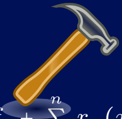
(Semi)Algebraic Proofs over $\{\pm 1\}$ Variables



$$\sum_{u=1}^{a} p_u f_u + \sum_{w=1}^{n} r_w (x^2 - 1) + \sum_{v=1}^{b} q_v^2 h_v = -1$$

Dmitry Sokolov

Banff International Research Station
January 20, 2020

LUND UNIVERSITY

KØBENHAVNS UNIVERSITET

Notation

$$(\mathcal{F}, \mathcal{H}) = \begin{cases} f_1(x_1, \dots, x_n) = 0 \\ f_2(x_1, \dots, x_n) = 0 \\ \dots \\ f_a(x_1, \dots, x_n) = 0 \\ \hline h_1(x_1, \dots, x_n) > 0 \\ h_2(x_1, \dots, x_n) > 0 \\ \dots \\ h_s(x_1, \dots, x_n) > 0 \end{cases}$$

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Range axiom R_i for a variable x_i:

- $\{\mathbf{0}, \mathbf{1}\}$ basis: $x_i^2 - x_i$;
- $\{\pm\mathbf{1}\}$ basis: $x_i^2 - 1$.

Proof Systems

The **Sum-of-Squares** (SOS) proof of $(\mathcal{F}, \mathcal{H})$:

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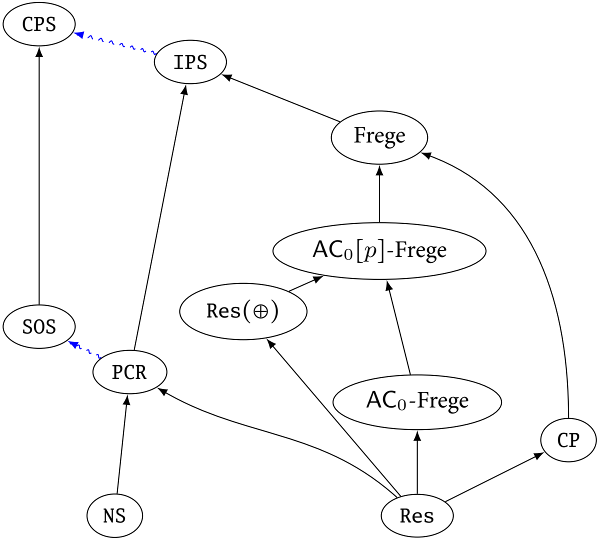
$$\dfrac{\dfrac{\dfrac{\dfrac{x + z - 2}{xy + yz - 2y} \qquad \dfrac{xy - 1 \qquad yz + 1}{xy + yz}}{2y}}{2y^2} \qquad y^2 - 1}{1}$$

Hierarchy



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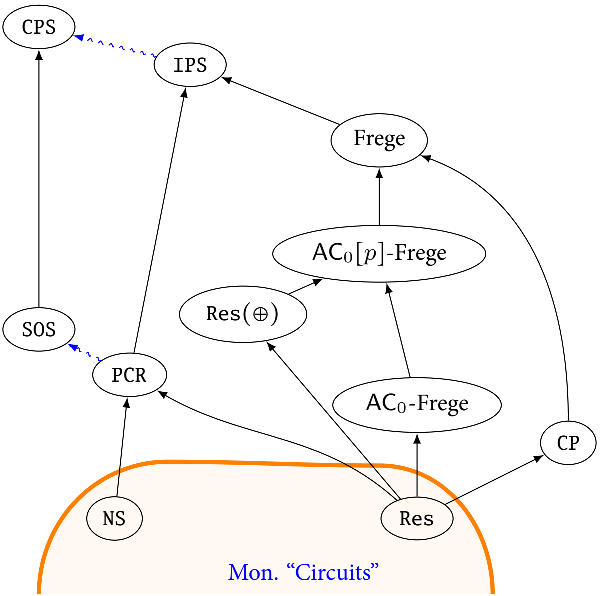

CPS
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$AC_0[p]$-Frege
Res(⊕)
SOS
PCR
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CP
NS
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Mon. "Circuits"

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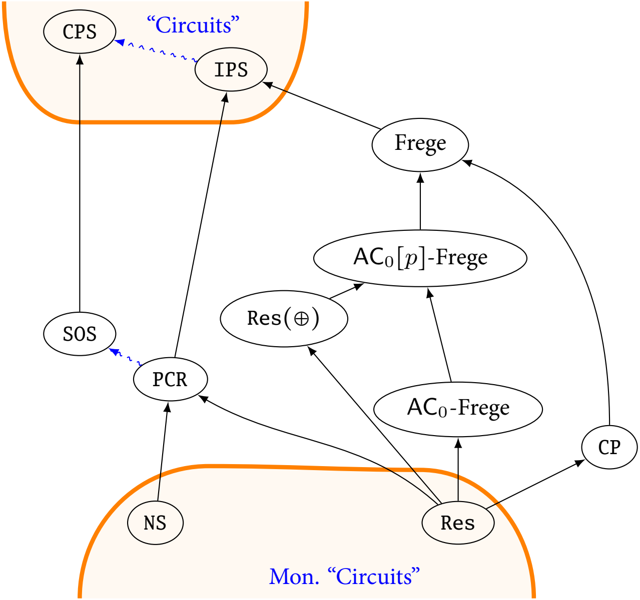
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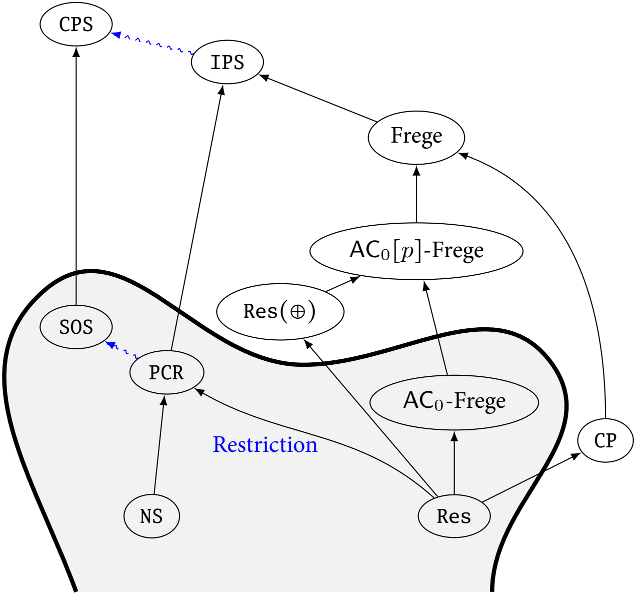
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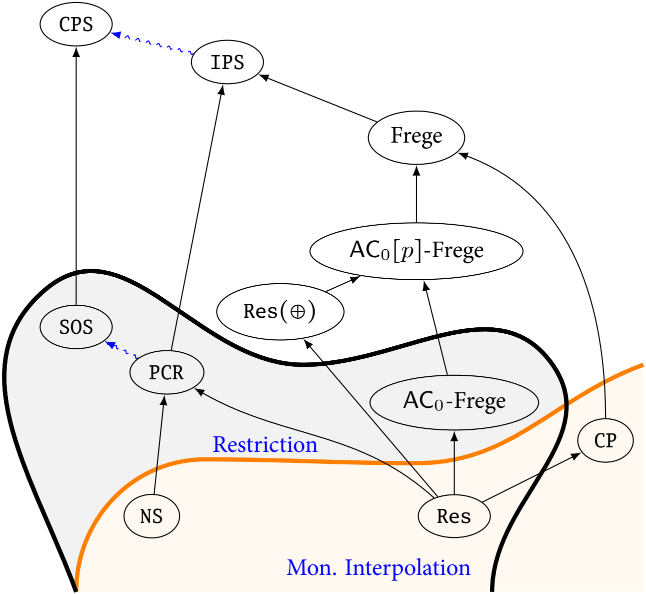
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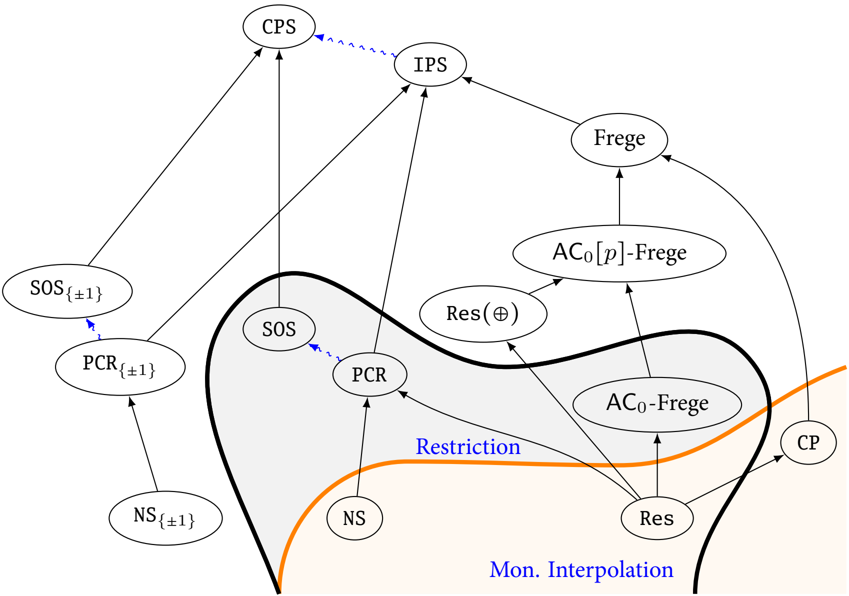
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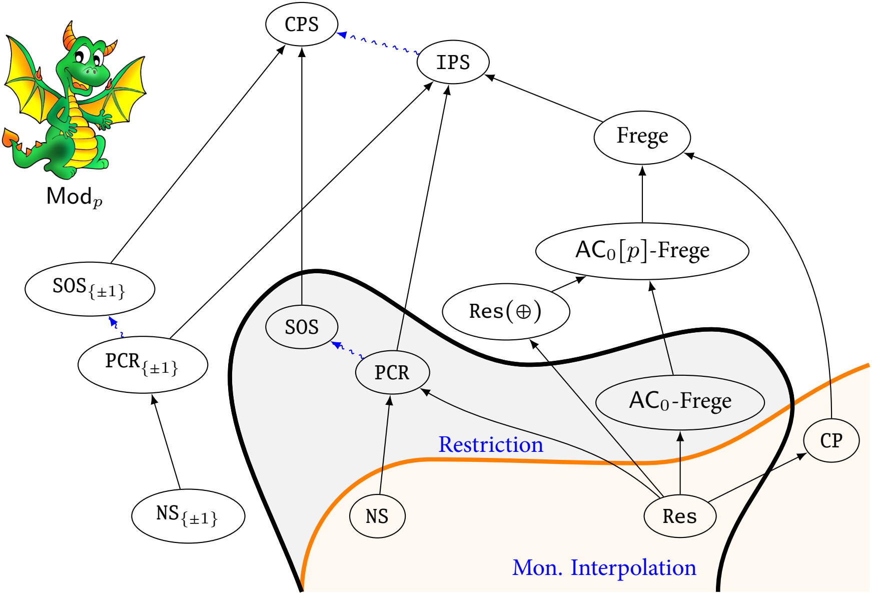

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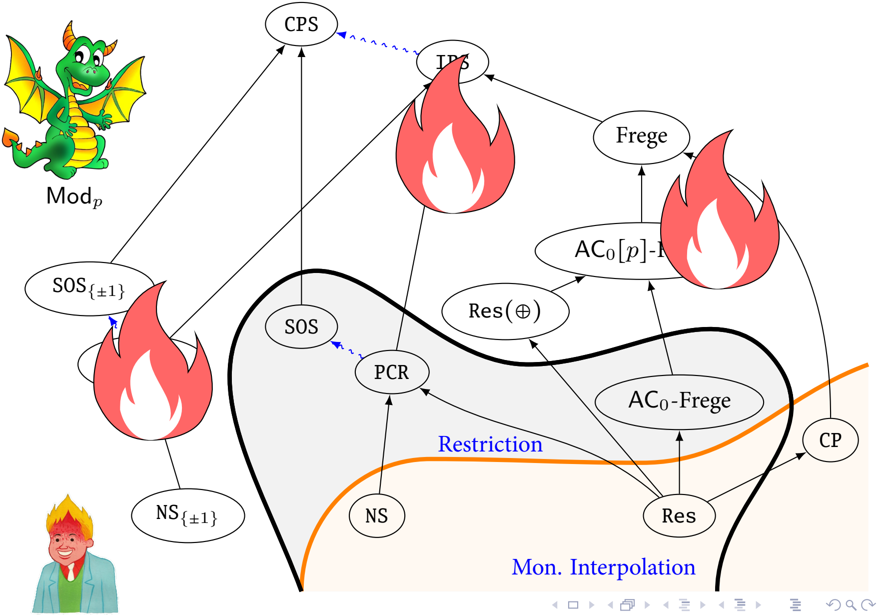

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Results

d_0 is the degree of $(\mathcal{F}, \mathcal{H})$. n is the number of variables of $(\mathcal{F}, \mathcal{H})$.

Theorem

Any $\texttt{SOS}_{\{\pm 1\}}$-proof of $(\mathcal{F}, \mathcal{H}) \circ \texttt{MAJ}(z_1, z_2, z_3)$ has size $\exp(\Omega(\frac{(d-d_0)^2}{n}))$.
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If φ is a random 11-CNF formula then whp any $\mathtt{SOS}_{\{\pm 1\}}$-proof or $\mathtt{PCR}^{\mathbb{F}}_{\{\pm 1\}}$-proof of φ has size $\exp(\Omega(n))$.

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Any $\texttt{PCR}^{\mathbb{F}}_{\{\pm 1\}}$-proof of Pigeonhole Principle has size $\exp(\Omega(n))$.

$\texttt{SOS}_{\{\pm 1\}}$ is strictly stronger than $\texttt{PCR}^{\mathbb{R}}_{\{\pm 1\}}$.

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$$\text{Size} := \sum_{u=1}^{a} (\texttt{MSize}(p_u) + \texttt{MSize}(f_u)) + \sum_{v=1}^{b} \texttt{MSize}(q_v) + \sum_{h \in \mathcal{H}} \texttt{MSize}(h)$$

$$\texttt{PCR}^{\mathbb{F}} : (p_1, \ldots, p_\ell)$$

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Strategy for the $\{0,1\}$ basis ($\texttt{PCR}^{\mathbb{F}}$)

$\pi \coloneqq (p_1, \ldots, p_\ell)$ is a proof of $\mathcal{F}$. $H \coloneqq \{t \mid t \in p_i, \text{deg}(t) \text{ is big}\}$.

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4. $\pi \upharpoonright (x = 0)$ is still a proof of $\mathcal{F} \upharpoonright (x = 0)$.
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Grigoriev 98; Buss, Grigoriev, Impagliazzo, Pitassi 01; Grigoriev 01

1. Tseitin formulas has small $\mathrm{PCR}^{\mathbb{F}}_{\{\pm 1\}}$ and $\mathrm{SOS}_{\{\pm 1\}}$-proofs.
2. There are Tseitin formulas that has $\mathrm{PCR}^{\mathbb{F}}$ or SOS-degree $\Omega(n)$.

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$p_i \coloneqq \sum_j t_j$. Degree of the symmetric differences between t_j's is the new source of hardness.

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$\texttt{Split}_x(\pi)$ is a proof of **damaged** version of $\mathcal{F}$.

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$\pi \coloneqq (p_1, \ldots, p_\ell)$ is a proof of $\mathcal{F}$. $H \coloneqq \{t \mid t \in \text{ QR of } \pi, \deg(t) \text{ is big}\}$.

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This is wrong Lemma, we need to change definition of QR to fix it.

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   $s_i = \sum_j t_{i,1} t_{i,j}$. Wlog $t_{i,1} \coloneqq t_{a,k}$ hence $s_i = t_{a,k} t_{a,1} q$.

Open problems

1. Lower (or upper!) bound on $\mathtt{PCR}_{\{\pm 1\}}$-proofs of Functional Pigeonhole Principle.
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