Catastrophes and Queueing Systems with Time-Varying Periodic Transition Rates

Barbara Haas Margolius
Cleveland State University
Cleveland, Ohio USA

Sherif Ammar
Menofia University
Egypt

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Hello

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Overview

1. Background
   - Key equation for QBDs with periodic transition rates
   - Key equation for system with catastrophes
   - Exponential Generating Function for the Bernoulli numbers (an aside to motivate the method)
2. Asymptotic Periodic solution for the Single Server Queue
   - Solution for the single server queue without catastrophes
   - Solution for the single server queue with catastrophes
3. Asymptotic periodic solution for Erlang arrivals with exponential service
   - Solution for Erlang arrivals with exponential service without catastrophes
   - Solution for Erlang arrivals with exponential service with catastrophes

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The asymptotic periodic distribution of the queue-length process is analogous to the steady state distribution for a system with constant transition rates.

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Catastrophes, when they occur, take the process to an un-numbered repair level.

The processes we consider have periodic transition rates with period one.

The infinitesimal generator for a QBD with time-varying periodic transition rates:

$$\mathbf{Q}(t) = \begin{bmatrix} \mathbf{B}(t) & \mathbf{A_1}(t) & & & \\ \mathbf{A}_{-1}(t) & \mathbf{A}_0(t) & \mathbf{A}_1(t) & & \\ & \mathbf{A}_{-1}(t) & \mathbf{A}_0(t) & \mathbf{A}_1(t) & \\ & & \ddots & \ddots & \end{bmatrix}$$

This leads to the system of differential equations:

$$\begin{aligned} \dot{\mathbf{p}}_0(t) &= \mathbf{p}_0(t)\mathbf{B}(t) + \mathbf{p}_1(t)\mathbf{A}_{-1}(t) \\ \dot{\mathbf{p}}_n(t) &= \mathbf{p}_{n-1}(t)\mathbf{A}_1(t) + \mathbf{p}_n(t)\mathbf{A}_0(t) + \mathbf{p}_{n+1}(t)\mathbf{A}_{-1}(t),\ n > 0. \end{aligned}$$

We can use this system of ordinary differential equations to solve for the generating function for the asymptotic periodic distribution.

The *key* equation for the generating function is given by

$$\mathbf{P}(z,t) = \sum_{j=0}^{\infty} \mathbf{p}_j(t) z^j = \int_{t-1}^{t} \mathbf{p}_0(u)\left(\mathbf{B}(u) - \mathbf{A}_0(u) - z^{-1}\mathbf{A}_{-1}(u)\right)\Phi(z,u,t)du \times \left(\mathbf{I} - \Phi(z,t-1,t)\right)^{-1}.$$

where $\Phi(z,s,t)$ is the generating function for the unbounded process.

Now consider a quasi-birth-death process with catastrophes and repairs. Catastrophes occur at rate $\gamma(t)$. These can occur from any level and from any phase within the level. When a catastrophe occurs, then the system is repaired at rate $\eta(t)$. These rates are also assumed to be periodic with period 1. The probability of being in the failure state at time t is given by $q(t)$.

For this system with catastrophes and repairs, we have the infinitesimal generator:

$$\mathbf{Q}(t) = \begin{bmatrix} -\eta(t) & \eta(t)\mathbf{a} & & & \\ \gamma(t)\mathbf{1} & \mathbf{B}(t)-\gamma(t)\mathbf{I} & \mathbf{A}_1(t) & & \\ \gamma(t)\mathbf{1} & \mathbf{A}_{-1}(t) & \mathbf{A}_0(t)-\gamma(t)\mathbf{I} & \mathbf{A}_1(t) & \\ \gamma(t)\mathbf{1} & & \mathbf{A}_{-1}(t) & \mathbf{A}_0(t)-\gamma(t)\mathbf{I} & \mathbf{A}_1(t) \\ \vdots & & & \ddots & \ddots \end{bmatrix}$$

with $\mathbf{1} = \mathbf{1}_{N\times 1}$ a column vector of ones and $\mathbf{p}_{1\times N}$ a row vector representing the probability distribution of which phase the system returns to after a repair. $\eta(t)$ is the repair rate and $\gamma(t)$ is the rate at which catastrophes occur.

This process has the following system of ordinary differential equations:
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$$\begin{aligned}\dot{q}(t) &= -\eta(t)q(t) + \gamma(t)(1 - q(t)) \\ \dot{\mathbf{p}}_0(t) &= q(t)\eta(t)\mathbf{a} + \mathbf{p}_0(t)(\mathbf{B}(t) - \gamma(t)\mathbf{I}) + \mathbf{p}_1(t)\mathbf{A}_{-1}(t) \\ \dot{\mathbf{p}}_n(t) &= \mathbf{p}_{n-1}(t)\mathbf{A}_1(t) + \mathbf{p}_n(t)(\mathbf{A}_0(t) - \gamma(t)\mathbf{I}) + \mathbf{p}_{n+1}(t)\mathbf{A}_{-1}(t), \; n > 0.\end{aligned}$$

Let $\mathbf{P}(z,t) = \sum_{j=0}^{\infty} \mathbf{p}_j(t) z^j$, as above, then the *key* equation for the generating function is

$$\mathbf{P}(z,t) = \int_{t-1}^{t} \left[\mathbf{p}_0(u)\left(\mathbf{B}(u) - \mathbf{A}_0(u) - z^{-1}\mathbf{A}_{-1}(u)\right) + q(u)\eta(u)\mathbf{a}\right] \Phi(z,u,t)du \times \left(\mathbf{I} - \Phi(z,t-1,t)\right)^{-1}.$$

Here $\Phi(z,s,t)$ is the same as that given for a QBD without catastrophes except that it is multiplied by $\mathrm{e}^{-\int_s^t \gamma(u)du}$.

Bernoulli numbers (an aside to motivate the method)

Recall that an exponential generating function (EGF) for a sequence $\{a_n\}$ is defined as

$$A(z) = \sum_{n=0}^{\infty} a_n \frac{z^n}{n!}.$$

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The EGF for the Bernoulli numbers is:

$$B(z) = \frac{z}{e^z - 1} \sim \frac{\chi_k}{z - \chi_k}, \quad z \to \chi_k$$

The poles of $B(z)$ are $\chi_k = 2\pi ik$, $k \in \mathbb{Z}\setminus\{0\}$, and the residue at χ_k is equal to χ_k.

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The expansion theorem for meromorphic functions holds that

$$f_n \equiv [z^n] f(z) = \sum_{j=1}^{m} c_j \alpha_j^{-n} + O(R^{-n})$$

where the α_j are poles of the generating function $f(z)$ and $f(z)$ is meromorphic at all points of the closed disc $|z| \leq R$ and analytic at all points $|z| = R$ and at $z = 0$.

In the case of the Bernoulli numbers, we take contours that pass between two poles. As the contour $R \to \infty$ (for $n \geq 0$) because the Cauchy kernel z^{-n-1} decreases as an inverse power of R while the exponential generating function (EGF) remains $O(R)$. In the limit, the coefficient integral is equal to the sum of the residues of the meromorphic function over the whole complex plane. .

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Elias Wegert
Technische Universität Bergakademie Freiberg
Institute of Applied Analysisnull, Germany

Figure: Increasing contour of integration, R as we zoom out

$$[z^n]\frac{z}{\mathrm{e}^z - 1}$$

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\begin{aligned}
[z^n]\frac{z}{e^z - 1} &= \frac{B_n}{n!} \\
&= -\sum_{k \in \mathbb{Z} \setminus \{0\}}^{\infty} \chi_k^{-n} = -\sum_{k=1}^{\infty} \left(\frac{1}{(-2\pi i k)^n} + \frac{1}{(2\pi i k)^n} \right)
\end{aligned}
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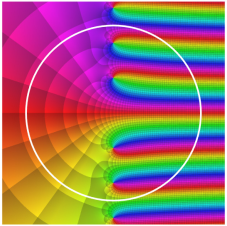

$$= -\frac{1}{(2\pi i)^n}\sum_{k=1}^{\infty}\frac{1}{k^n}\left(1 + (-1)^n\right)$$

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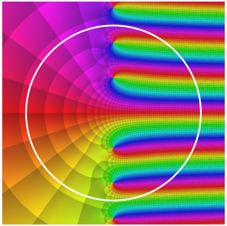

$$= -\frac{1}{(2\pi i)^n}\sum_{k=1}^{\infty}\frac{1}{k^n}\left(1+(-1)^n\right)$$

$$= \begin{cases} 0 & n \text{ odd, } n \geq 3 \\ -\frac{2}{(2\pi i)^n}\sum_{k=1}^{\infty}\frac{1}{k^n} & n \text{ even} \end{cases}$$

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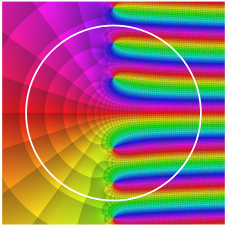

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$$B_{2n} = (-1)^{n-1}(2n)!2^{1-2n}\pi^{-2n}\zeta(2n).$$

The *key* equation for the generating function is

$$P(z,t) = \int_{t-1}^{t} p_0(u)\mu(u)(1 - z^{-1})\mathrm{e}^{\int_u^t \lambda(\xi)(z-1)+\mu(\xi)(z^{-1}-1)d\xi} du \times (1 - \mathrm{e}^{\bar{\lambda}(z-1)+\bar{\mu}(z^{-1}-1)})^{-1}$$

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The poles are solutions to

$$\bar{\lambda}(z-1) + \bar{\mu}(z^{-1} - 1) = 2\pi i k, \;\; k \in \mathbb{Z},$$

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$$\bar{\lambda}(z-1) + \bar{\mu}(z^{-1}-1) = 2\pi i k,\ \ k \in \mathbb{Z},$$

that is,

$$\chi_k = \frac{1}{2\bar{\lambda}}\left(\bar{\lambda} + \bar{\mu} + 2\pi i k + \sqrt{(\bar{\lambda} + \bar{\mu} + 2\pi i k)^2 - 4\bar{\lambda}\bar{\mu}}\right).$$

Figure: Increasing contour of integration, R (shown in white), as we zoom out, $\bar{\lambda} = 2$, $\bar{\mu} = 5$. Unit circle shown in black. For this example,

$$\lambda(t) = 2 + \tfrac{2}{3}\cos(2\pi t),$$
$$\mu(t) = 5 + \tfrac{5}{2}\sin(2\pi t).$$

An exact formula for the level probabilities is then given by

$$p_\ell(t) = \int_{t-1}^{t} p_0(u)\mu(u) \sum_{k=-\infty}^{\infty} \frac{(1-\chi_k^{-1})\chi_k^{-\ell} \mathrm{e}^{\int_u^t \left(\lambda(\xi)(\chi_k-1)+\mu(\xi)(\chi_k^{-1}-1)\right)d\xi}}{\sqrt{(\bar{\lambda}+\bar{\mu}+2\pi ik)^2-4\bar{\lambda}\bar{\mu}}} du.$$

The *key* equation for the generating function for the single server queue with catastrophes is

$$P(z,t) = \int_{t-1}^{t} \left(\mu(u)(1-z^{-1})p_0(u) + \eta(u)q(u)\right)\Phi(z,u,t)du \times \left(1 - e^{\bar{\lambda}(z-1)-\bar{\mu}(z^{-1}-1)-\bar{\gamma}}\right)^{-1}$$

and

$$q(t) = \int_{t-1}^{t} \gamma(u)e^{-\int_u^t(\eta(\nu)+\gamma(\nu))d\nu}du(1-e^{-\bar{\eta}-\bar{\gamma}})^{-1}$$

where $\bar{\gamma} = \int_{t-1}^{t}\gamma(u)du$ is the average catastrophe rate over the period, $\bar{\lambda} = \int_{t-1}^{t}\lambda(u)du$ is the average arrival rate, and so on.

The poles are solutions to

$$\bar{\lambda}(z-1)+\bar{\mu}(z^{-1}-1)-\bar{\gamma}=2\pi ik,\ k\in\mathbb{Z},$$

that is,

$$\chi_k=\frac{1}{2\bar{\lambda}}\left(\bar{\lambda}+\bar{\mu}+\bar{\gamma}+2\pi ik+\sqrt{(\bar{\lambda}+\bar{\mu}+\bar{\gamma}+2\pi ik)^2-4\bar{\lambda}\bar{\mu}}\right).$$

Applying the same argument as for the Bernoulli numbers, the exact formula is the sum of the residues in the complex plane:

$$p_\ell(t)=\int_{t-1}^{t}\sum_{k=-\infty}^{\infty}\left(p_0(u)\mu(u)(1-\chi_k^{-1})+\eta(u)q(u)\right)$$
$$\times\frac{\chi_k^{-\ell}\mathrm{e}^{\int_u^t\left(\lambda(\xi)(\chi_k-1)+\mu(\xi)(\chi_k^{-1}-1)-\gamma(\xi)\right)d\xi}}{\sqrt{(\bar{\lambda}+\bar{\mu}+\bar{\gamma}+2\pi ik)^2-4\bar{\lambda}\bar{\mu}}}du.$$

Figure: Increasing contour of integration, R (shown in white), as we zoom out, $\bar{\lambda} = 2$, $\bar{\mu} = 5$. Unit circle shown in black. For this example,

$$\lambda(t) = 2 + \frac{2}{3}\cos(2\pi t),$$
$$\mu(t) = 5 + \frac{5}{2}\sin(2\pi t),$$
$$\gamma(t) = 4 + \frac{18}{5}\sin(2\pi t),$$
$$\eta(t) = 3 + 2\sin(2\pi t).$$

The *key* equation for the generating function is

$$
\begin{aligned}
P(z,t) &= \int_{t-1}^{t} \mathbf{p}_0(u)\left(\mathbf{B}(u) - \mathbf{A}_0(u) - z^{-1}\mathbf{A}_{-1}(u)\right)\Phi(z,u,t)du \\
&\quad \times (\mathbf{I} - \Phi(z,t-1,t))^{-1} \\
&= \int_{t-1}^{t} \mathbf{p}_0(u)\mu(u)(1-z^{-1})H\mathrm{diag}\left[\frac{\mathrm{e}^{\int_u^t \epsilon_\ell(z,q)dq}}{1-\mathrm{e}^{\int_{t-1}^t \epsilon_\ell(z,q)dq}}\right] duH^{-1}.
\end{aligned}
$$

where

$$\left[H \mathrm{diag}\left[\frac{\mathrm{e}^{\int_u^t \epsilon_\ell(z,q)dq}}{1-\mathrm{e}^{\int_{t-1}^t \epsilon_\ell(z,q)dq}} \right] H^{-1} \right]_{j,j+m} = \frac{z^{m/K}}{K} \sum_{\ell=0}^{K-1} \frac{\mathrm{e}^{\int_u^t \epsilon_\ell(z,q)dq}}{1-\mathrm{e}^{\int_{t-1}^t \epsilon_\ell(z,q)dq}} \omega_K^{-m\ell}.$$

is a Toeplitz matrix and $\omega_K = \exp\{2\pi i/K\}$ are the Kth roots of unity.

The poles occur at those values of z outside of the unit circle such that

$$-\bar{\mu} - \bar{\nu} + \bar{\mu} z^{-1} + z^{1/K} \bar{\nu} = 2\pi i m, \ m \in \mathbb{Z}.$$

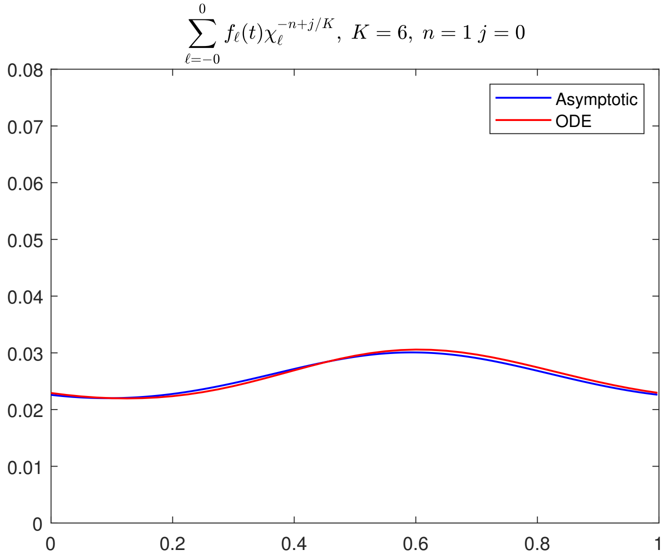
The exact solution is given by the sum of the residues. Taking the limit as $z \to \chi_m$, we have the residue at χ_m given by

$$\frac{1}{1 - \frac{z}{\chi_m}} \left(\frac{\chi_m - 1}{\bar{\nu} \chi_m^{(K+1)/K} - K\bar{\mu}} \right)$$
$$\times \int_{t-1}^{t} \mathbf{p}_0(u) \mu(u) \mathrm{e}^{\int_u^t (\nu(s)\chi_m^{1/K} - \nu(s) - \mu(s) + \mu(s)\chi_m^{-1}) ds} du$$
$$\times \begin{bmatrix} 1 & \chi_m^{1/K} & & \cdots & \chi_m^{(K-1)/K} \\ \chi_m^{-1/K} & \ddots & \chi_m^{1/K} & \cdots & \chi_m^{(K-2)/K} \\ \vdots & & \ddots & \cdots & \vdots \\ \vdots & \cdots & & \ddots & \vdots \\ \chi_m^{(1-K)/K} & \cdots & & \chi_m^{-1/K} & 1 \end{bmatrix}.$$

Let $q_m(t) = \sum_{j=0}^{K-1} \chi_m^{-j/K} p_{0,j}(t)$ and $c_m = \frac{\chi_m - 1}{\bar{\nu}\chi_m^{(K+1)/K} - K\bar{\mu}}$, then

$$p_{n,j}(t) = \sum_{m=-\infty}^{\infty} c_m \chi_m^{-n+j/K} \int_{t-1}^{t} q_m(u)\mu(t) \mathrm{e}^{\int_u^t (\nu(s)\chi_m^{1/K} - \nu(s) - \mu(s) + \mu(s)\chi_m^{-1}) ds} du$$

$\mu(t) = 2 + 2/3\cos(2\pi t),\ \nu(t) = 5 + 2.5\sin(2\pi t)$



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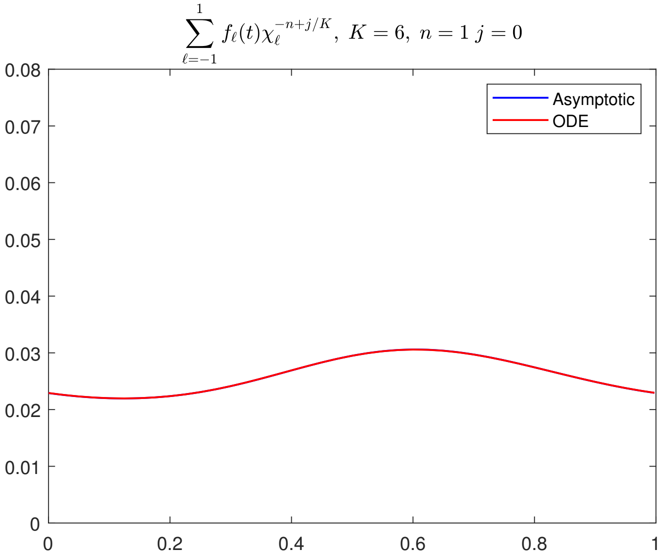

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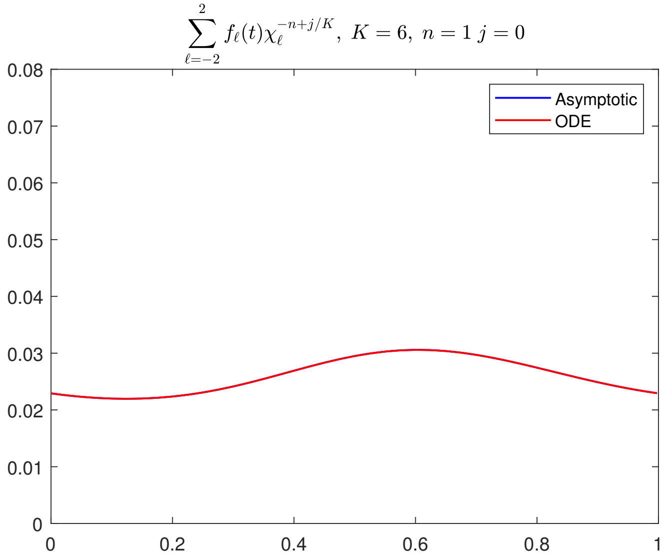

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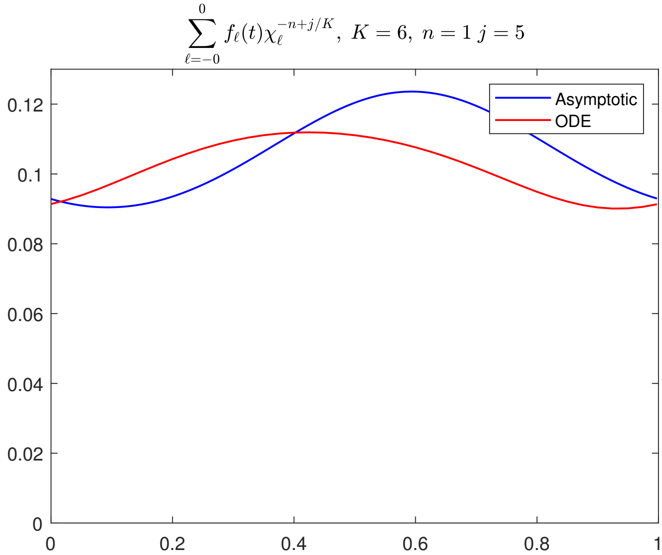

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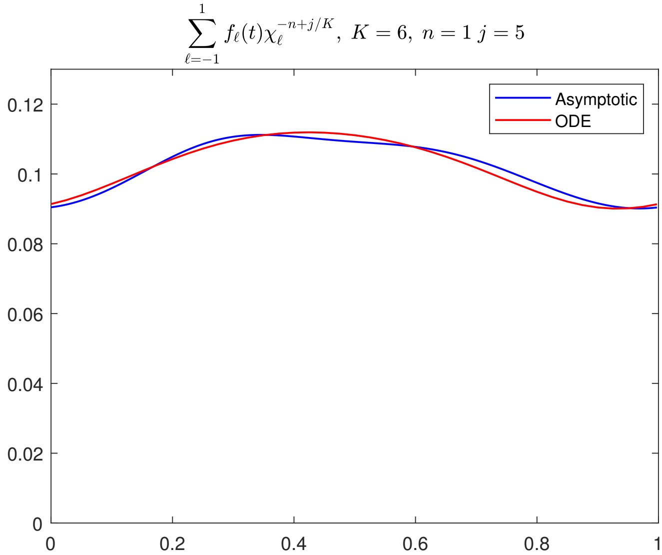

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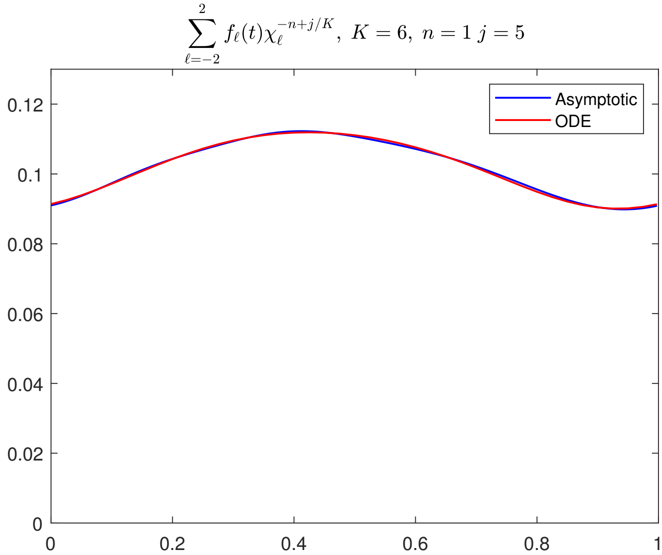

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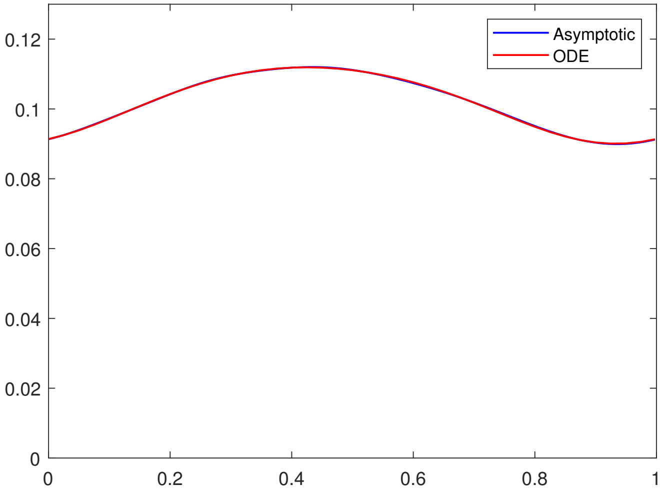
$$\sum_{\ell=-3}^{3} f_\ell(t)\chi_\ell^{-n+j/K},\ K=6,\ n=1\ j=5$$



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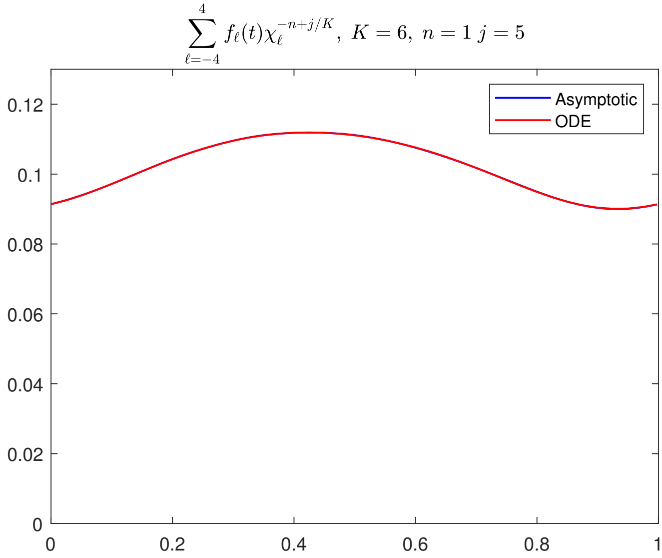

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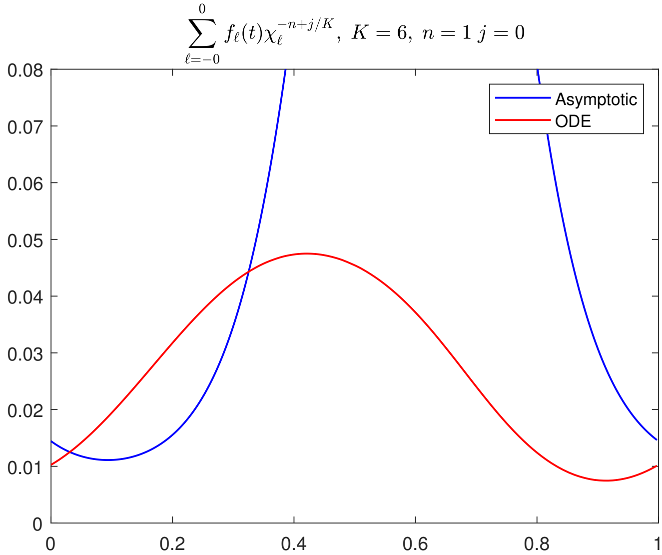

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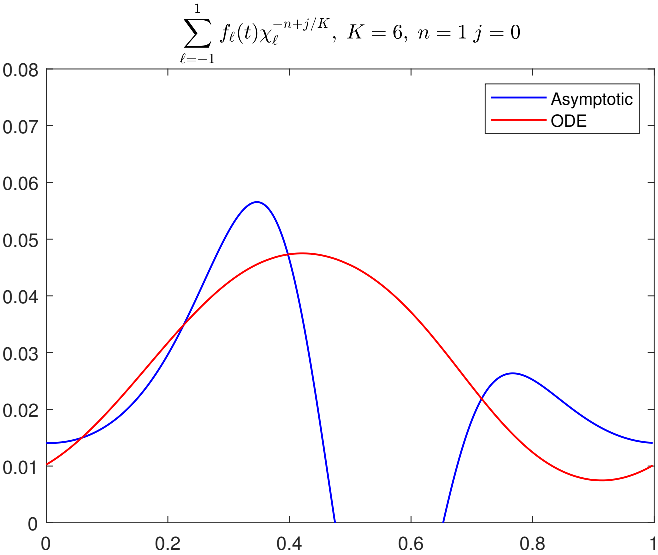

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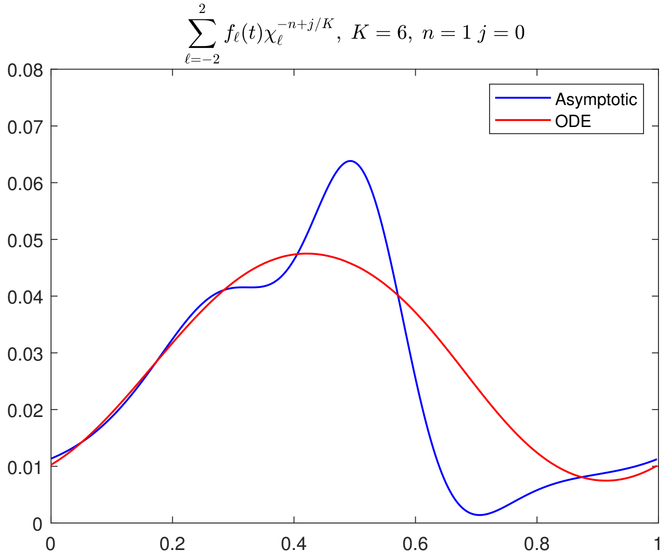
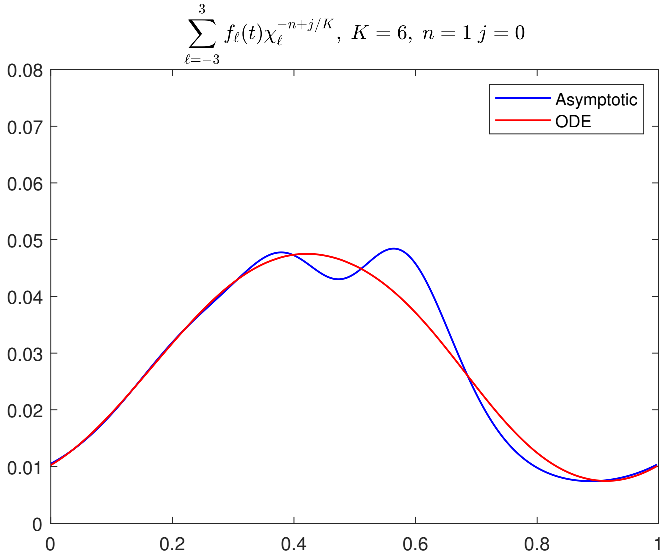
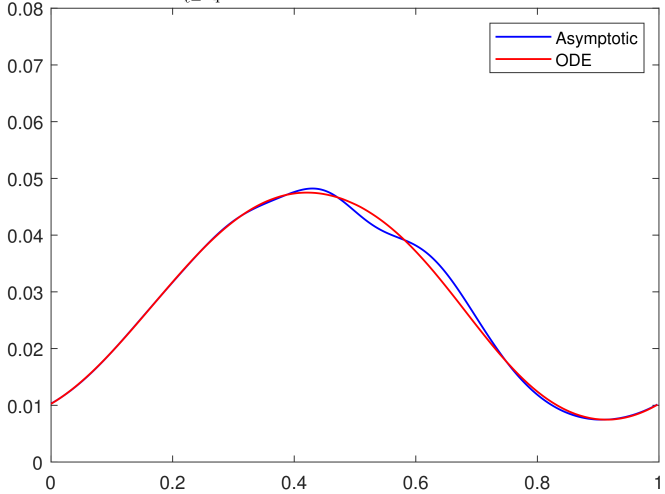


$\mu(t) = 20 + 20/3\cos(2\pi t),\ \nu(t) = 50 + 25\sin(2\pi t)$

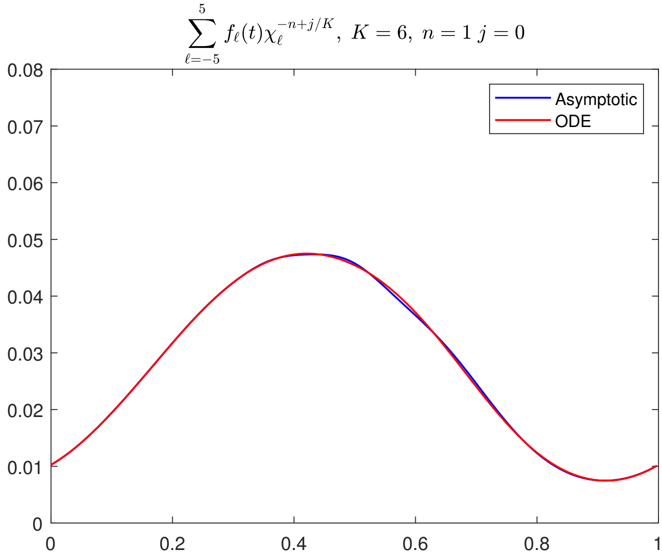


$\mu(t) = 20 + 20/3\cos(2\pi t),\ \nu(t) = 50 + 25\sin(2\pi t)$

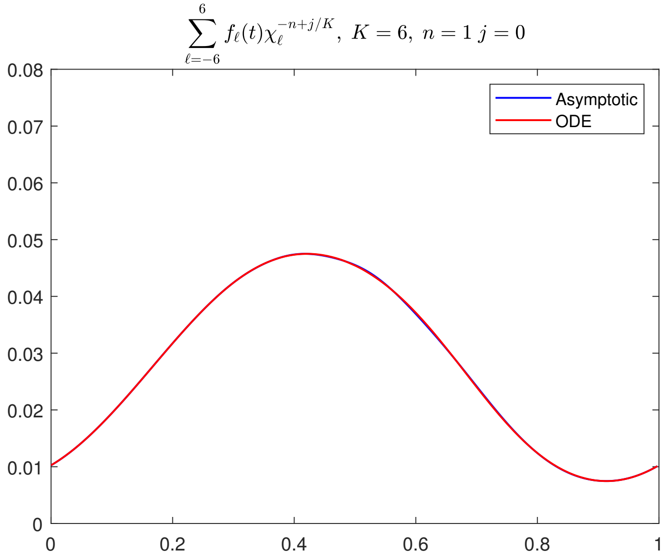
$$\sum_{\ell=-4}^{4} f_\ell(t)\chi_\ell^{-n+j/K},\ K=6,\ n=1\ j=0$$

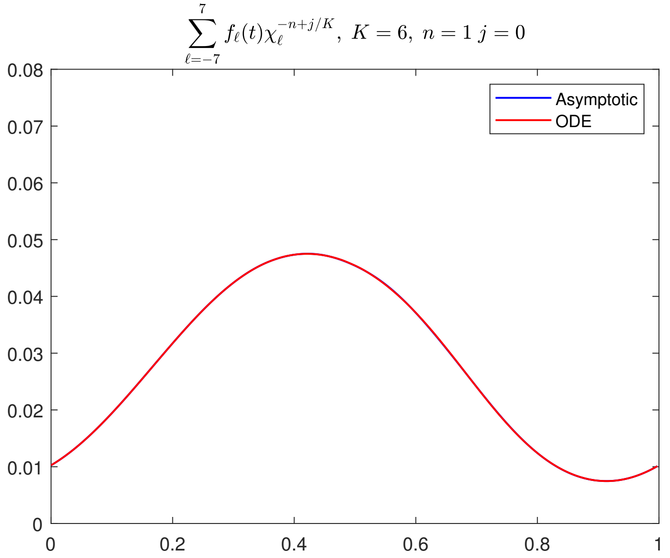


$\mu(t) = 20 + 20/3\cos(2\pi t),\ \nu(t) = 50 + 25\sin(2\pi t)$

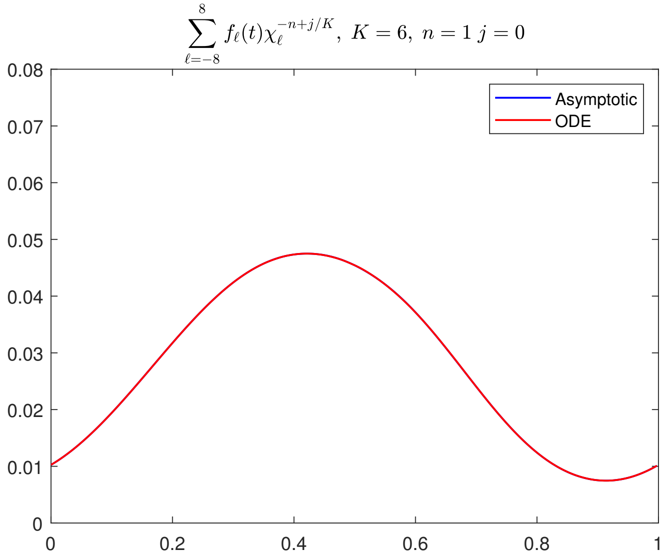


$\mu(t) = 20 + 20/3\cos(2\pi t),\ \nu(t) = 50 + 25\sin(2\pi t)$

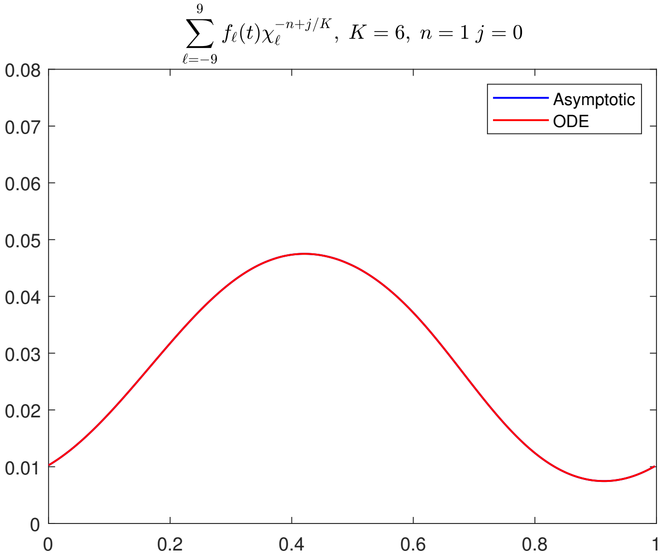


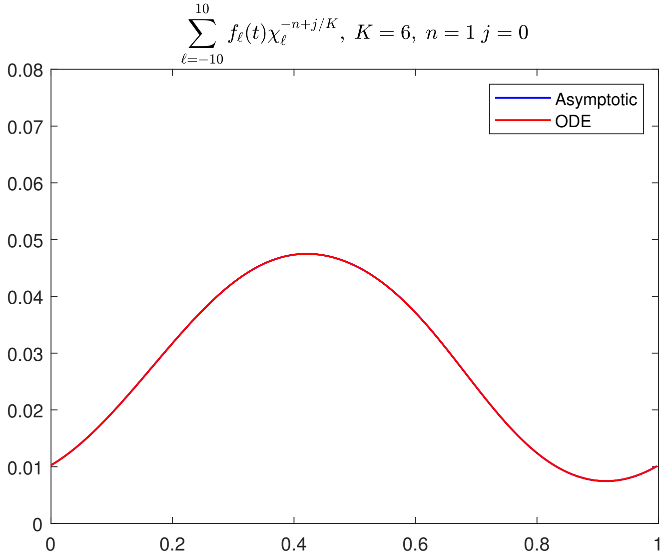

$\sum_{\ell=-7}^{7} f_\ell(t)\chi_\ell^{-n+j/K},\ K = 6,\ n = 1\ j = 0$
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
0.2
0.4
0.6
0.8
1
Asymptotic
ODE

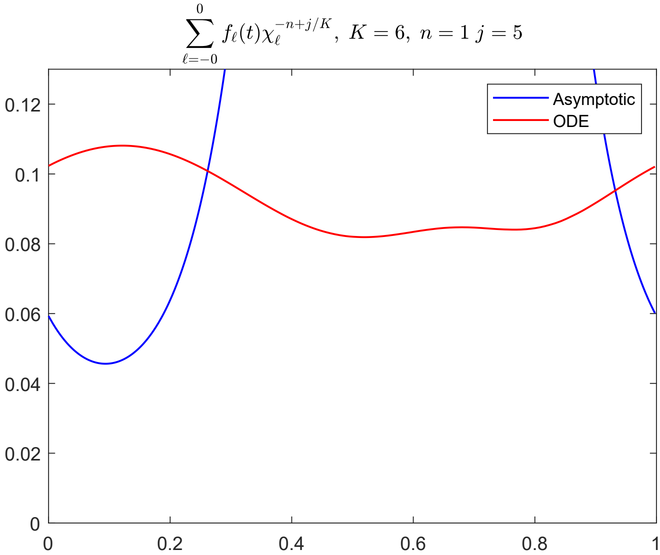
$\mu(t) = 20 + 20/3\cos(2\pi t),\ \nu(t) = 50 + 25\sin(2\pi t)$



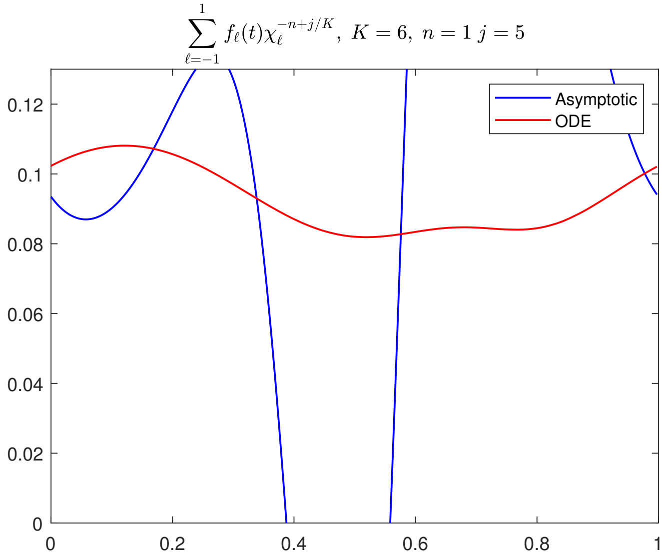
$\mu(t) = 20 + 20/3\cos(2\pi t),\ \nu(t) = 50 + 25\sin(2\pi t)$



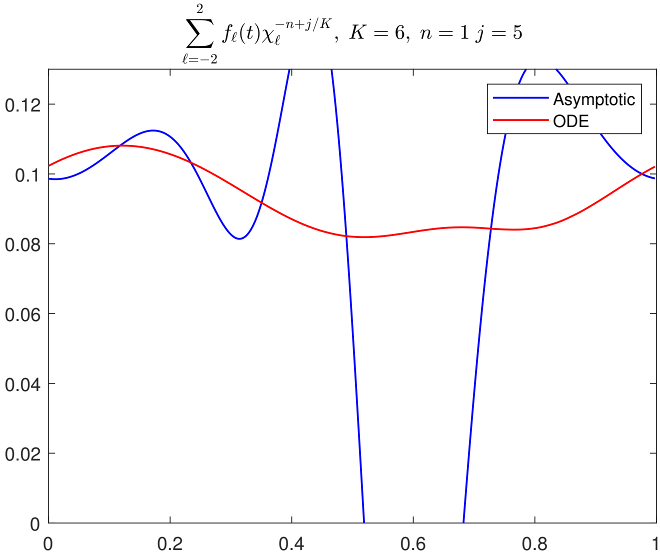

$\sum_{\ell=-10}^{10} f_\ell(t)\chi_\ell^{-n+j/K},\ K = 6,\ n = 1\ j = 0$
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
0
0.2
0.4
0.6
0.8
1
Asymptotic
ODE


$\sum_{\ell=-0}^{0} f_\ell(t)\chi_\ell^{-n+j/K},\ K = 6,\ n = 1\ j = 5$
0.12
0.1
0.08
0.06
0.04
0.02
0
0
0.2
0.4
0.6
0.8
1
Asymptotic
ODE

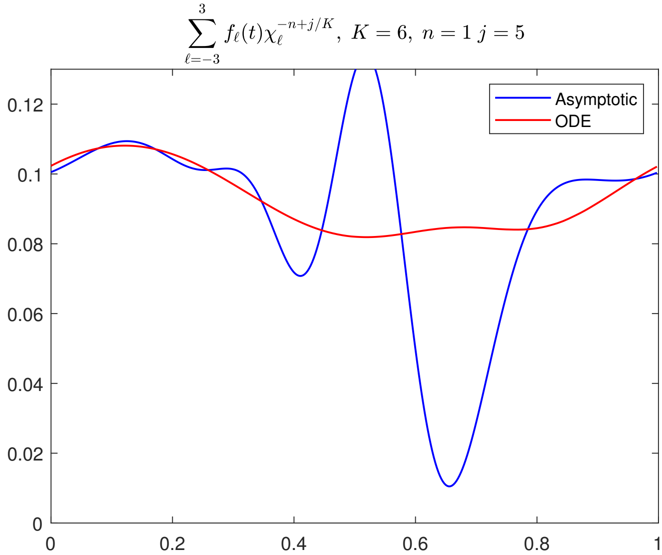
$\mu(t) = 20 + 20/3\cos(2\pi t),\ \nu(t) = 50 + 25\sin(2\pi t)$



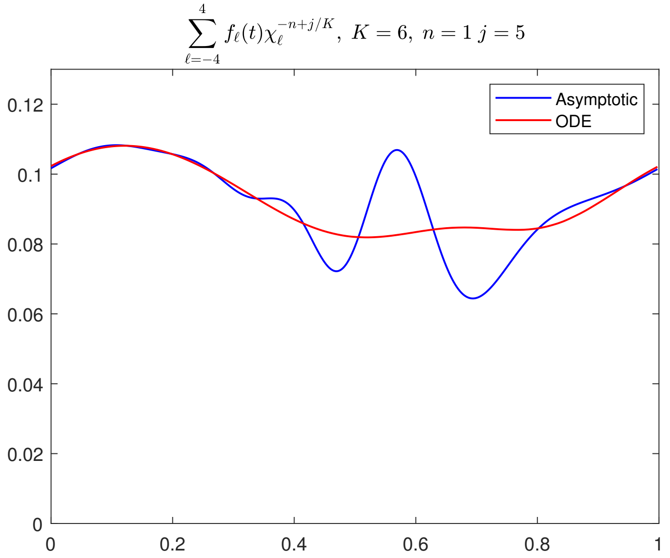
$\mu(t) = 20 + 20/3\cos(2\pi t),\ \nu(t) = 50 + 25\sin(2\pi t)$



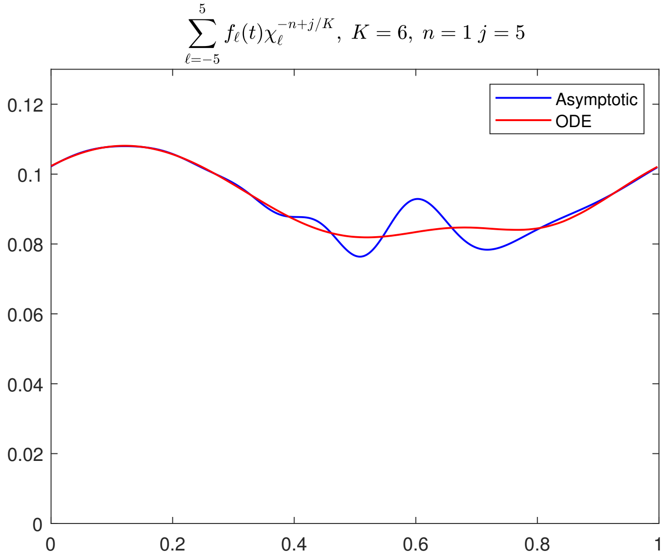
$\mu(t) = 20 + 20/3\cos(2\pi t),\ \nu(t) = 50 + 25\sin(2\pi t)$



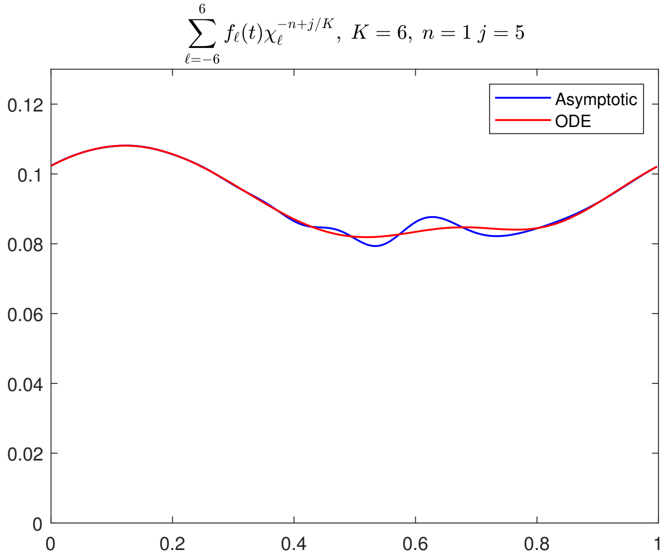
$\mu(t) = 20 + 20/3\cos(2\pi t),\ \nu(t) = 50 + 25\sin(2\pi t)$



$\mu(t) = 20 + 20/3\cos(2\pi t),\ \nu(t) = 50 + 25\sin(2\pi t)$

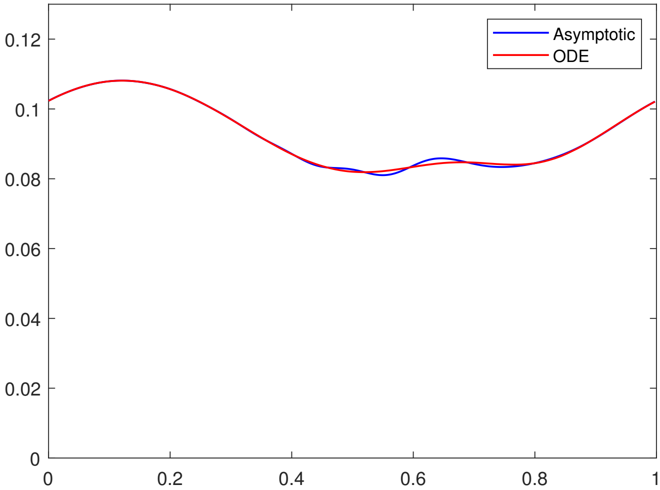


$\mu(t) = 20 + 20/3\cos(2\pi t),\ \nu(t) = 50 + 25\sin(2\pi t)$

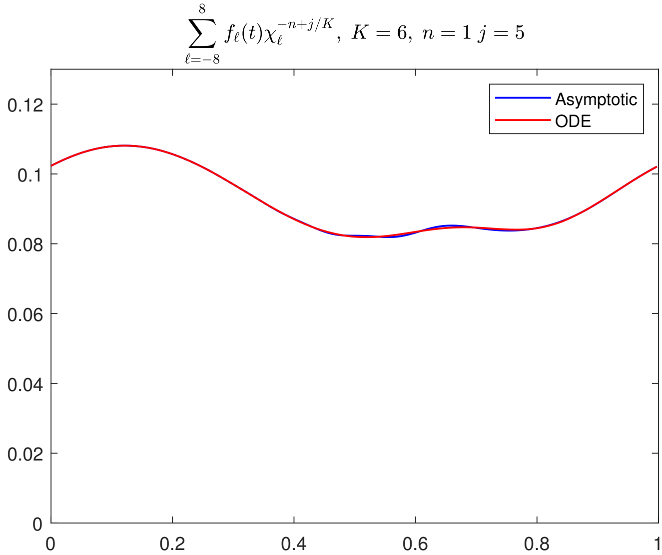


$\mu(t) = 20 + 20/3\cos(2\pi t),\ \nu(t) = 50 + 25\sin(2\pi t)$

$$\sum_{\ell=-7}^{7} f_\ell(t)\chi_\ell^{-n+j/K},\ K=6,\ n=1\ j=5$$

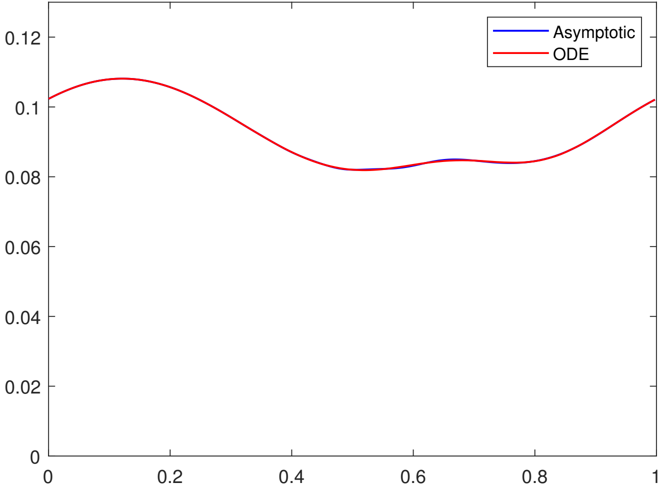


$\mu(t) = 20 + 20/3\cos(2\pi t),\ \nu(t) = 50 + 25\sin(2\pi t)$

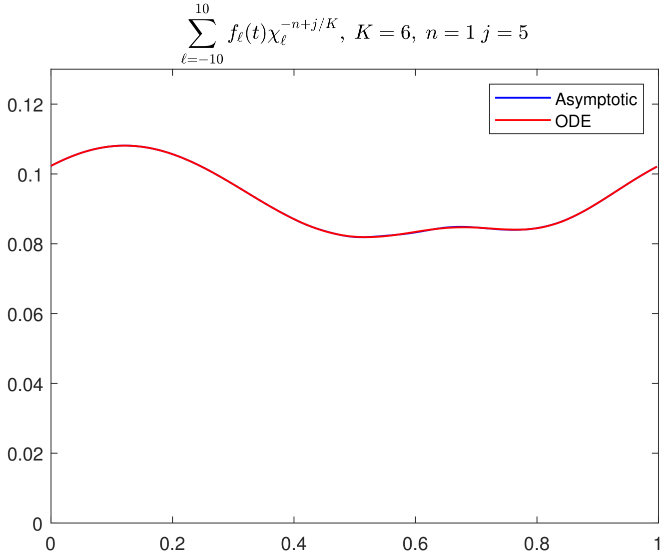


$\mu(t) = 20 + 20/3\cos(2\pi t),\ \nu(t) = 50 + 25\sin(2\pi t)$

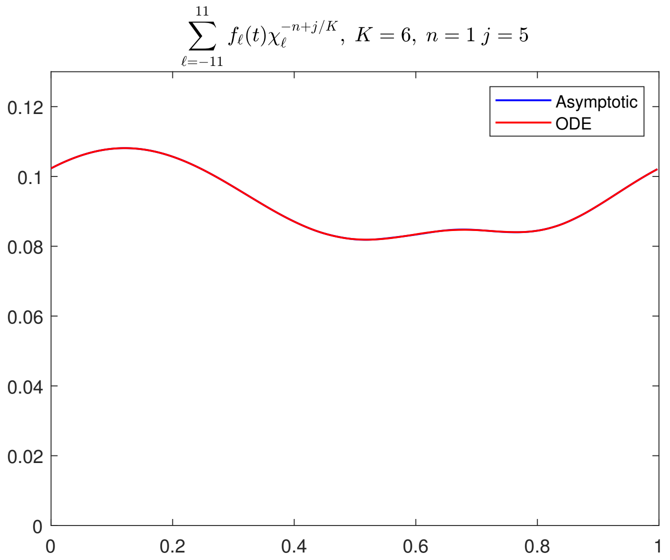
$$\sum_{\ell=-9}^{9} f_\ell(t)\chi_\ell^{-n+j/K},\ K=6,\ n=1\ j=5$$



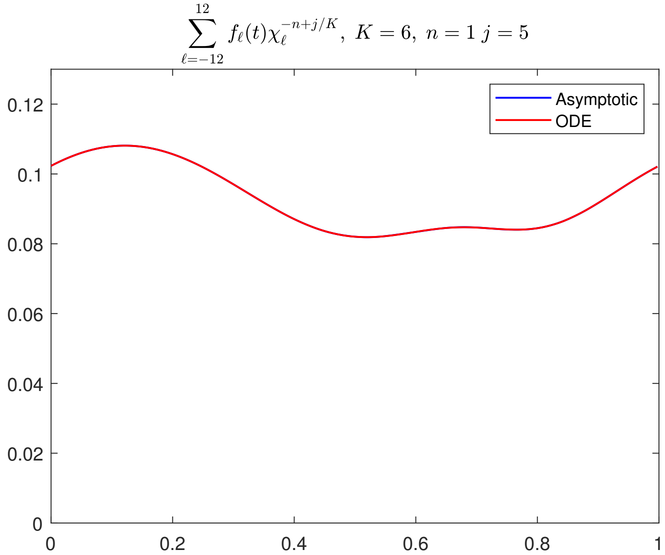
$\mu(t) = 20 + 20/3\cos(2\pi t),\ \nu(t) = 50 + 25\sin(2\pi t)$



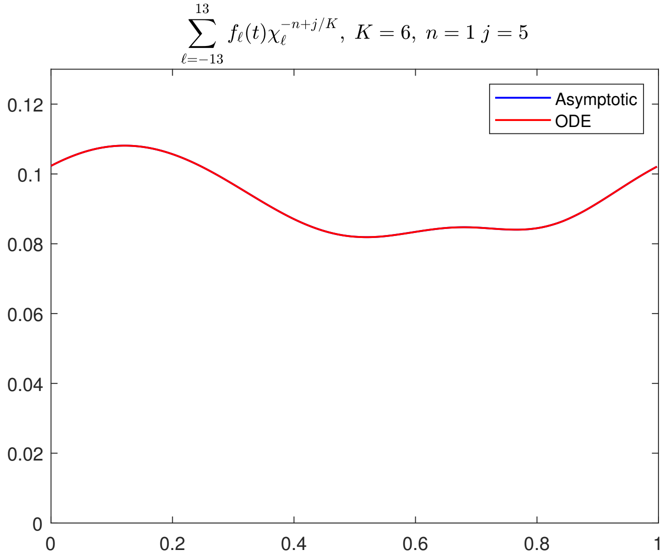
$\mu(t) = 20 + 20/3\cos(2\pi t),\ \nu(t) = 50 + 25\sin(2\pi t)$



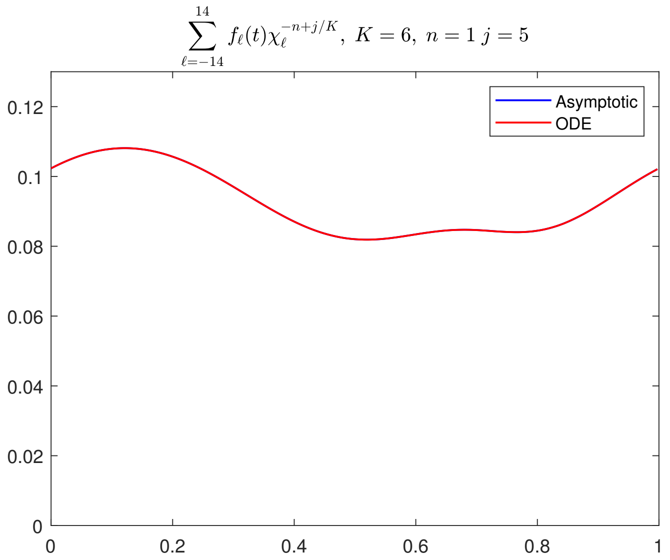
$\mu(t) = 20 + 20/3\cos(2\pi t),\ \nu(t) = 50 + 25\sin(2\pi t)$



$\mu(t) = 20 + 20/3\cos(2\pi t),\ \nu(t) = 50 + 25\sin(2\pi t)$



$\mu(t) = 20 + 20/3\cos(2\pi t),\ \nu(t) = 50 + 25\sin(2\pi t)$



The *key* equation for generating function is

$$\mathbf{P}(z,t) = \int_{t-1}^{t} \left[\mathbf{p}_0(u)\mu(u)(1-z^{-1}) + \eta(u)q(u)\mathbf{a}\right] \Phi(z,u,t)du \times (\mathbf{I} - \Phi(z,t-1,t))^{-1}.$$

The poles occur at those values of z outside of the unit circle such that

$$-\bar{\mu} - \bar{\nu} - \bar{\gamma} + \bar{\mu}z^{-1} + z^{1/K}\bar{\nu} = 2\pi i m, \ m \in \mathbb{Z}.$$

The exact solution is given by the sum of the residues. Taking the limit as $z \to \chi_m$, we have the residue at χ_m given by

$$\frac{1}{1 - \frac{z}{\chi_m}} \left(\frac{1}{\bar{\nu}\chi_m^{(K+1)/K} - K\bar{\mu}} \right) \times \int_{t-1}^{t} (\mathbf{p}_0(u)\mu(u)(\chi_m - 1) + \chi_m q(u)\eta(u)\mathbf{a}) \mathrm{e}^{\int_u^t (\nu(s)\chi_m^{1/K} - \nu(s) - \mu(s) - \gamma(s) + \mu(s)\chi_m^{-1}) ds} du \times \begin{bmatrix} 1 & \chi_m^{1/K} & & \cdots & \chi_m^{(K-1)/K} \\ \chi_m^{-1/K} & \ddots & \chi_m^{1/K} & \cdots & \chi_m^{(K-2)/K} \\ \vdots & & \ddots & \cdots & \vdots \\ \vdots & \cdots & & \ddots & \vdots \\ \chi_m^{(1-K)/K} & \cdots & & \chi_m^{-1/K} & 1 \end{bmatrix}.$$

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Thank You!!!

CanQueue 2011, Banff

Summary: We can find exact solutions of the asymptotic periodic distribution for some queueing systems using singularity analysis.

Poles of $M/M/1$ with catastrophes