Non-negative rank of ϵ-perturbed matrices

Pavel Hrubeš

Institute of Mathematics, Prague

January 24, 2020

The model

$f : \{0, 1\}^n \rightarrow \{0, 1\}$

The model

$f : \{0,1\}^n \to \{0,1\}$

- $\mathcal{L}(x,y)$ a system of inequalities

$$L_1(x,y) \geq b_1,\ \ldots,\ L_m(x,y) \geq b_m$$

$x = (x_1, \ldots x_n)$, $y = (y_1, \ldots, y_k)$

The model

$f : \{0,1\}^n \to \{0,1\}$

- $\mathcal{L}(x,y)$ a system of inequalities

$$L_1(x,y) \geq b_1, \ \ldots , \ L_m(x,y) \geq b_m$$

  $x = (x_1, \ldots x_n)$, $y = (y_1, \ldots, y_k)$
- $\forall x \in \{0,1\}^n$

$$f(x) = 1 \text{ iff } \exists y \in \mathbb{R}^k \ \mathcal{L}(x,y) \text{ holds}$$

The model

$f : \{0,1\}^n \to \{0,1\}$

- $\mathcal{L}(x,y)$ a system of inequalities

$$L_1(x,y) \geq b_1,\ \dots,\ L_m(x,y) \geq b_m$$

  $x = (x_1, \dots x_n)$, $y = (y_1, \dots, y_k)$
- $\forall x \in \{0,1\}^n$

$$f(x) = 1 \text{ iff } \exists y \in \mathbb{R}^k\ \mathcal{L}(x,y) \text{ holds}$$

- Complexity= number of inequalities in $\mathcal{L}$

The model

$f : \{0,1\}^n \to \{0,1\}$

- $\mathcal{L}(x,y)$ a system of inequalities

$$L_1(x,y) \geq b_1, \ \ldots, \ L_m(x,y) \geq b_m$$

  $x = (x_1, \ldots x_n)$, $y = (y_1, \ldots, y_k)$
- $\forall x \in \{0,1\}^n$

$$f(x) = 1 \text{ iff } \exists y \in \mathbb{R}^k \ \mathcal{L}(x,y) \text{ holds}$$

- Complexity= number of inequalities in $\mathcal{L}$

Monotone version: require x to have non-negative coefficients

The model

$P \subseteq \mathbb{R}^n$ a polyhedron

The model

$P \subseteq \mathbb{R}^n$ a polyhedron

- Extension complexity of P, $\mathrm{xc}(P)$:= the smallest r s.t. P is a projection of some $Q \subseteq \mathbb{R}^{n+k}$ with r facets.

$$x \in P \text{ iff } \exists y \in \mathbb{R}^k\ (x, y) \in Q\,,\ \ Q \text{ given by } r \text{ inequalities.}$$

The model

$P \subseteq \mathbb{R}^n$ a polyhedron

- Extension complexity of P, $\mathrm{xc}(P)$:= the smallest r s.t. P is a projection of some $Q \subseteq \mathbb{R}^{n+k}$ with r facets.

  $$x \in P \text{ iff } \exists y \in \mathbb{R}^k \, (x, y) \in Q\,, \;\; Q \text{ given by } r \text{ inequalities.}$$

- Separation complexity of $f : \{0,1\}^n \to \{0,1\}$, $\mathrm{sep}(f)$:= minimum extension complexity of $P \subseteq \mathbb{R}^n$ with

  $$f^{-1}(1) \subseteq P\,,\;\; f^{-1}(0) \cap P = \emptyset\,.$$

The model

$P \subseteq \mathbb{R}^n$ a polyhedron

- Extension complexity of P, $\mathrm{xc}(P)$:= the smallest r s.t. P is a projection of some $Q \subseteq \mathbb{R}^{n+k}$ with r facets.

  $$x \in P \text{ iff } \exists y \in \mathbb{R}^k\,(x, y) \in Q\,,\ Q \text{ given by } r \text{ inequalities.}$$

- Separation complexity of $f : \{0,1\}^n \to \{0,1\}$, $\mathrm{sep}(f)$:= minimum extension complexity of $P \subseteq \mathbb{R}^n$ with

  $$f^{-1}(1) \subseteq P\,,\ f^{-1}(0) \cap P = \emptyset\,.$$

- Monotone separation complexity of f, $\mathrm{sep}_+(f)$:= minimum extension complexity of P with

  $$f^{-1}(1) \subseteq P^*\,,\ f^{-1}(0) \cap P^* = \emptyset\,,$$

  $P^* := \{z \in \mathbb{R}^n : \exists x \in P\,,\ x \leq z\}$.

- Considered in [Valiant' 82], [Yannakakis'91]

- Considered in [Valiant' 82], [Yannakakis'91]

$$\text{Circuit-Size}(f) \geq \frac{1}{10}\text{sep}(f)\,.$$

- Considered in [Valiant' 82], [Yannakakis'91]

$$\text{Circuit-Size}(f) \geq \frac{1}{10}\text{sep}(f)\,.$$

- [Oliveira, Pudlák'17]

$$\text{Monotone-Circuit-Size}(f) \geq \frac{1}{10}\text{sep}_{+}(f)\,.$$

- Considered in [Valiant' 82], [Yannakakis'91]

$$\text{Circuit-Size}(f) \geq \frac{1}{10}\text{sep}(f)\,.$$

- [Oliveira, Pudlák'17]

$$\text{Monotone-Circuit-Size}(f) \geq \frac{1}{10}\text{sep}_+(f)\,.$$

- There exists a non-explicit f with $\text{sep}(f) \geq 2^{\Omega(n)}$ [H'19]

Lovász-Schrijver connection

- Exponential l.b. for LS* [Beame et al.'07], [LS'08, Sherstov'12,..]

Lovász-Schrijver connection

- Exponential l.b. for LS^* [Beame et al.'07], [LS'08, Sherstov'12,..]
- LS has feasible interpolation via general Boolean circuits [Pudlák'98]

Lovász-Schrijver connection

- Exponential l.b. for LS^* [Beame et al.'07], [LS'08, Sherstov'12,..]
- LS has feasible interpolation via general Boolean circuits [Pudlák'98]
- a modification of LS has feasible interpolation via monotone linear programs [Oliveira, Pudlák'98]

Non-negative rank

$M \in \mathbb{R}^{n \times m}$, a matrix with *non-negative* entries

Non-negative rank

$M \in \mathbb{R}^{n \times m}$, a matrix with *non-negative* entries

Non-negative rank of M, $\text{rk}_+(f)$:= minimum r s.t.

$$M = A \cdot B\,,$$

for some $A \in \mathbb{R}^{n \times r}$, $B \in \mathbb{R}^{r \times m}$ with *non-negative* entries.

$P_0 \subseteq P_1 \subseteq \mathbb{R}^n$

$P_0 := \mathrm{conv}(v_1, \ldots, v_{m_0})$
P_1 defined by inequalities $\ell_1(x) \geq b_1, \ldots, \ell_{m_1}(x) \geq b_{m_1}$

Slack matrix $S \in \mathbb{R}^{m_1 \times m_0}$:

$P_0 \subseteq P_1 \subseteq \mathbb{R}^n$

$P_0 := \mathrm{conv}(v_1, \ldots, v_{m_0})$
P_1 defined by inequalities $\ell_1(x) \geq b_1, \ldots, \ell_{m_1}(x) \geq b_{m_1}$

Slack matrix $S \in \mathbb{R}^{m_1 \times m_0}$:

- Rows indexed by the constraints of P_1, columns by the points v_j.

$P_0 \subseteq P_1 \subseteq \mathbb{R}^n$

$P_0 := \text{conv}(v_1, \ldots, v_{m_0})$
P_1 defined by inequalities $\ell_1(x) \geq b_1, \ldots, \ell_{m_1}(x) \geq b_{m_1}$

Slack matrix $S \in \mathbb{R}^{m_1 \times m_0}$:

- Rows indexed by the constraints of P_1, columns by the points v_j.
- $S_{i,j} = \ell_i(v_j) - b_i$.

$P_0 \subseteq P_1 \subseteq \mathbb{R}^n$

$P_0 := \text{conv}(v_1, \ldots, v_{m_0})$
P_1 defined by inequalities $\ell_1(x) \geq b_1, \ldots, \ell_{m_1}(x) \geq b_{m_1}$

Slack matrix $S \in \mathbb{R}^{m_1 \times m_0}$:

- Rows indexed by the constraints of P_1, columns by the points v_j.
- $S_{i,j} = \ell_i(v_j) - b_i$.

Theorem (Yannakakis, Fiorini et al.)

$$rk_+(S) - 1 \leq \min_{P_0 \subseteq P \subseteq P_1} xc(P) \leq rk_+(S)\,.$$

$f : \{0, 1\}^n \to \{0, 1\}$ a monotone Boolean function.

$f : \{0,1\}^n \to \{0,1\}$ a monotone Boolean function.

Define $M_+(f)$ as $|f^{-1}(0)| \times |f^{-1}(1)|$ matrix s.t.:

$f : \{0,1\}^n \to \{0,1\}$ a monotone Boolean function.

Define $M_+(f)$ as $|f^{-1}(0)| \times |f^{-1}(1)|$ matrix s.t.:

- rows indexed by rejecting inputs $f(y) = 0$, columns by accepting inputs $f(x) = 1$,

$f : \{0,1\}^n \to \{0,1\}$ a monotone Boolean function.

Define $M_+(f)$ as $|f^{-1}(0)| \times |f^{-1}(1)|$ matrix s.t.:

- rows indexed by rejecting inputs $f(y) = 0$, columns by accepting inputs $f(x) = 1$,
- $M_+(f)_{y,x} = \sum_{i=1}^{n} x_i(1 - y_i)$.

$f : \{0,1\}^n \to \{0,1\}$ a monotone Boolean function.

Define $M_+(f)$ as $|f^{-1}(0)| \times |f^{-1}(1)|$ matrix s.t.:

- rows indexed by rejecting inputs $f(y) = 0$, columns by accepting inputs $f(x) = 1$,
- $M_+(f)_{y,x} = \sum_{i=1}^{n} x_i(1 - y_i)$.

Properties:

$f : \{0,1\}^n \to \{0,1\}$ a monotone Boolean function.

Define $M_+(f)$ as $|f^{-1}(0)| \times |f^{-1}(1)|$ matrix s.t.:

- rows indexed by rejecting inputs $f(y) = 0$, columns by accepting inputs $f(x) = 1$,
- $M_+(f)_{y,x} = \sum_{i=1}^{n} x_i(1 - y_i)$.

Properties:

- every entry is ≥ 1,

$f : \{0,1\}^n \to \{0,1\}$ a monotone Boolean function.

Define $M_+(f)$ as $|f^{-1}(0)| \times |f^{-1}(1)|$ matrix s.t.:

- rows indexed by rejecting inputs $f(y) = 0$, columns by accepting inputs $f(x) = 1$,
- $M_+(f)_{y,x} = \sum_{i=1}^{n} x_i(1 - y_i)$.

Properties:

- every entry is ≥ 1,
- $\mathrm{rk}_+(M_+(f)) \leq n$.

$f : \{0,1\}^n \to \{0,1\}$ a monotone Boolean function.

Define $M_+(f)$ as $|f^{-1}(0)| \times |f^{-1}(1)|$ matrix s.t.:

- rows indexed by rejecting inputs $f(y) = 0$, columns by accepting inputs $f(x) = 1$,
- $M_+(f)_{y,x} = \sum_{i=1}^{n} x_i(1 - y_i)$.

Properties:

- every entry is ≥ 1,
- $\text{rk}_+(M_+(f)) \leq n$.

$M(f)$:

- $M(f)_{y,x}$ = Hamming distance of y and x.

$R_\epsilon(f) := \mathrm{rk}_+(M_+(f) - \epsilon J)\,,$ where J is the all-ones matrix.

$R_\epsilon(f) := \mathrm{rk}_+(M_+(f) - \epsilon J)$, where J is the all-ones matrix.

- Monotone-Fla-Size$(f) \geq \frac{R_1(f)}{n-1}$ [H'12, Goos et al.'18].

$R_\epsilon(f) := \mathrm{rk}_+(M_+(f) - \epsilon J)\,,$ where J is the all-ones matrix.

- Monotone-Fla-Size$(f) \geq \frac{R_1(f)}{n-1}$ [H'12, Goos et al.'18].
- Monotone-Circuit-Size$(f) \geq \frac{\min_{\epsilon>0} R_\epsilon(f)}{10} - n$.

$$R_\epsilon(f) := \mathrm{rk}_+(M_+(f) - \epsilon J)\,,\ \text{where } J \text{ is the all-ones matrix.}$$

- Monotone-Fla-Size$(f) \geq \frac{R_1(f)}{n-1}$ [H'12, Goos et al.'18].
- Monotone-Circuit-Size$(f) \geq \frac{\min_{\epsilon>0} R_\epsilon(f)}{10} - n$.

Theorem

$$|sep_+(f) - \min_{\epsilon>0} R_\epsilon(f)| \leq O(n)\,.$$

$$R_\epsilon(f) := \mathrm{rk}_+(M_+(f) - \epsilon J)\,,$$ where J is the all-ones matrix.

- Monotone-Fla-Size$(f) \geq \frac{R_1(f)}{n-1}$ [H'12, Goos et al.'18].
- Monotone-Circuit-Size$(f) \geq \frac{\min_{\epsilon>0} R_\epsilon(f)}{10} - n$.

Theorem

$$|sep_+(f) - \min_{\epsilon>0} R_\epsilon(f)| \leq O(n)\,.$$

- Replacing $M_+(f)$ by $M(f)$, the above hold for non-monotone computations.

Strictly positive rank

$M \in \mathbb{R}^{n \times m}$, a matrix with *positive* entries

Strictly positive rank

$M \in \mathbb{R}^{n \times m}$, a matrix with *positive* entries

Strictly positive rank of M, $\text{rk}_*(f)$:= minimum r s.t.

$$M = A \cdot B\,,$$

for some $A \in \mathbb{R}^{n \times r}$, $B \in \mathbb{R}^{r \times m}$ with *positive* entries

Strictly positive rank

$M \in \mathbb{R}^{n \times m}$, a matrix with *positive* entries

Strictly positive rank of M, $\mathrm{rk}_*(f)$:= minimum r s.t.

$$M = A \cdot B\,,$$

for some $A \in \mathbb{R}^{n \times r}$, $B \in \mathbb{R}^{r \times m}$ with *positive* entries

Theorem

$$|sep_+(f) - rk_*(M_+(f))| \leq O(n)\,.$$

Open problem 1

Find an explicit monotone f with $\mathrm{sep}_+(f)$ superpolynomial in n.

Open problem 2

Find an explicit M with positive entries such that $\min_{\epsilon>0} \mathrm{rk}_+(M_+(f) - \epsilon J)$ is superpolynomial in $\mathrm{rk}_+(M)$.

THANK YOU