A Stable Layer Stripping Algorithm for Electrical Impedance Tomography

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EIT inverse problem

- Let $\Omega \subset \mathbb{R}^n$, $\sigma \in C^1(\Omega) \cap C(\overline{\Omega})$, and let $u \in H^2(\Omega)$ be the solution of

$$\begin{aligned} \nabla \cdot (\sigma \nabla u) &= 0 \text{ in } \Omega, \\ \sigma \frac{\partial u}{\partial n}\Big|_{\partial\Omega} &= f \in H_0^{1/2}(\partial\Omega), \end{aligned}$$

where

$$H_0^{1/2}(\partial\Omega) = \{f \in H^{1/2}(\partial\Omega) \mid \int_{\partial\Omega} f dS = 0\}.$$

- Neumann-to-Dirichlet map,

$$W = W[\sigma] : H^{1/2}(\partial\Omega) \to H^{3/2}(\partial\Omega), \quad \sigma \frac{\partial u}{\partial n}\Big|_{\partial\Omega} \mapsto u\big|_{\partial\Omega}$$

- **Calderón problem:** From the knowledge of W_σ, reconstruct σ

Layer Stripping and Invariant Imbedding

Background: One-dimensional (zero energy) Schrödinger equation:

$$-\psi''(x) + V(x)\psi(x) = 0.$$

Define the *wave impedance* $\eta(x)$,

$$\eta(x) = \frac{\psi'(x)}{\psi(x)}.$$

Differentiate:

$$\eta'(x) = \frac{\psi''(x)}{\psi(x)} - \left(\frac{\psi'(x)}{\psi(x)}\right)^2 = V(x) - \eta(x)^2.$$

Layer Stripping and Invariant Imbedding

- The wave impedance satisfies a Riccati equation

$$\eta'(x) = V(x) - \eta(x)^2.$$

- The wave impedance represents the Dirichlet-to-Neumann map:

$$\psi'\big|_{x=a} = \eta(a)\psi\big|_{x=a}.$$

- Moving the boundary $\{x = a\}$ is tantamount to solving the Riccati equation.
- The Neumann-to-Dirichlet map $\nu = 1/\eta$ (wave admittance) satisfies also a Riccati equation:

$$\nu'(x) = 1 - V(x)\nu(x)^2.$$

Layer Stripping and Invariant Imbedding

Extension to EIT: Let

$$D = \{(r,\theta) \mid 0 \leq r < 1\} \subset \mathbb{R}^2,$$

and u satisfies

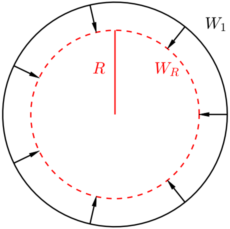
$$\begin{aligned} \nabla \cdot (\sigma \nabla u) &= 0 \text{ in } D, \\ \sigma \frac{\partial u}{\partial r}\bigg|_{r=1} &= f. \end{aligned}$$

Neumann-to-Dirichlet map

$$W_1 : H^s(\partial D) \to H^{s+1}(\partial D), \quad f \mapsto u\big|_{r=1},$$

where $s \geq -1/2$.

Layer Stripping and Invariant Imbedding



1. Define a family W_R of NtD operators over circles of radius R
2. Propagate the boundary data W_1
3. Evaluate the conductivity while marching in

Layer Stripping and Invariant Imbedding

1. Extension of the boundary data: Define

$$u_R(t,\theta) = u(Rt,\theta), \quad \sigma_R(t,\theta) = \sigma(Rt,\theta), \quad 0 < R \leq 1,$$

satisfying

$$\nabla_\xi \cdot (\sigma_R \nabla_\xi u_R) = 0 \text{ in } D,$$

where $\xi = (t\cos\theta, t\sin\theta)$.

$$\boxed{W_R : H^{1/2}(\partial D) \to H^{3/2}(\partial D), \quad W_R \left[\sigma_R \frac{\partial u_R}{\partial t}\right]_{t=1} = u_R(1).}$$

Layer Stripping and Invariant Imbedding

2. Propagation: Define

$$U = U(R,\theta) = \begin{bmatrix} v \\ w \end{bmatrix} \in H^{3/2}(\partial D) \times H^{1/2}(\partial D),$$

where

$$v = [u_R]_{t=1}, \quad w = \left[\sigma_R \frac{\partial u_R}{\partial t}\right]_{t=1}.$$

Differentiate with respect to R:

$$\dot{v} = \frac{\partial v}{\partial R} = \frac{1}{R\sigma(R,\theta)} \left[\sigma(Rt,\theta)\frac{\partial}{\partial t}u(tR,\theta)\right]_{t=1} = \frac{1}{R\sigma(R,\theta)} w,$$

or

$$\boxed{\dot{v} = \frac{1}{R} G w, \quad G : H^{1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega).}$$

Layer Stripping and Invariant Imbedding

$$\begin{aligned}\dot{w} &= \frac{\partial w}{\partial R} = \frac{\partial}{\partial R}\left[\sigma(R,\theta)R\frac{\partial u}{\partial R}(R,\theta)\right]\dot{w} \\ &= -\frac{1}{R}\frac{\partial}{\partial \theta}\left[\sigma(R,\theta)\frac{\partial}{\partial \theta}u(R,\theta)\right] \\ &= -\frac{1}{R}\frac{\partial}{\partial \theta}\left[\sigma(R,\theta)\frac{\partial}{\partial \theta}v\right],\end{aligned}$$

or

$$\boxed{\dot{w} = -\frac{1}{R}Sv, \quad S : H^{3/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega).}$$

Hence,

$$\boxed{R\dot{U} = \begin{bmatrix} 0 & G \\ -S & 0 \end{bmatrix} U.}$$

Layer Stripping and Invariant Imbedding

Differentiate

$$v = W_R w,$$

giving

$$\begin{aligned} \dot{W}_R w &= \dot{v} - W_R \dot{w} = \frac{1}{R}(Gw + W_R S v) \\ &= \frac{1}{R}(G + W_R S W_R) w \end{aligned}$$

Riccati equation:

$$\boxed{R\dot{W}_R = G + W_R S W_R.}$$

$$H_0^{1/2}(\partial D) \xrightarrow{W_R} H_0^{3/2}(\partial D) \xrightarrow{S_R} H_0^{-1/2}(\partial D) \xrightarrow{W_R} H_0^{1/2}(\partial D),$$

Layer Stripping and Invariant Imbedding

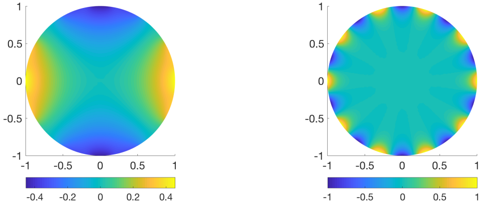
3. Reconstruction: High frequency asymptotics:

$$(W_R)_{jk} = \langle e^{ij\theta}, W_R e^{ik\theta} \rangle,$$

we have

$$\boxed{\lim_{|k|\to\infty} |k| (W_R)_{n+k,k} = \frac{1}{2\pi R} \int e^{in\theta} \frac{1}{\sigma(R,\theta)} d\theta = \widehat{\rho}_n,}$$

where $\rho = 1/\sigma$ is the resistivity.



Layer Stripping

Layer stripping algorithm:

- Fix radii $1 = r_0 > r_1 > \ldots > r_J > 0$. Denote

$$A_j = \{(r,\theta) \mid r_j < r < r_{j-1}\}, \quad 1 \leq j \leq J.$$

- Set $j = 0$
- For $j = 1 : J$ repeat
  - (a) Estimate $\sigma\big|_{A_j}$ from the high frequency components of $W_{r_{j-1}}$;
  - (b) Propagate $W_{r_{j-1}} \to W_{r_j}$ using the Riccati equation.

Somersalo E, Isaacson D, Cheney M and Isaacson E (1991) Layer stripping: A direct numerical method for impedance imaging. Inverse Problems **7** 899–926.
Cheney M, Isaacson D, Somersalo E and Isaacson E (1995) Layer stripping process for impedance imaging. U.S. Patent no. 5 390 110, February 14, 1995.

Layer Stripping

Ill-posedness: The ill-posed nature of the EIT inverse problem shows up as instability of the (backwards) Riccati equation.

Special case: Radial conductivity $\sigma = \sigma(r)$: The Fourier modes of W_R decouple. With $\sigma = 1$, we have

$$(W_R)_{kk} = w_k(R) = \frac{1}{|k|}.$$

The backwards Riccati initial value problem:

$$R\frac{dw_k}{dR} = 1 - k^2 w_k^2, \quad w_k(1) = b = \mathrm{data}.$$

Layer Stripping

Explicit solution:

$$w_k(R) = \frac{1}{|k|}\frac{R^{2|k|} - M(|k|b)}{R^{2|k|} + M(|k|b)}, \quad M(|k|b) = \frac{1-|k|b}{1+|k|b}.$$

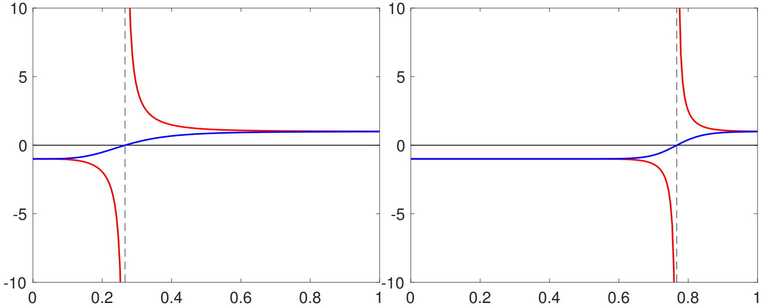
- Exact solution is recovered if the data are noiseless, that is, $b = 1/|k|$, implying that $M(|k|b) = 0$.
- Noisy data with relative error $\varepsilon \neq 0$,

$$b = \frac{1}{|k|}(1 \pm \varepsilon), \quad |\varepsilon| < 1.$$

The solution either becomes singular ($\varepsilon > 0$) or negative ($\varepsilon < 0$) at a radial value

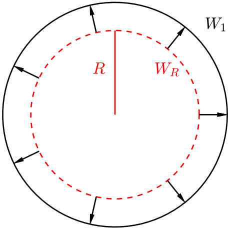
$$R \propto \frac{1}{|\varepsilon|^{1/2|k|}} \to 1, \text{ as } |k| \to \infty.$$

Layer Stripping



Solutions $|k|w_k(R)$ of the backwards Riccati equation, with positive (red) and negative (blue) error. $|\varepsilon| = 0.01$. On the left, $k = 2$, and on the right, $k = 10$. The correct solution is $|k|w_k(R) = 1$.

Reformulation



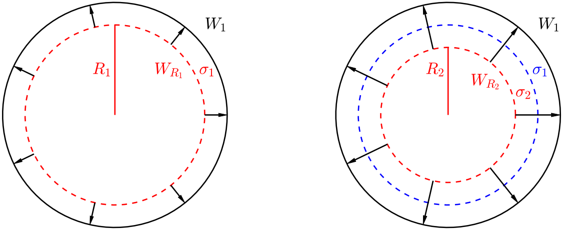
Forward problem in an annular domain: Given an $R < 1$, denote
$A_R = \{(r,\theta) \mid R < r < 1\}$.
Consider the mapping

$$\Psi_R : (W_R, \sigma\big|_{A_R}) \mapsto W_1.$$

The forward problem is well-posed, as the forward Riccati propagation is stable.

Inverse problem: Given a noisy observation of W_1, estimate $(W_R, \sigma\big|_{A_R})$.

Sequential estimation



- Sequence of radii $1 > R_1 > R_2 > \dots$
- Using W_1, estimate (W_{R_1}, σ_1), where $\sigma_1 = \sigma\big|_{A_{R_1}}$
- Using W_1 and the estimate for σ_1, estimate (W_{R_2}, σ_2), where $\sigma_2 = \sigma\big|_{A_{R_2}}$
- $\dots$

Discretization

- Radii

$$1 = R_0 > R_1 > \ldots > R_J > 0,$$

defining rings

$$A_j = \{(r,\theta) \mid R_j < r < R_{j-1}\}, \quad 1 \leq j \leq J.$$

- Approximation:

$$\sigma\big|_{A_j}(r,\theta) = \sigma_j(\theta), \quad R_j < r < R_{j-1}.$$

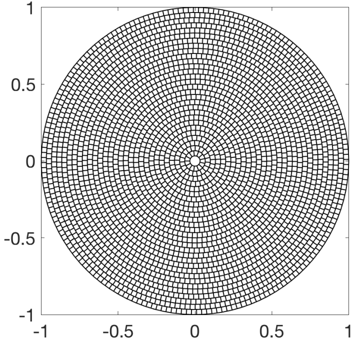
- Logarithmic parametrization of each σ_j:

$$(\lambda_j)_\ell = \log\frac{\sigma_j(\theta_\ell)}{\sigma_0}, \quad 1 \leq \ell \leq n_j, \ \lambda_{J+1} = \log\frac{\sigma(0)}{\sigma_0},$$

where

$$n_j = \left\lfloor \frac{2\pi r_j}{h} \right\rfloor.$$

Discretization



Discretization

- Parametrization of the conductivity in $R_j \leq r \leq 1$:

$$\lambda_{(j)} = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_j \end{bmatrix} \in \mathbb{R}^{N_j}, \quad N_j = n_1 + \ldots + n_j,$$

  where $1 \leq j \leq J$.
- Interior boundary value:

$$w_j = \mathrm{vec}(\mathsf{W}_{R_j}) \quad \text{(stack the columns in a vector)}$$

- Exterior boundary value: Numerical Riccati solver,

$$\psi_j(w_j, \lambda_{(j)}) = w_0.$$

Sequential estimation

State vectors, evolution model

$$x_0 \to x_1 \to x_2 \to \dots$$

Indirect observations,

$$b_j = F(x_j) + \varepsilon_j.$$

1. Given the probability distribution $\pi_k(x_k)$, propagate (push forward)

$$\pi_k(x_k) \to \widetilde{\pi}_{k+1}(x_{k+1})$$

2. Using $\widetilde{\pi}_{k+1}$ as prior, use Bayes' formula to update

$$\pi_{k+1}(x_{k+1}) \propto \widetilde{\pi}_{k+1}(x_{k+1})\pi(b_{k+1} \mid x_{k+1})$$

3. Repeat

Sequential estimation

Define the *state vector*

$$x_k = \begin{bmatrix} w_0 \\ w_k \\ \lambda_{(k)} \end{bmatrix}.$$

For Bayesian filtering algorithm, we need

1. **State evolution model:** A stochastic model

$$x_k \to x_{k+1}.$$

2. **Observation model:** A stochastic model

$$x_k \to b_k.$$

Sequential estimation

1. State evolution model:

$$x_k = \begin{bmatrix} w_0 \\ w_k \\ \lambda_{(k)} \end{bmatrix} \xrightarrow{(a)} \begin{bmatrix} w_{k+1} \\ \lambda_{(k+1)} \end{bmatrix} \xrightarrow{(b)} \begin{bmatrix} w_0 \\ w_{k+1} \\ \lambda_{(k+1)} \end{bmatrix} = x_{k+1}.$$

(a) Given the current $\lambda_{(k)}$, draw λ_{k+1} from the conditional prior distribution $\pi_{\rm pr}(\lambda_{k+1} \mid \lambda_{(k)})$ (smoothness prior for σ),
Draw w_{k+1} from the prior distribution $\pi'_{\rm pr}(w_{k+1})$.

(b) Propagate w_{k+1} through the $k+1$ layers using the first order Möbius propagator.
Add innovation with variance estimated from the second order Möbius propagator (numerical modeling error).

Sequential estimation

2. Observation model:

$$b_k = \mathsf{P}x_k + \varepsilon_k,$$

where

$$\mathsf{P} = \begin{bmatrix} \mathsf{I} & \mathsf{O} & \mathsf{O} \end{bmatrix} : \begin{bmatrix} w_0 \\ w_k \\ \lambda_{(k)} \end{bmatrix} \mapsto w_0.$$

Gaussian observation error:

$$\varepsilon_k \sim \mathcal{N}(0, \Sigma).$$

Ensemble Kalman Filtering (EnKF)

1. **Propagation step:** Given a sample
$$\{x_k^{(1)}, x_k^{(2)}, \ldots, x_k^{(N)}\}$$
from the current posterior, generate a predictive sample using the propagation model,
$$\{\widehat{x}_{k+1}^{(1)}, \widehat{x}_{k+1}^{(2)}, \ldots, \widehat{x}_{k+1}^{(N)}\}$$
2. Calculate the empirical mean $\overline{x}_{k+1}$ and covariance G_{k+1}.
3. Given the observation b_{k+1}, generate a data ensemble
$$\{b_{k+1}^{(1)}, b_{k+1}^{(2)}, \ldots, b_{k+1}^{(N)}\}, \quad b_{k+1}^{(j)} = b_{k+1} + e^{(j)}, \quad e^{(j)} \sim \mathcal{N}(0, \Sigma).$$
4. **Analysis step:** Generate a sample from the posterior by setting
$$x_{k+1}^{(j)} = \mathrm{argmin}\left\{\|x - \widehat{x}_{k+1}^{(j)}\|_{\mathsf{G}_{k+1}}^2 + \|\mathsf{P}x - b_{k+1}^{(j)}\|_{\Sigma}^2\right\}.$$

Ensemble Kalman Filtering (EnKF)

Some observations:

- The forward model P is linear, and therefore the solution $x_{k+1}^{(j)}$ is obtained by a linear operation,

$$x_{k+1}^{(j)} = \widehat{x}_{k+1}^{(j)} + \mathsf{K}(b_{k+1}^{(j)} - \mathsf{P}\widehat{x}_{k+1}^{(j)}),$$

  where K is the Kalman gain matrix
- The information about $b_{k+1}^{(j)}$ is passed to the parameter $\lambda_{(k+1)}$ through the cross covariance matrix

$$\mathrm{cov}\big(w_0, \lambda_{(k+1)}\big).$$

- The algorithm, despite of nonlinearity, is **derivative-free**.

Likelihood model revisited

- The data b are formally used J times.
- A family of forward models,

$$b = \psi_j(\lambda_{(j)}, w_j) + \varepsilon_j, \quad \varepsilon_j \sim \mathcal{N}(0, \mathrm{C}).$$

- Mean likelihood model (cf. Sequential Monte Carlo),

$$\begin{aligned} \pi_{\rm lkh}(b \mid \lambda_{(J)}, w_{(J)}) \quad &\propto \quad \exp\left(-\frac{1}{2J}\sum_{j=1}^{J}\|b - \psi_j(\lambda_{(j)}, w_j)\|_{\mathrm{C}}^2\right) \\ &\propto \quad \prod_{j=1}^{J} \pi_{\rm lkh}^{j}(b \mid \lambda_{(j)}, w_j), \end{aligned}$$

where

$$\pi_{\rm lkh}^{j}(b \mid \lambda_{(j)}, w_j) \propto \exp\left(-\frac{1}{2J}\|b - \psi_j(\lambda_{(j)}, w_j)\|_{\mathrm{C}}^2\right).$$

Posterior model

Bayes' theorem: (posterior $\propto$ prior $\times$ likelihood):

$$\pi_{\mathrm{post}}(w_{(J)}, \lambda_{(J)} \mid b) \propto \pi_{\mathrm{pr}}(w_{(J)}, \lambda_{(J)}) \prod_{j=1}^{J} \pi_{\mathrm{lkh}}^{j}(b \mid \lambda_{(j)}, w_j).$$

Introduce the kth approximation of the posterior:

$$\pi_{\mathrm{post}}^{k}(w_{(k)}, \lambda_{(k)} \mid b) \propto \pi_{\mathrm{pr}}(w_{(k)}, \lambda_{(k)}) \prod_{j=1}^{k} \pi_{\mathrm{lkh}}^{j}(b \mid \lambda_{(j)}, w_j).$$

Posterior model

(a) A priori, w_{k+1} is independent of $w_{(k)}$ and $\lambda_{(k+1)}$.

(b) A priori, λ_{k+1} is independent of $w_{(k)}$ but may not be independent of $\lambda_{(k)}$.

$$\pi_{\rm pr}(w_{(k+1)}, \lambda_{(k+1)}) = \pi'(w_{k+1})\pi(\lambda_k \mid \lambda_{(k)})\pi_{\rm pr}(w_{(k)}, \lambda_{(k)}),$$

Posterior model

Recursive updating:

$$\pi_{\rm post}^{k+1}(w_{(k+1)}, \lambda_{(k+1)} \mid b) \propto \pi_{\rm pr}(w_{(k+1)}, \lambda_{(k+1)}) \prod_{j=1}^{k+1} \pi_{\rm lkh}^{j}(b \mid \lambda_{(j)}, w_j)$$

$$= \pi'_{\rm pr}(w_{k+1})\pi_{\rm pr}(\lambda_{k+1} \mid \lambda_{(k)})\pi_{\rm pr}(w_{(k)}, \lambda_{(k)}) \prod_{j=1}^{k+1} \pi_{\rm lkh}^{j}(b \mid \lambda_{(j)}, w_j)$$

$$= \pi'_{\rm pr}(w_{k+1})\pi_{\rm pr}(\lambda_{k+1} \mid \lambda_{(k)})\pi_{\rm lkh}^{j}(b \mid \lambda_{(k+1)}, w_{k+1})\pi_{\rm post}^{k}(w_{(k)}, \lambda_{(k)} \mid b)$$

$$= \underbrace{\left\{\pi'_{\rm pr}(w_{k+1})\pi_{\rm pr}(\lambda_{k+1} \mid \lambda_{(k)})\pi_{\rm post}^{k}(w_{(k)}, \lambda_{(k)} \mid b)\right\}}_{(*)} \pi_{\rm lkh}^{j}(b \mid \lambda_{(k+1)}, w_{k+1}).$$

where $(*)$ can be thought of as an updated prior for the next round.

Posterior model

Marginalize with respect to the initial values:

$$\overline{\pi}^{k}_{\rm post}(\lambda_{(k)} \mid b) = \int \pi^{k}_{\rm post}(w_{(k)}, \lambda_{(k)} \mid b) dw_{(k)}.$$

Integrating the recursive formula with respect to w_j:

$$\boxed{\begin{aligned}\overline{\pi}^{k+1}_{\rm post}(\lambda_{(k+1)} \mid b) &= \pi_{\rm pr}(\lambda_{k+1} \mid \lambda_{(k)})\overline{\pi}^{k}_{\rm post}(\lambda_{(k)} \mid b) \\ &\quad \times \left(\int \pi'_{\rm pr}(w_{k+1})\pi^{j}_{\rm lkh}(b \mid \lambda_{(k+1)}, w_{k+1}) dw_{k+1}\right).\end{aligned}}$$

Basis of the EnKF updating.

Solver for the Riccati equation

Action of $GL(2n)$ in the symplectic space $\mathbb{R}^{2n}$,

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \mapsto z' = \mathsf{A}z = \begin{bmatrix} \mathsf{A}_{11} & \mathsf{A}_{12} \\ \mathsf{A}_{21} & \mathsf{A}_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}.$$

Assume that

$$z_1 = \mathsf{W} z_2.$$

We have

$$\begin{aligned} z_1' &= \mathsf{A}_{11} z_1 + \mathsf{A}_{12} z_2 = (\mathsf{A}_{11}\mathsf{W} + \mathsf{A}_{12}) z_2 \\ z_2' &= \mathsf{A}_{21} z_1 + \mathsf{A}_{22} z_2 = (\mathsf{A}_{21}\mathsf{W} + \mathsf{A}_{22}) z_2, \end{aligned}$$

implying that

$$z_1' = (\mathsf{A}_{11}\mathsf{W} + \mathsf{A}_{12})(\mathsf{A}_{21}\mathsf{W} + \mathsf{A}_{22})^{-1} z_2'.$$

Solver for the Riccati equation

Conclusion: The action of $\mathsf{A} \in GL(2n)$ induces a transformation

$$\boxed{\mathsf{W} \mapsto \mathsf{W}' = (\mathsf{A}_{11}\mathsf{W} + \mathsf{A}_{12})(\mathsf{A}_{21}\mathsf{W} + \mathsf{A}_{22})^{-1}}$$

on the Grassmannian $\mathrm{Gr}_n(2n)$.

Differential equations: Assume that $z = z(R) \in \mathbb{R}^{2n}$ satisfies

$$\dot{z} = \mathsf{C}z = \begin{bmatrix} \mathsf{C}_{11} & \mathsf{C}_{12} \\ \mathsf{C}_{21} & \mathsf{C}_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \quad z_1 = \mathsf{W}z_2.$$

Then,

$$\dot{z}_1 = \dot{\mathsf{W}}z_2 + \mathsf{W}\dot{z}_2,$$

or

$$\dot{\mathsf{W}}z_2 = \dot{z}_1 - \mathsf{W}\dot{z}_2.$$

Solver for the Riccati equation

Substitution:

$$\begin{aligned}\dot{W} z_2 &= \dot{z}_1 - W\dot{z}_2 \\ &= C_{11} z_1 + C_{12} z_2 - W(C_{21} z_1 + C_{22} z_2) \\ &= (C_{12} + C_{11} W - W C_{22} - W C_{21} W)\, z_2,\end{aligned}$$

that is, W satisfies the Riccati equation

$$\boxed{\dot{W} = C_{12} + C_{11} W - W C_{22} - W C_{21} W.}$$

Conclusion: A linear evolution model in $\mathbb{R}^{2n}$ induces a Riccati a flow on the Grassmannian $\mathrm{Gr}_n(2n)$.

Solver for the Riccati equation

First order scheme: Set

$$z(R+h) \approx z(R) + \mathsf{C}z(R)h = \underbrace{(\mathsf{I} + h\mathsf{C})}_{=\mathsf{A}} z(R).$$

Then, by the previous analysis, the first order propagation of W is obtained by

$$\begin{aligned} \mathsf{W}(h) &\approx (\mathsf{A}_{12} + \mathsf{A}_{11}\mathsf{W})(\mathsf{A}_{22} + \mathsf{A}_{21}\mathsf{W})^{-1} \\ &= (h\mathsf{C}_{12} + (\mathsf{I} + h\mathsf{C}_{11})W(R))(\mathsf{I} + h\mathsf{C}_{22} + h\mathsf{C}_{21}\mathsf{W}(R))^{-1} \end{aligned}$$

For the NtD problem, comparing the Riccati equations,

$$\mathsf{C} = \frac{1}{R}\begin{bmatrix} 0 & \mathsf{G} \\ -\mathsf{S} & 0 \end{bmatrix}.$$

Solver for the Riccati equation

First order Möbius propagation scheme:

$$\mathrm{W}(R+h) = \left(W(R) + \frac{h}{R}\mathrm{G}(R)\right)\left(\mathrm{I} - \frac{h}{R}\mathrm{S}(R)\mathrm{W}(R)\right)^{-1}.$$

Theorem

The eigenvalues of the matrix matrix $\mathrm{S}(R)\mathrm{W}(R)$ *are all real and negative.*

Forward propagation ($h > 0$) stable, while backwards propagation ($h < 0$) may encounter singularities.

Solver for the Riccati equation

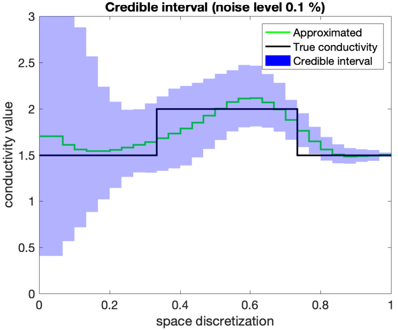
- The Grassmannian is a compact manifold, so the singularities of the Riccati equation are removable coordinate singularities.
- The Möbius solver, unlike standard RK of LMM solvers, have no problems going through the singularities.
- It is easy to define higher order solvers (error control for modeling approximation errors).

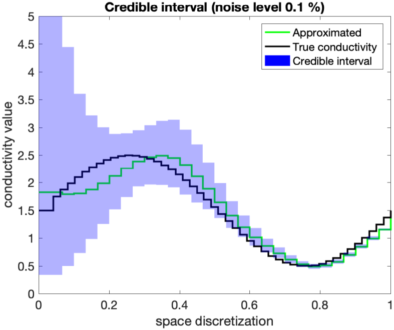
Computed examples

- EnKF with 1000 particles
- Number of frequencies $= 60$ (30 sines, 30 cosines)
- NtD map generated using a FEM approximation
- Additive white noise added
- Forward map: First order Möbius solver, accuracy controlled by using a second order Möbius solver.
- Radial case: Second order AR model to generate $\lambda_{j+1} \mid \lambda_{(j)}$.
- Non-Radial case: Use second order Gaussian smoothness prior, conditioning.

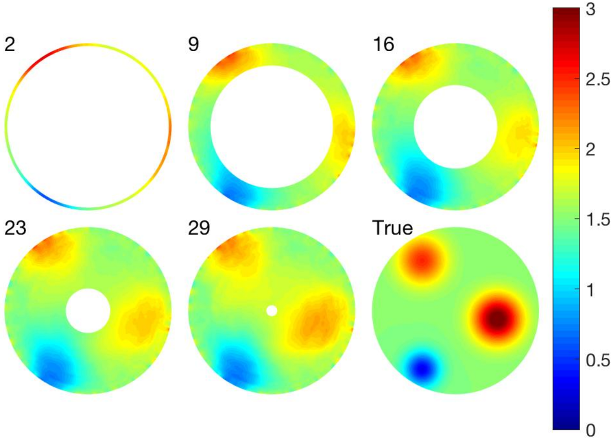
Computed examples

Radial conductivity

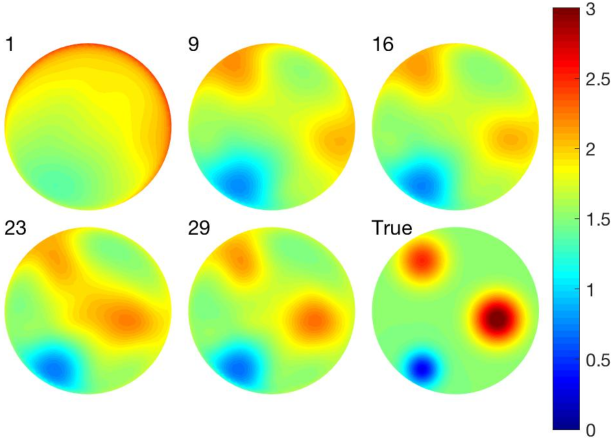




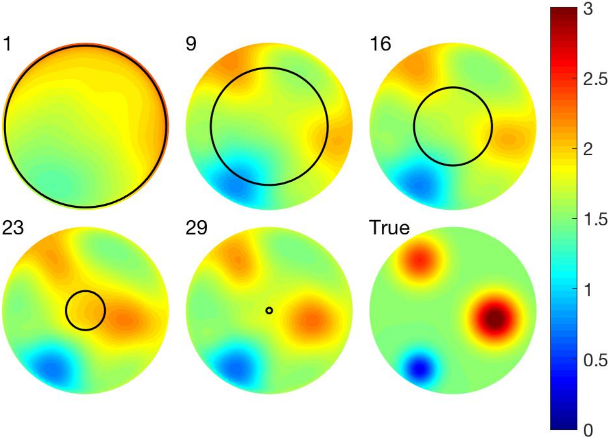
Results



Results



Results



Adding electrodes

- The layer stripping assumes that the data consist of the continuous Neumann-to-Dirichlet operator
- In reality, the EIT data is collected by using a finite number of contact electrodes
- Passing from electrode data to continuous data is in itself an ill-posed problem.

Complete Electrode Model (CEM)

$$\begin{cases} \nabla \cdot (\sigma \nabla v) = 0 & \text{in } \Omega, \\ \sigma \dfrac{\partial v}{\partial n} = 0 & \text{on } S_1 \setminus \cup e_\ell \text{ and } S_2 \\ v + z_\ell \sigma \dfrac{\partial v}{\partial n} = V_\ell & \text{on } e_\ell,\ 1 \leq \ell \leq L \\ \int_{e_\ell} \sigma \dfrac{\partial v}{\partial n} dS = J_\ell, & 1 \leq \ell \leq L, \end{cases} \tag{1}$$

Conservation of charge requires

$$\sum_{\ell=1}^{L} J_\ell = 0. \tag{2}$$

Data: *Give* $J \in \mathbb{R}^L$, *measure* $V \in \mathbb{R}^L$. *Resistance matrix* $\mathrm{R}_\sigma : J \mapsto V$.

Connection with DtN

Theorem

Given $J \in \mathbb{R}^L_0$ and $f \in H^{1/2}(\partial\Omega)$, let $(v, V) \in \mathscr{H} = H^1(\Omega) \times \mathbb{R}^L_0$ be the solution of the CEM problem with applied current pattern J. Then

$$\int_{\partial\Omega} v\Lambda_\sigma f dS + \sum_{\ell=1}^{L} \frac{1}{z_\ell} \int_{e_\ell} (f - W_\ell)(v - V_\ell) dS - \sum_{\ell=1}^{L} J_\ell W_\ell = 0, \tag{3}$$

for all $W \in \mathbb{R}^L_0$.

Connection with DtN, Matrix form

Orthonormal basis in $H^{1/2}(\partial\Omega)$:

$$\varphi_0(\theta) = \frac{1}{\sqrt{2\pi}}, \quad \varphi_{2j}(\theta) = \frac{1}{\sqrt{\pi j}}\cos j\theta, \quad \varphi_{2j-1}\theta = \frac{1}{\sqrt{\pi j}}\sin j\theta, \quad j = 1, 2, \ldots .$$

Matrix representation of Λ_σ:

$$(\mathsf{L}_\sigma)_{jk} = \int_{\partial\Omega} \varphi_j \Lambda_\sigma \varphi_k dS = \langle \varphi_j, \Lambda_\sigma \varphi_k \rangle, \quad 0 \leq j, k < \infty,$$

where

$$\mathsf{L}_\sigma : \ell^2 \to \ell^2.$$

Connection with DtN, Matrix form

Orthonormal basis for current/voltage patterns:

$$(\Phi_m)_\ell = \sqrt{\frac{(2-\delta_{m,L/2})}{L}} \cos \frac{2\pi}{L} m(\ell - 1), \quad 1 \leq \ell \leq L,$$

Representation of the resistance map in the basis Φ:

$$\widetilde{\mathrm{R}}_\sigma = \Phi^{\mathsf{T}} \mathrm{R}_\sigma \Phi \in \mathbb{R}^{(L-1)\times(L-1)},$$

Connection with DtN, Matrix form

Some auxiliary matrices:

$$\mathsf{D}_{\ell\ell} = \frac{|e_\ell|}{z_\ell}, \quad 1 \leq \ell \leq L,$$

$$\mathsf{Y}_{j\ell} = \frac{1}{|e_\ell|} \int_{e_\ell} \varphi_j dS, \quad 1 \leq \ell \leq L, \quad 0 \leq j < \infty,$$

$$\mathsf{M}_{jk} = \sum_{\ell=1}^{L} \frac{1}{z_\ell} \int_{e_\ell} \varphi_j \varphi_k dS, \quad 0 \leq j, k < \infty.$$

Connection with DtN, Matrix form

Theorem

The matrices $\mathrm{L}_\sigma : \ell^2 \to \ell^2$ *and* $\mathrm{R}_\sigma \in \mathbb{R}^{L\times L}$ *satisfy the identity*

$$\Phi^{\mathsf{T}}\mathrm{D}\Phi - (\mathrm{YD}\Phi)^{\mathsf{T}}(\mathrm{L}_\sigma + \mathrm{M})^{-1}\mathrm{YD}\Phi = \widetilde{\mathrm{R}}_\sigma^{-1}, \tag{4}$$

where $\widetilde{\mathrm{R}}_\sigma$ *is the representation of the resistance map in the basis* Φ,

$$\widetilde{\mathrm{R}}_\sigma = \Phi^{\mathsf{T}}\mathrm{R}_\sigma\Phi \in \mathbb{R}^{(L-1)\times(L-1)},$$

- Computing R_σ from L_σ or its inverse is a well-posed problem
- The converse, estimating L_σ from R_σ is an ill-posed problem.
- For the stable layer stripping algorithm, only the stable form is necessary:

$$(\lambda_{(k)}, w_k) \mapsto w_0 \mapsto \mathrm{L}_\sigma \mapsto \mathrm{R}_\sigma.$$

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