Complement.

Key Point

The kappa map is the analog of the Kreweras Complement, for the class of finite semidistributive lattices.

Semidistributive Lattices

Definition

A semidistributive lattice L satisfies a weakening of the distributive law. For any x, y, and z in L:

$$\text{If } x \vee y = x \vee z, \text{ then } x \vee (y \wedge z) = x \vee y$$

$$\text{If } x \wedge y = x \wedge z, \text{ then } x \wedge (y \vee z) = x \wedge y$$

Important Examples

- the Tamari lattices and c-Cambrian lattices
- the weak order for any finite Coxeter group W
- the poset of regions from a simplicial hyperplane arrangement
- the lattice of torsion classes*

Torsion Classes

Definition

- Let Λ be a finite dimensional, basic algebra over an arbitrary field K.
- Denote by Λ the category of finitely generate (right) modules.

A **torsion class** $\mathcal{T}$ is a class of modules that is closed under quotients, isomorphisms, and extensions.

Running Example

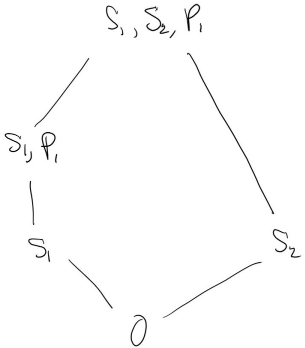
A **torsion class** $\mathcal{T}$ is a class of modules that is closed under quotients, isomorphisms, and extensions. Consider the set of modules over the path algebra with quiver $Q = 1 \to 2$.

- S_1 - Simple (no submodules or quotients)
- S_2 - Simple (no submodules or quotients)
- P_1 - Projective modules which is an extension of S_1 and S_2.

$$S_2 \hookrightarrow P_1 \twoheadrightarrow S_1$$

Lattice of Torsion classes

We study the lattice (poset) of torsion classes also denoted $\operatorname{tors}\Lambda$ in which $\mathcal{S} \leqslant \mathcal{T}$ whenever $\mathcal{S} \subseteq \mathcal{T}$.



Main Result A

Main Theorem A [B., Todorov, Zhu]

Let Λ be a finite dimensional algebra, and let M be a Λ-brick. (A **brick** is a module M whose endomorphism ring is a division ring.)

- Each completely join-irreducible torsion class has the form $\mathscr{F}ilt(\mathrm{Gen}(M))$, where M is a brick.
- $\kappa : \mathrm{CJI}(\mathrm{tors}\,\Lambda) \to \mathrm{CMI}(\mathrm{tors}\,\Lambda)$ is a bijection with

$$\kappa(\mathscr{F}ilt(\mathrm{Gen}(M))) = {}^{\perp}M$$

where ${}^{\perp}M$ denotes the set $\{X \in \mathrm{mod}\,\Lambda \mid \mathrm{Hom}_{\Lambda}(X, M) = 0\}$.

Remark

The kappa-map is well defined for *finite* semidistributive lattices, but the lattice of torsion classes is rarely finite. What makes this result interesting is that we show that κ is well-defined even when the lattice of torsion classes is infinite.

Extending the kappa map

I want to build the case that κ is the analog of the Kreweras complement. The Kreweras complement is defined for all elements in $NC(W)$. Now we extend κ to all element of L.

The canonical join representation

Each element x in a finite semidistributive lattice has a unique "factorization" in terms of the join operation which is irredundant and lowest, called the **canonical join representation** and denoted by $x = \text{CJR}(x) = \bigvee A$.

- $|\text{CJR}(x)|$ is equal to the number of lower-covers of x.
- Each element in $\text{CJR}(x)$ is join-irreducible.
- There is an analogous "factorization" using the meet called the **canonical meet representation**.

Extending the kappa map

Theorem [B.]

The κ-map sends canonical join representations to canonical meet representations.

Definition

Let L be a finite semidistributive lattice. Define

$$\bar{\kappa}(x) = \bigwedge \{\kappa(j) : j \in \mathrm{CJR}(x)\}.$$

Extending the kappa map

Recall that each completley join-irreducible torsion class is determined by a brick M. When the lattice of torsion classes is finite, the canonical join representation of a torsion class can be read off from the bricks.

Theorem [B.]

The join of any collection of *hom-orthogonal* bricks is the canonical join representation for some torsion class, and each CJR takes this form.

Extending the kappa map

Definition

Let L be a finite semidistributive lattice. Let x be an element which has a canonical join representation such that $\kappa(j)$ is defined for each $j \in \mathrm{CJR}(x)$. Define

$$\bar{\kappa}(x) = \bigwedge \{\kappa(j) : j \in \mathrm{CJR}(x)\}.$$

Corollary

[B., Todorov, Zhu] Let Λ be a finite dimensional algebra. Let $\mathcal{T}$ be a torsion class which has a canonical join representation of the following form: $\mathrm{CJR}(\mathcal{T}) = \bigvee_{\alpha \in A} \mathscr{F}ilt(\mathrm{Gen}(M_\alpha))$, where M_α are Λ-bricks. Then $\bar{\kappa}(\mathcal{T})$ is defined and is of the form:

$$\bar{\kappa}(\mathcal{T}) = \bigcap_{\alpha \in A} {}^{\perp} M_\alpha.$$

Iterative Compositions of κ

Theorem

Let $\operatorname{tors}\Lambda$ be finite, and let r be the number of vertices in the corresponding quiver Q. For any $\mathcal{T} \in \operatorname{tors}\Lambda$ let $|\mathcal{T}| := |\operatorname{CJR}(T)|$ denote the number of canonical joinands of $\mathcal{T}$. Then for any $\bar{\kappa}$-orbit $\mathcal{O}$ we have

$$\frac{1}{|\mathcal{O}|}\sum_{\mathcal{T}\in\mathcal{O}} |\mathcal{T}| = r/2$$

Orbit of the Kreweraw Complement

For any orbit $\mathcal{O}$ of the Kreweras complement on the generalized noncrossing partition lattice $NC(W)$ satisfies

$$\frac{1}{|\mathcal{O}|}\sum_{P\in\mathcal{O}} |P| = r/2$$

where P is a noncronssing partition and r is the rank of W.

A commutative Diagram

- We saw that certain torsion classes correspond to the (type A) Tamari lattice. This is not an accident!
- For each W, and any orientation c, there is an algebra whose lattice of torsion classes is the corresponding c-Cambrian lattice.

A commutative Diagram

- We saw that certain torsion classes correspond to the (type A) Tamari lattice. This is not an accident!
- For each W, and any orientation c, there is an algebra whose lattice of torsion classes is the corresponding c-Cambrian lattice.
- There is similarly a representation theoretic analog of the $NC(W)$, the generalized noncrossing partition lattice.
- Work of Thomas, Engle and Ringel establishes that a certain poset of subcategories called "Wide subcategories" also ordered by inclusion is isomorphic to NC(W), and they describe a representation theoretic formula for the Kreweras complement, which we call ϵ.

A commutative Diagram

- We saw that certain torsion classes correspond to the (type A) Tamari lattice. This is not an accident!
- For each W, and any orientation c, there is an algebra whose lattice of torsion classes is the corresponding c-Cambrian lattice.
- There is similarly a representation theoretic analog of the $NC(W)$, the generalized noncrossing partition lattice.
- Work of Thomas, Engle and Ringel establishes that a certain poset of subcategories called "Wide subcategories" also ordered by inclusion is isomorphic to NC(W), and they describe a representation theoretic formula for the Kreweras complement, which we call ϵ.
- Just as there is a map from the Tamari lattice to the NC(W), so too there is a map from the lattice of torsion classes to the lattice of wide subcategories, which we call α.

A commutative Diagram

Combinatorics	**Representation Theory**
Tamari Lattice	Lattice of torsion classes
Noncrossing Partitions	Wide Subcategories
Kreweras Complement	ϵ-map
"Bump": Tamari $\rightarrow$ NC	α-map: $\operatorname{tors}\Lambda \rightarrow \operatorname{wide}\Lambda$

A commutative Diagram

Combinatorics	**Representation Theory**
Tamari Lattice	Lattice of torsion classes
Noncrossing Partitions	Wide Subcategories
Kreweras Complement	ϵ-map
"Bump": Tamari $\to$ NC	α-map: $\operatorname{tors}\Lambda \to \operatorname{wide}\Lambda$

When Λ is hereditary...

$$\begin{array}{ccc} \operatorname{tors}_0 \Lambda & \xrightarrow{\bar{\kappa}} & \operatorname{tors}\Lambda \\ \downarrow \alpha & & \downarrow \alpha \\ \operatorname{wide}\Lambda & \xrightarrow{\epsilon} & \operatorname{wide}\Lambda. \end{array}$$

Iterative Compositions of κ

Theorem C

Recall that each join-irreducible torsion class is $\mathscr{F}ilt(\text{Gen}(M))$, where M is a brick. When Λ is hereditary, then applying $\bar{\kappa}$ twice corresponds to applying the (inverse of the) Auslander-Reiten translation to M.

$$\bar{\kappa}^2(\mathcal{T}_M) = \mathcal{T}_{\bar{\tau}^{-1}M}.$$

Here $\bar{\tau}^{-1}M = \tau^{-1}M$ for non-injective modules M and $\bar{\tau}^{-1}I(S) = P(S)$ where $I(S)$ and $P(S)$ are the injective envelope and projective cover of the same simple S.

Thank you!