Locally bound preserving limiters on conforming and nonconforming meshes for hyperbolic problems

Lilia Krivodonova
University of Waterloo

with Andrew Giuliani (Courant Institute) and Krishna Dutt (Waterloo)

May 14, 2019

Hyperbolic Conservation Laws

We consider hyperbolic conservation laws in two and three spatial dimentions

$$\frac{\partial}{\partial t}\mathbf{u} + \nabla \cdot F(\mathbf{u}) = 0,$$

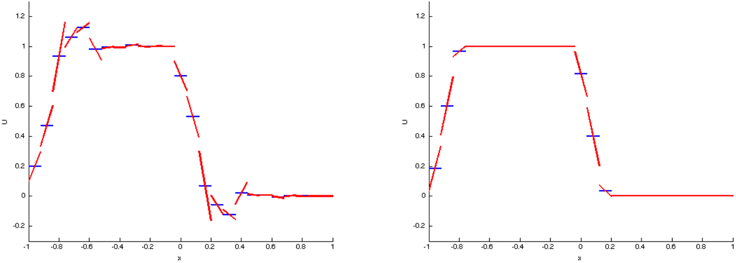
where

$\mathbf{u} = (u_1, u_2, ..., u_M)^\mathsf{T}$	conserved variables
$\mathbf{F} = (\mathbf{F}_1, \mathbf{F}_2)$	problem dependent flux function

Examples: Euler, MHD, Maxwell,...

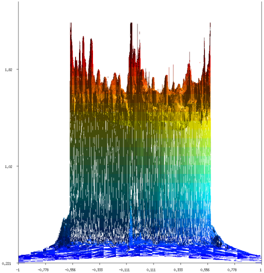
Oscillatory solutions

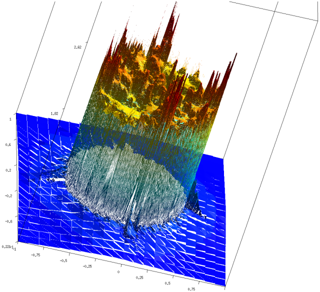
- Solutions of nonlinear hyperbolic conservation laws can develop discontinuities
- Near discontinuities, numerical solution produces oscillations and may become unstable
- Oscillations are characterized by steep slopes that we reduce, i.e. limit



Propagation of a pulse. Left: no limiting, Right: with a limiter.
Blue: averages, Red: linear solutions

Oscillatory solution to Noh problem





- The exact solution is constant in the center of the domain, the numerical solution is highly oscillatory due to poor limiting.
  Left: side view, Right: tilted view.

Limiters in 1D

- In one dimension, we have a fully developed theory for stability and second order accuracy using total variation

$$TV(U) = \sum_i |U_i - U_{i-1}|$$

- Limiters enforce total variation diminishing property (TVD)

$$TV(U^{n+1}) \leq TV(U^n)$$

- The standard DG limiter for

$$U_i = c_{i,0} + c_{i,1}\xi$$

  compares the solution slope $c_{i,1}$ to the backward $c_{i,0} - c_{i-1,0}$ and forward $c_{i+1,0} - c_{i,0}$ differences in averages

$$\tilde{c}_{i,1} = minmod(c_{i,1}, c_{i,0} - c_{i-1,0}, c_{i+1,0} - c_{i,0})$$

Limiters in 2D and 3D

- A lot of limiters have been proposed
- Many are too complicated to be practical
- Most are ad-hoc

Goal: limiters for DG that are

- easy to implement
- fast to execute
- with some theoretical backing: conditions on second order accuracy and stability
- robust

Total variation in 2D

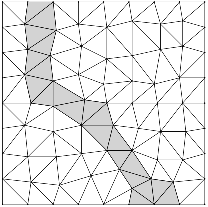
- Total variation in 1D

$$TV(U) = \sum_i |\bar{U}_i - \bar{U}_{i-1}|$$

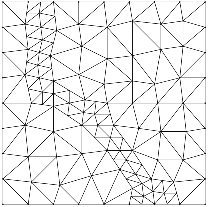
- Total variation on Cartesian grids, where TVD schemes are at most 1st order accurate

$$TV(U) = \sum_i |\bar{U}_{i,k} - \bar{U}_{i-1,k}| + \sum_k |\bar{U}_{i,k} - \bar{U}_{i,k-1}|$$

- Total variation on triangular grids: difficult to define



Original mesh



Refined mesh.

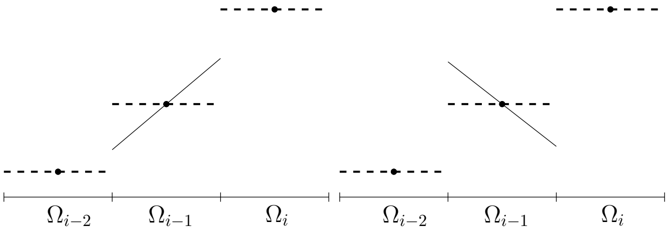
Local Maximum Principle (LMP)

The numerical solution satisfies the local maximum principle (LMP) if

$$U_{min} = \min_{j \in N_i} \overline{U}_j^n \leq \overline{U}_i^{n+1} \leq \max_{j \in N_i} \overline{U}_j^n = U_{max}$$

where N_i is a set containing the indices of elements in the neighborhood of Ω_i and $\overline{U}_i^n$ is the cell average.

- LMP guarantees that the solution will not grow in amplitude
- LMP (below, right) is weaker than TVD (below, left)



Scalar limiters

- Numerical solution on Ω_i in terms of orthonormal basis functions φ_j and degrees of freedom $c_{i,j}$

$$\tilde{U}_i = c_{i,0} + c_{i,1}\varphi_1 + c_{i,2}\varphi_2$$

- Simple idea: require values of U_i at some limit points (ξ_l, η_l) to be within a range defined by averages its neighbors

$$U_{min} \leq U_i(\xi_l, \eta_l) \leq U_{max}$$

where $U_{min} = \min_j \bar{U}_j$ and $U_{max} = \max_j \bar{U}_j$.

- If the condition is failed, scale the slope of U_i by $0 \leq \alpha \leq 1$

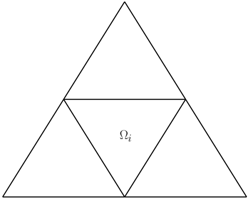
$$\tilde{U}_i = c_{i,0} + \alpha(c_{i,1}\varphi_1 + c_{i,2}\varphi_2)$$

until it is satisfied.

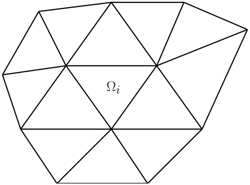
Limiting neighborhoods

Who are the neighbors?

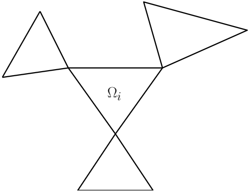
- Elements sharing edges with Ω_i (edge neighborhood)
- Elements sharing vertices with Ω_i (vertex neighborhood)
- Other choices



Edge-neighborhood N_i^e: Ω_i and all elements that share an edge with Ω_i.



Vertex-neighborhood N_i^v: Ω_i and all elements that share a vertex with Ω_i.

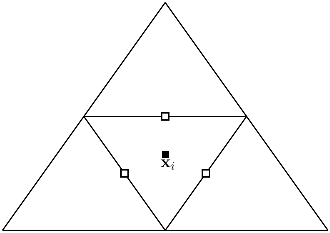


Reduced neighborhood N_i^r: Ω_i and three vertex neighbors.

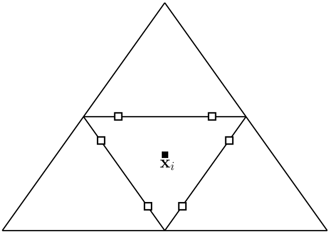
Limiting points

Would should limiting points be?

- Edge midpoints (as in FVM)
- Quadrature points (as input values in propagation of U_i)
- Vertices (as local min and max values)



Edge midpoints rule.

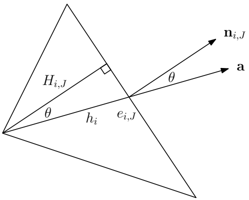


Two-point Gauss-Legendre quadrature rule.

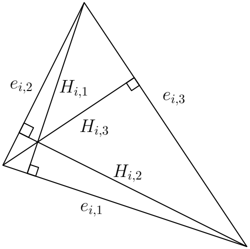
Local maximum principle

Result: Solution will satisfy local maximum principle when we limit at quadrature points (or closer to vertices than quadrature points) and

$$\Delta t \le \frac{1}{6} \min_i \frac{h_i}{||\mathbf{a}||}$$



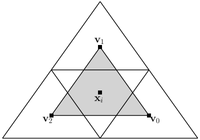
Cell size in the direction of flow h_i.



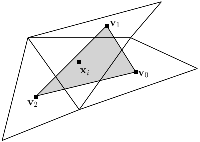
Cell size for systems $h_i = \min(H_{i,1}, H_{i,2}, H_{i,3})$.

- A. Giuliani and L. Krivodonova. Analysis of slope limiters on unstructured triangular meshes.JCP'18

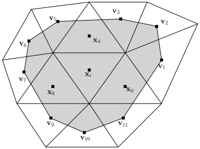
Result: Solution will be 2nd order accurate when limit points belong to the admissibility region, which is the convex hull of centroids of elements involved in limiting (shaded regions)



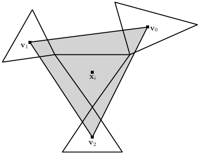
Edge-neighborhood, equilateral triangles.



Edge-neighborhood, deformed triangles.

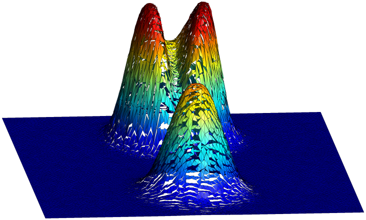


Vertex-neighborhood.

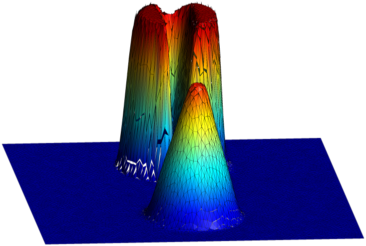


Reduced-neighborhood.

Rotating slotted cylinder and cone



Two-point limiting, edge neighborhood.



Two-point limiting, vertex neighborhood.

Moment Limiter / Directional Derivatives

Limit each linear coefficient separately

$$\tilde{U}_i = c_{i,0} + l_1 c_{i,1}\varphi_1 + l_2 c_{i,2}\varphi_2$$

Consider the directional derivative of $U_i^n(\mathbf{r})$ in the direction of $\mathbf{w}$, in the canonical coordinate system $\mathbf{r} = (r, s)$

$$D_{\mathbf{w}} U_i^n(\mathbf{r}) = \nabla_{rs} U_i \cdot \mathbf{w} = \left(c_{i,1}^n \nabla_{rs}\varphi_1 + c_{i,2}^n \nabla_{rs}\varphi_2\right) \cdot \mathbf{w}$$

or

$$D_{\mathbf{w}} U_i^n(\mathbf{r}) = \left(c_{i,1}^n \left(6\ ,\ 0\right) + c_{i,2}^n \left(2\sqrt{3}\ ,\ 4\sqrt{3}\right)\right) \cdot \mathbf{w}$$

In the directions $\mathbf{w}_1 = \frac{2}{\sqrt{5}}\left(1, -\frac{1}{2}\right)$ and $\mathbf{w}_2 = (0, 1)$, the directional derivatives decouple $c_{i,1}^n$ and $c_{i,2}^n$

$$D_{\mathbf{w}_1} U_i^n = 6\left(\frac{2}{\sqrt{5}}\right) c_{i,1}^n \quad \text{and} \quad D_{\mathbf{w}_2} U_i^n = 4\sqrt{3} c_{i,2}^n$$

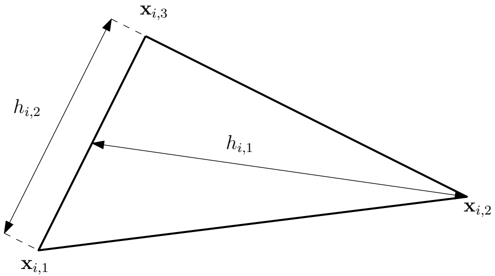
and we can try to use the ideas of 1D limiting along lines $\mathbf{w}_1$ and $\mathbf{w}_2$.

Directional Derivatives

We map $\mathbf{w}_1$ and $\mathbf{w}_2$ from the canonical to the physical space: $\mathbf{w}_1 \to \mathbf{v}_{i,1}$ and $\mathbf{w}_2 \to \mathbf{v}_{i,2}$ and solve for $c_{i,1}$ and $c_{i,2}$.

$$c_{i,1} = \frac{h_{i,1}}{6} D_{\mathbf{v}_{i,1}} U_i \quad \text{and} \quad c_{i,2} = \frac{h_{i,2}}{4\sqrt{3}} D_{\mathbf{v}_{i,2}} U_i$$

$c_{i,1}$ is a scaled derivative in direction $\mathbf{v}_{i,1}$ and $c_{i,2}$ is a scaled derivative in direction $\mathbf{v}_{i,2}$



Directional limiting

- We limit the solution in the directions $\mathbf{v}_{i,1}$ and $\mathbf{v}_{i,2}$ by comparing it to forward and backward differences using reconstructed solution values.
- Unlimited coefficients:

$$c_{i,1} = \frac{h_{i,1}}{6} D_{\mathbf{v}_{i,1}} U_i \qquad \text{and} \qquad c_{i,2} = \frac{h_{i,2}}{4\sqrt{3}} D_{\mathbf{v}_{i,2}} U_i$$

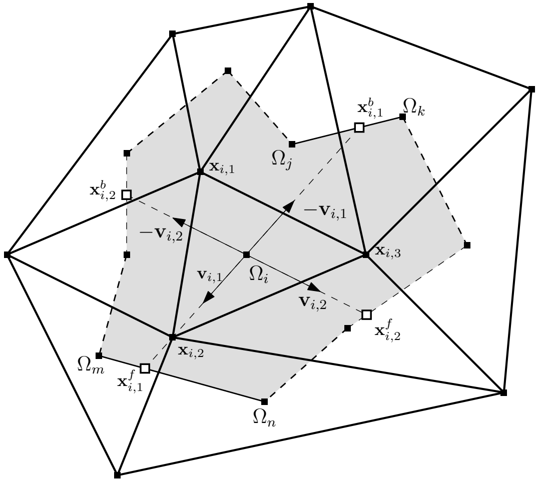
- Limited coefficients

$$\tilde{c}_{i,1} = l^f_{i,1} \frac{h_{i,1}}{6} \frac{U^f_{i,1} - \overline{U}_i}{d^f_{i,1}} \qquad \text{or} \qquad \tilde{c}_{i,1} = l^b_{i,1} \frac{h_{i,1}}{6} \frac{\overline{U}_i - U^b_{i,1}}{d^b_{i,1}},$$

$$\tilde{c}_{i,2} = l^f_{i,2} \frac{h_{i,2}}{4\sqrt{3}} \frac{U^f_{i,2} - \overline{U}_i}{d^f_{i,2}} \qquad \text{or} \qquad \tilde{c}_{i,2} = l^b_{i,2} \frac{h_{i,2}}{4\sqrt{3}} \frac{\overline{U}_i - U^b_{i,2}}{d^b_{i,2}},$$

where $d^f_{i,1}$ is the distance from $\mathbf{x}^f_{i,1}$ to $\mathbf{x}_i$, etc.

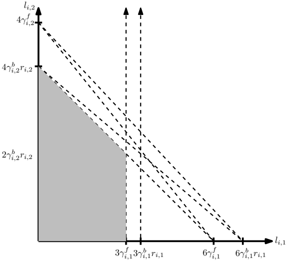
Reconstruction directions, points and values



$$U^f_{i,1} = \beta^f_{i,1}\bar{U}_m + (1 - \beta^f_{i,1})\bar{U}_n \qquad U^b_{i,1} = \beta^b_{i,1}\bar{U}_j + (1 - \beta^b_{i,1})\bar{U}_k$$

Admissible choices for limiting coefficients $l_{i,1}, l_{i,2}$

- Derive a set of inequalities $l_{i,1}$ and $l_{i,2}$ should satisfy for solution to satisfy LMP and be 2nd order accurate (dashed lines correspond to the equal sign)
- Any point in the shaded will result in a valid limiter
- Points close to the boundaries give less diffusive solutions
- Choose the simplest among less restrictive limiters



- A. Giuliani and L. Krivodonova. A moment limiter for the discontinuous Galerkin method on unstructured triangular meshes. SISC'19

Moment Limiter in 2D

$$\tilde{c}_{i,1} = \text{minmod}\left(\frac{U^f_{i,1} - \overline{U}^n_i}{2}, c_{i,1}, \frac{\overline{U}^n_i - U^b_{i,1}}{2}\right)$$

$$\tilde{c}_{i,2} = \text{minmod}\left(\frac{U^f_{i,2} - \overline{U}^n_i}{2\sqrt{3}}, c_{i,2}, \frac{\overline{U}^n_i - U^b_{i,2}}{2\sqrt{3}}\right)$$

$$U^f_{i,1} = \beta^b_{i,1}\overline{U}^n_j + (1 - \beta^b_{i,1})\overline{U}_k$$

$$U^b_{i,1} = \beta^f_{i,1}\overline{U}^n_m + (1 - \beta^f_{i,1})\overline{U}^n_n$$

$$U^f_{i,2} = \beta^b_{i,2}\overline{U}^n_s + (1 - \beta^b_{i,2})\overline{U}^n_t$$

$$U^b_{i,2} = \beta^f_{i,2}\overline{U}^n_u + (1 - \beta^f_{i,2})\overline{U}^n_w$$

We store for each element Ω_i

- Pointers to the eight elements involved in the stencil: $c_{j,0}, c_{k,0}, c_{m,0}, c_{n,0}, c_{s,0}, c_{t,0}, c_{u,0}, c_{w,0}$
- four coefficients $\beta^f_{i,1}, \beta^b_{i,1}, \beta^f_{i,2}, \beta^b_{i,2}$

Moment Limiter in 3D

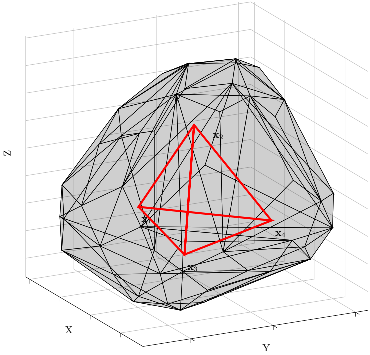
$$\tilde{c}_{i,1} = \text{minmod}\left(\frac{U^f_{i,1} - \overline{U}^n_i}{\sqrt{10}}, c_{i,1}, \frac{\overline{U}^n_i - U^b_{i,1}}{\sqrt{10}}\right)$$
$$\tilde{c}_{i,2} = \text{minmod}\left(\frac{U^f_{i,2} - \overline{U}^n_i}{2\sqrt{5}}, c_{i,2}, \frac{\overline{U}^n_i - U^b_{i,2}}{2\sqrt{5}}\right)$$
$$\tilde{c}_{i,3} = \text{minmod}\left(\frac{U^f_{i,3} - \overline{U}^n_i}{2\sqrt{15}}, c_{i,3}, \frac{\overline{U}^n_i - U^b_{i,3}}{2\sqrt{15}}\right)$$

We store for each element Ω_i

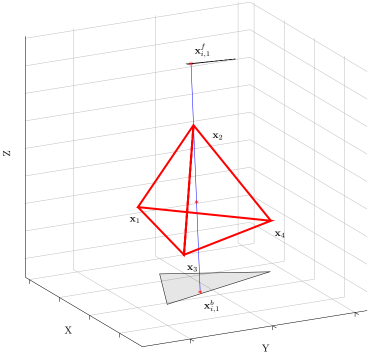
- Pointers to the 18 elements involved in the stencil
- 18 weights

Moment limiter in 3D

Same idea as in 2D: Find three directions in which directional derivatives of U uncouple the solution coefficients

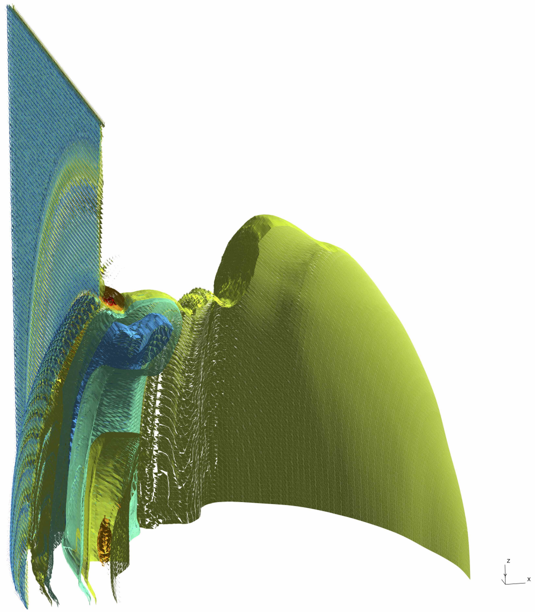


Triangulation of the convex hull of neighboring centroids.



Interpolation planes for $\mathbf{v}_{i,1}$.

Mach 10 shock interracting with a bubble

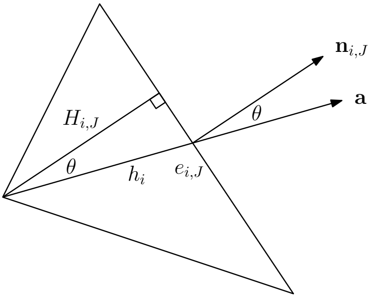


Time step restriction in 2D

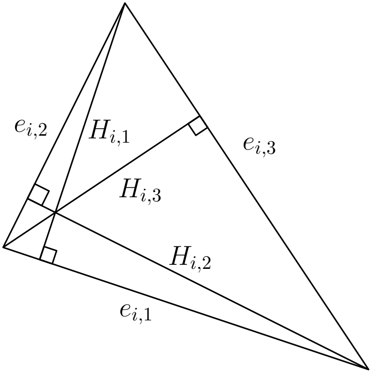
Bounds of solution are guaranteed under the following CFL condition:

$$\Delta t \leq \frac{1}{4} \min_i \frac{h_i}{||\mathbf{a}||}$$

where h_i is the width in the direction of the flow.



Cell size in the direction of flow h_i.



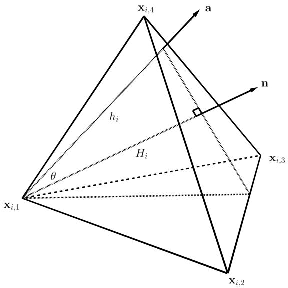
Cell size for systems $h_i = \min(H_{i,1}, H_{i,2}, H_{i,3})$.

Time step restrictions in 3D

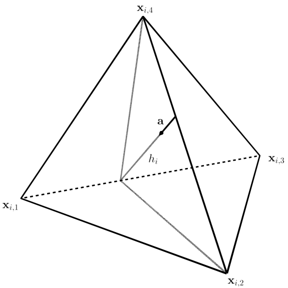
Bounds of solution are guaranteed under the following CFL condition:

$$\Delta t \leq \frac{1}{6} \frac{h_i}{||\mathbf{a}||}$$

where h_i is the width in the direction of the flow.



Three inflow faces and one outflow face.



Two inflow faces and two outflow faces.

Verification of CFL number in 3D

We solve the linear advection equation with

$$u_0(x, y, z) = \begin{cases} 1 \text{ if } \sqrt{x^2 + y^2 + z^2} \leq \frac{1}{4} \\ 0 \text{ otherwise,} \end{cases}$$

1/CFL	Minimum	Maximum
3	-0.8727	2.0377
4	-0.3134	1.3169
5.5	-0.001642	1.0004169
6	**0**	**1**

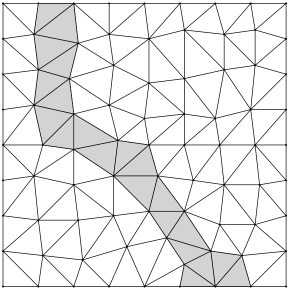
(a) Forward Euler.

1/CFL	Minimum	Maximum
3	-0.01077	1.0122
4	-8.271e-7	1
5.5	0	0.9999
6	**0**	**0.9999**

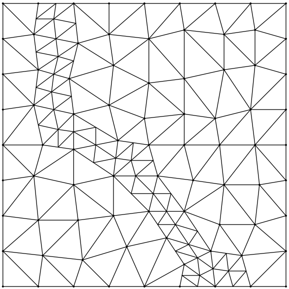
(b) RK2.

Table 1: Minimum and maximum cell averages for various CFL numbers.

Adaptive refinement and limiting on nonconforming meshes



Shaded triangles are flagged for refinement.

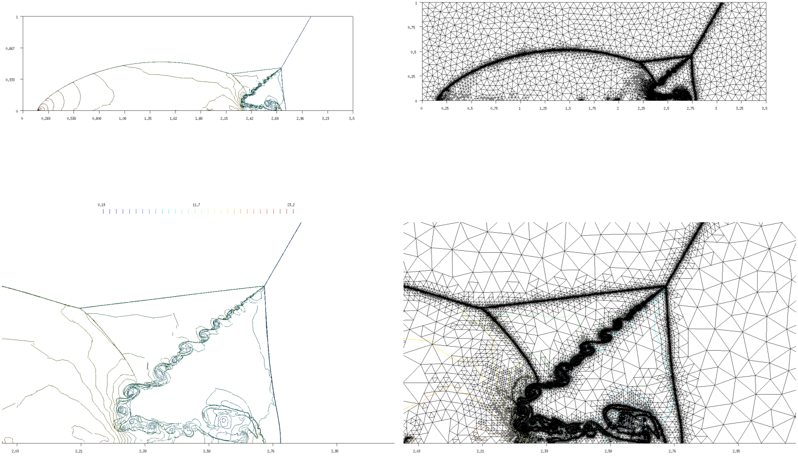


Cell-based refinement.

Cell-based refinement on an unstructured mesh of triangles.

- Extend both limiters to noncomforming meshes where neighboring elements might be of different sizes.

Double Mach reflection with 7 levels of refinement, t=0.2



Runtime comparison moment vs scalar limiters

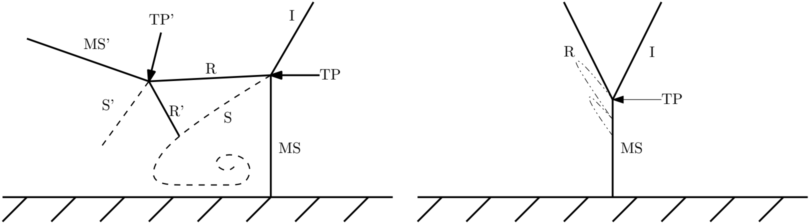
	moment	scalar
Total run time	1453.59s	1963.3s
Limiting time	51.58s	273.13s
Setting up neighborhood from vertex database	165.8s	81.5s
Average number of elements in the domain	140780.98	160602.9
Number of refinement cycles	24495	30647

Refining every 2 time steps.

	Moment	scalar
Total run time	1688.56s	1984.4s
Limiting time	68.98s	311.5s
Setting up neighborhood from vertex database	46.92s	18.94s
Average number of elements in the domain	189347.3	190121.87
Number of refinement cycles	5792	5970

Refining every 10 time steps.

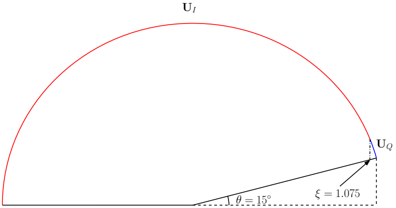
- Moment limiter: find new points and weights. More expensive to set up but cheaper to run.
- Scalar limiter: use immediate neighbors (avoid projections). Easier to set up but more expensive to run.



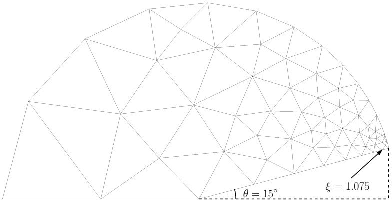
Double Mach reflection. Guderley Mach reflection.

Figure 1: Double Mach and Guderley reflections. The incident (I), primary and secondary reflected shocks (R, R'), Mach stems (MS, MS') and sliplines (S, S') are indicated. The sonic line in the Guderley Mach reflection case is indicated by the dashed-dotted line.

Domain and initial mesh for self-similar computations

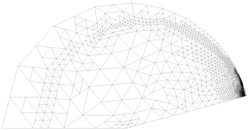


Computational domain. The incident $\mathbf{U}_I$ and quiescent $\mathbf{U}_Q$ states are imposed on the red and blue boundaries, respectively. The bottom boundary is reflecting.

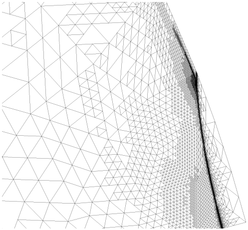


Initial, conforming mesh.

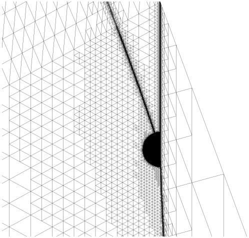
Final mesh of about 6.2 million cells



Adaptively refined mesh.

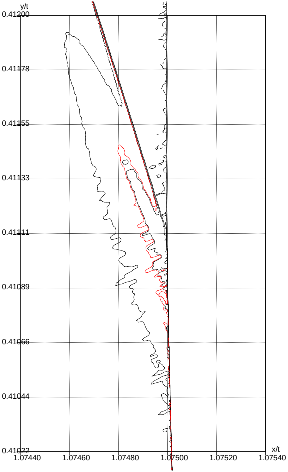


First zoom of Mach stem and triple point region.

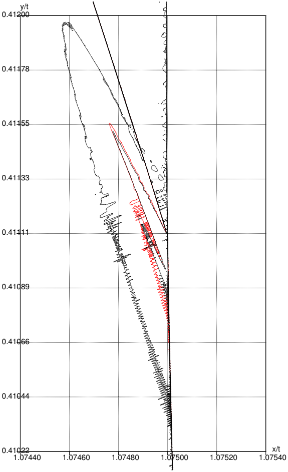


Second zoom of Mach stem and triple point region.

Zoom of solution near triple point



$h \approx 4.3 \cdot 10^{-6}$



$h \approx 5.8 \cdot 10^{-7}$

- A. Giuliani and L. Krivodonova. Adaptive mesh refinement on graphics processing units for applications in gas dynamics.JCP'19

Conclusion

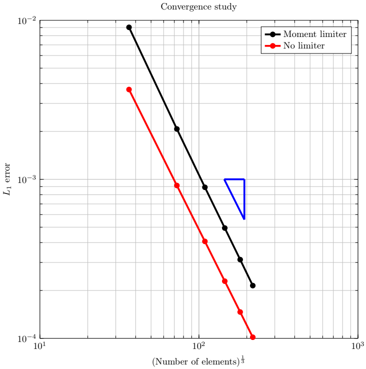
- We introduced a new approach to second order limiting for DG in 2D and 3D and analyzed old limiters
- We can prove 2nd order accuracy and tight bounds on the solution
  - Limiter acts like a TVD-type limiter in two (in 2D) or 3 (in 3D) separate directions
  - Bounds are proven using local maximum principle
  - Conditions for 2nd order accuracy derived
- Limiter is easy to implement and very efficient
  - Stencil is small (for 2D and 3D problems)
  - Stencil is fixed and does not depend on unstructured mesh configuration
  - Stencil and weights are computed as a preprosessing step
- Limiters (scalar and moment) extended to non-conforming meshes

Accuracy

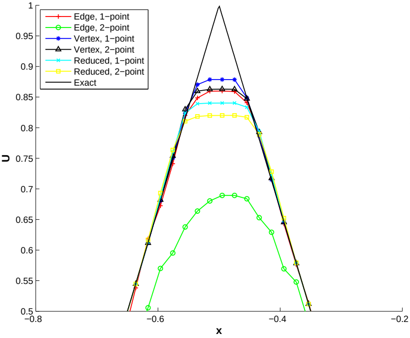
We solve the linear advection equation with the flux $\mathbf{F}(u) = [u, u, u]$ and the initial condition

$$u_0(x, y, z) = \begin{cases} \cos^2\left(\frac{\pi}{2} r\right) & \text{if } r \leq 1, \\ 0 \text{ elsewhere.} \end{cases}$$

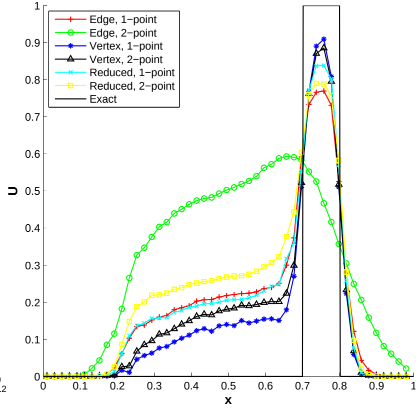
where $r = 2\sqrt{x^2 + y^2 + z^2}$ until the final time $T = 0.25$.



Rotating slotted cylinder and cone



(h) Profile at the cone.



(i) Profile at the slotted cylinder.

Full set of limiting inequalities on outflow edges

One condition imposed by Ω_i and one by its edge neighbor:

$$\frac{l_{i,1}}{6\gamma^f_{i,1}} + \frac{l_{i,2}}{4\gamma^b_{i,2} r_{i,2}} \le 1 \quad \text{and} \quad \frac{l_{i,1}}{6\gamma^b_{i,1} r_{i,1}} + \frac{l_{i,2}}{4\gamma^f_{i,2}} \le 1 \quad \text{if} \quad k = 1$$

$$\frac{l_{i,1}}{6\gamma^f_{i,1}} + \frac{l_{i,2}}{4\gamma^f_{i,2}} \le 1 \quad \text{and} \quad \frac{l_{i,1}}{6\gamma^b_{i,1} r_{i,1}} + \frac{l_{i,2}}{4\gamma^b_{i,1} r_{i,2}} \le 1 \quad \text{if} \quad k = 2$$

$$\frac{l_{i,1}}{3\gamma^b_{i,1} r_{i,1}} \le 1 \quad \text{and} \quad \frac{l_{i,1}}{3\gamma^f_{i,1}} \le 1 \quad \text{if} \quad k = 3$$