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Ill-Posedness of the Third Order NLS Equation with Raman Scattering Term

Yoshio TSUTSUMI (Kyoto University),

with Nobu KISHIMOTO (Kyoto University, RIMS)

1 Introduction

- 3rd Order NLS with Raman Scatteing Term

$$\partial_t u = \alpha_1 \partial_x^3 u + i\alpha_2 \partial_x^2 u + i\gamma_1 |u|^2 u + \gamma_2 \partial_x \left(|u|^2 u \right) - i\Gamma u \partial_x \left(|u|^2 \right), \quad (1)$$

$$t \in [-T, T], \quad x \in \mathbf{T},$$

$$u(0, x) = u_0(x), \quad x \in \mathbf{T}. \quad (2)$$

α_j, γ_j, Γ; real constants, $\alpha_1^2 + \alpha_2^2 \neq 0$, $\Gamma \neq 0$.

$u : [-T, T] \times \mathbf{T} \to \mathbf{C}$; slowly varying envelope of electric field,
The last term on RHS of (1) represents the effect of Raman scattering.
Assume that

$$\alpha_1 \neq 0 \implies 2\alpha_2/3\alpha_1 \notin \mathbf{Z}. \qquad \text{(NR)}$$

Pulse with slowly varying envelope in photonic crystal fiber

V. Agrawal, *"Nonlinear Fiber Optics "*, Fourth Edition, Acadmic Press, 2007.

Problem: Is the Cauchy problem of (1) well-posed in Sobolev spaces H^s, in analytic function space or in the Gevrey class?

- Main Theorems on Ill-Posedness

Theorem 1 (Kishimoto-Y.T, 2018)
$2\alpha_2/3\alpha_1 \notin \mathbf{Z}$ $(\alpha_1 \neq 0)$, $1 \leq s_1 \leq s < s_1 + 1.$

Then, $\exists u_0 \in H^s(\mathbf{T})$ such that for any $T > 0$ the Cauchy problem of (1) with $u(0) = u_0$ has no solution $u \in C([0,T); H^{s_1}(\mathbf{T}))$, nor solution $u \in C((-T,0]; H^{s_1}(\mathbf{T}))$.

Remark 1 (i) Instead of $\mathbf{T}$, when we consider (1) on $\mathbf{R}$, it is known that (LWP) holds in regular Sobolev spaces (Hayashi and Ozawa (1994), Chihara (1994)). The spectrum of the Laplacian on $\mathbf{T}$ is discrete while it is continuous on $\mathbf{R}$. The difference between $\mathbf{T}$

and $\mathbf{R}$ comes from the nature of the spectrum of the Laplacian.
(ii) Even if $\alpha_1 = 0$, Theorem 1 holds.

Theorem 2 (Kishimoto-Y.T, 2018)
$2\alpha_2/3\alpha_1 \notin \mathbf{Z}$ $(\alpha_1 \neq 0)$, $s \geq 1$,
$u^* \in C([0,T]; H^s(\mathbf{T}))$; solution to (1) on $[0,T]$ for some $T > 0$. Then,
$\forall \varepsilon > 0$, $0 < \forall \tau \leq T$, $\exists$ real analytic function ϕ on $\mathbf{T}$ with $\|\phi\|_{H^s} \leq \varepsilon$ such that either there is no solution u to (1) in $C([0,\tau]; H^s(\mathbf{T}))$

with initial condition $u(0) = u^*(0) + \phi$, or such a solution exists but

$$\sup_{t \in [0,\tau]} \|u(t) - u^*(t)\|_{H^s} \geq \varepsilon^{-1}.$$

Remark 2 Theorem 2 implies the breakdown of continuous dependence on initial data. The assertion of Theorem 2 is weaker than Theorem 1, while the former can cover a larger class of initial data than the latter.

- Idea of Proofs for Theorems 1 and 2

Conservation Law of Mass:

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2}^2, \quad t \in [-T, T].$$

Remark 3 Momentum and energy are not conserved because of Raman scattering.

(Translation and Gauge Transformation)

$$v(t,x) = u\Big(t, x - \frac{\gamma_2}{\pi} \int_0^t \|u(s)\|_{L^2}^2 \, ds\Big)$$
$$\times \, e^{-\frac{\gamma_1}{\pi} i \int_0^t \|u(s)\|_{L^2}^2 \, ds - \frac{\Gamma}{2\pi} i \int_0^t \mathrm{Im}(\partial_x u, u) \, ds}.$$

Then, (1) can be written as follows.

$$
\begin{aligned}
&\partial_t v + ia\partial_x v = \alpha_1 \partial_x^3 v + i\alpha_2 \partial_x^2 v \qquad (3)\\
&\quad + i\gamma_1 \Big(|v|^2 - \frac{1}{\pi} \|v(t)\|_{L^2}^2 \Big) v \\
&\quad + \gamma_2 \Big[2\Big(|v|^2 - \frac{1}{2\pi} \|v(t)\|_{L^2}^2 \Big) \partial_x v + v^2 \partial_x \bar{v} \Big] \\
&\quad + \frac{\Gamma}{(2\pi)^{3/2}} \sum_{k \in \mathbf{Z}} e^{-ikx} \\
&\qquad \times \sum_{(k_1+k_2)(k_2+k_3) \neq 0} (k_1 + k_2) \hat{v}(k_1) \hat{\bar{v}}(k_2) \hat{v}(k_3),
\end{aligned}
$$

where $\hat{v}(t,k)$ denotes the Fourier transform in x of $v(t,x)$ and

$$a = \frac{\Gamma}{2\pi} \|u_0\|_{L^2}^2.$$

The Cauchy-Riemann type elliptic operator $\partial_t + ia\partial_x$ appears due to the Raman scattering term, which gives rise to the ill-posedness of the Cauchy problem (1)-(2).

Remark 4 The elliptic regularity theorem for the Cauchy-Riemann type operator yields that no solution $u \in C((-T,T);H^{s_1})$ for any $T>0$, which is slightly weaker than Theorem 1. For the proof of Theorems 1 and 2, we need to use the dispersive nature of equation (3), which implies the smoothing type effect. This is why we need to assume (NR).

(Interaction representation)

$$w(t,x) = e^{-t(\alpha_1 \partial_x^3 + i\alpha_2 \partial_x^2)} v(t,x).$$

Apply Fourier transform in x to (3) $\Longrightarrow$

$$\partial_t \hat{w}(k) - \ a\, k \hat{w}(k) = \frac{i\gamma_1 + i\gamma_2 k}{2\pi} |\hat{w}(k)|^2 \hat{w}(k)$$

$$+ \frac{i\gamma_1}{2\pi} \sum_{\substack{k_1+k_2+k_3=k \\ (k_1+k_2)(k_2+k_3)\neq 0}} e^{it\Phi} \hat{w}(k_1) \bar{\hat{w}}(-k_2) \hat{w}(k_3)$$

$$+ \sum_{\substack{k_1+k_2+k_3=k \\ (k_1+k_2)(k_2+k_3)\neq 0}} \frac{i\gamma_2 k + \Gamma(k_1 + k_2)}{2\pi}$$

$$\times e^{it\Phi} \hat{w}(k_1) \bar{\hat{w}}(-k_2) \hat{w}(k_3)$$

$$=: \hat{F}_1(t,k) + \hat{F}_2(t,k) + \hat{F}_3(t,k).$$

Here,

$$\begin{aligned}\Phi(k_1,k_2,k_3) = &\big(\alpha_1 k^3 + \alpha_2 k^2\big) \\ &- (\alpha_1 k_1^3 + \alpha_2 k_1^2) + \big(\alpha_1(-k_2)^3 + \alpha_2(-k_2)^2\big) \\ &- (\alpha_1 k_3^3 + \alpha_2 k_3^2) \\ =& 3\alpha_1(k_1+k_2)(k_2+k_3)\Big(k_3+k_1+\frac{2\alpha_2}{3\alpha_1}\Big).\end{aligned}$$

Under the assumption $\frac{2\alpha_2}{3\alpha_1} \notin \mathbf{Z}$, it holds that

$$\Phi(k_1, k_2, k_3) = 0 \;\Leftrightarrow\; (k_1 + k_2)(k_2 + k_3) = 0,$$
$$\Phi(k_1, k_2, k_3) \neq 0 \;\Rightarrow\; |\Phi(k_1, k_2, k_3)| \sim |k_1 + k_2|\,|k_2 + k_3|\,|k_3 + k_1|.$$

(Resonant case) If $s \geq 1$,

$$|k|\big|\hat{F}_1(t,k)\big| \lesssim |k|^{-1}\big(|k|^s\big|\hat{w}(k)\big|\big)^3,$$

which is the smoothing type estimate.

(Nonresonant case) The time integration of $\hat{F}_2$ and $\hat{F}_3$ leads to the smoothing type effect thanks to the oscillation of $e^{it\Phi}$. Therefore, $\exists u_0 \in H^s$ such that if $T > 0$ (resp. $T < 0$),

$$\left| e^{aTk} \hat{u}_0(k) \right| \longrightarrow \infty,$$

$$\left| \frac{\int_0^T e^{a(T-t')k} \hat{F}_j(t', k)\, dt'}{e^{aTk} \hat{u}_0(k)} \right| \longrightarrow 0$$

as $k \to \infty$ (resp. $k \to -\infty$), $\quad j = 1, 2.$

$\Longrightarrow$ Thoerems 1 and 2

- Physical Literature Related to Ill-Posedness
  - M. Erkintalo, G. Genty, B. Wetzel and J.M. Dudley, *Optics Express*, **18**(24), 2010. Limitations of the linear Raman gain approximation

  - T.X. Tran and F. Biancalana, arXiv:1504.03865v3 [physics.optics] , 2015. Unphysical metastability of the fundamental Raman soliton

  - Fabio Biancalana, Heriot-Watt University

This approach is universally used amongst physicists, ...
(Private Communications)

Remark 5 The mathematical notion of ill-posedness is interpreted as the instability of a physical system at hand. But it is not very clear whether this instability accounts for some physical phenomena or it implies just the limitation of the model.

Remark 6 A large number of numerical simulations for the Cauchy problem (1)-(2) have been made though it is ill-posed in Sobolev spaces. In those numerical computations, such analytic functions as Gaussian and super-Gaussian pulses are chosen as initial data. So, it is natural to expect that the Cauchy problem (1)-(2) should be solvable in the analytic function space. Indeed, we can prove the result on the unique solvability in the analytic function space.

- Solvability in Analytic Function Spaces

$$
\|f\|_{\mathcal{A}(r)} := \|e^{r|k|}\hat{f}(k)\|_{\ell^1(\mathbf{Z})},\ r > 0,
$$

$$
\mathcal{A}(r) := \{f \in L^2(\mathbf{T}) \,\big|\, \|f\|_{\mathcal{A}(r)} < \infty\}.
$$

Remark 7 Functions in $\mathcal{A}(r)$ are real analytic and have analytic extensions on the strip $\{z \in \mathbf{C} | \, |\mathrm{Im}\, z| < r\}$. The function space $\mathcal{A}(r)$ was employed by Ukai (1984) for the Boltzmann equation, by Kato and Masuda (1986) for a class of nonlinear evolution equations and by Foias and Temam (1989) for the incompressible Navier-Stokes equations.

Theorem 3 Let α_j, $j = 1, 2$ be two real numbers and let $r > 0$. For any $u_0 \in \mathcal{A}(r)$, there exist $T > 0$ such that the Cauchy problem (1)-(2) has a unique solution $u \in C([-T, T]; \mathcal{A}(r/2))$ on $(-T, T)$. Moreover, T can be chosen as

$$T \gtrsim \min\{1, r\} \|u_0\|_{\mathcal{A}(r)}^{-2},$$

where the implicit constant does not depend on r and u_0.

Remark 8 Theorem 3 is a kind of the abstract Cauchy-Kowlevsky theorem. We do not have to assume $2\alpha_2/3\alpha_1 \notin \mathbf{Z}$ in Theorem 3. Even when $\alpha_1 = \alpha_2 = 0$, Theorem 3 holds.

Open Problem It is not known if the solution given by Theorem 3 exists globally in time. Some numerical simulations suggest that when the initial datum is Gaussian or super-Gaussian, the solution may exist globally in time or for a long period of time. What if the initial datum is a sech pulse of the cubic NLS?

- Ill-posedness in the Gevrey class

It is natural to ask if the Cauchy problem (1)-(2) is well-posed in the Gevrey class or not.

$\sigma \geq 1, \;\; s \geq 0, \;\; a > 0,$

$$G^\sigma_{s,a} = \{ f \in C^\infty(\mathbf{T};\mathbf{C}); \quad \hat{f}(k) = O(|k|^{-s} e^{-a|k|^{1/\sigma}}), \ |k| \to \infty \},$$

$$\|f\|_{G^\sigma_{s,a}} = \sup_{k \in \mathbf{Z}} e^{a|k|^{1/\sigma}} \langle k \rangle^s |\hat{f}_k|,$$

$$\langle k \rangle = \max\{1, |k|\}.$$

$$G^\sigma = \bigcup_{a>0} G^\sigma_{0,a} \ (\text{Gevrey class of order } \sigma).$$

Remark 9 The space $G^\sigma_{s,a}$ is the Banach space while G^σ is not the Banach space. The Gevrey space G^σ is the topological space equipped with the inductive limit topology.

Theorem 4 (Kishimoto-Y.T, 2019) Let $\sigma > 1$. For any $u_0 \in G^\sigma \setminus \bigcap_{a'>0} G^\sigma_{0,a'}$ there exists no $T > 0$ such that the Cauchy problem (1)–(2) has a solution in $C([-T,T];G^\sigma)$.

Theorem 4 follows from the following Gevrey smoothing effect.

Lemma 1 (Kishimoto-Y.T, 2019) Let $\sigma > 1$, and let $u(t) \in C([-T, T]; G^\sigma)$ be a solution to (1) on $(-T, T)$ for some $T > 0$. Then, $u(t) \in \bigcap_{a'>0} G^\sigma_{0,a'}$ for all $t \in (-T, T)$.

Thank you
for your attention!

(Example of initial datum) Let s, s_1 be such that $1 \leq s_1 \leq s < s_1 + 1$. We take any $s_0 \in (s, s_1 + 1)$ and choose initial data u_0 as follows.

$$\hat{u}_0(k) := \begin{cases} |k|^{-s_0} & \text{if } k = \pm 2^j \text{ for some } j \in \mathbf{N}, \\ 0 & \text{otherwise,} \end{cases}$$

which is clearly in $H^s(\mathbf{T})$.

(periodic Gaussian pulse)

$$g_\lambda(x) = \sum_{k=-\infty}^{\infty} \hat{g}_\lambda(k) e^{ikx},$$

$$\hat{g}_\lambda(k) = \lambda e^{-\lambda^2 k^2}, \quad k \in \mathbf{Z}, \; \lambda > 0.$$

(periodic hyperbolic secant pulse)

$$h_\lambda(x) = \sum_{k=-\infty}^{\infty} \hat{h}_\lambda(k) e^{ikx},$$

$$\hat{h}_\lambda(k) = \lambda\pi \operatorname{sech}\left(\frac{\pi k}{2\lambda}\right), \qquad k \in \mathbf{Z}, \; \lambda > 0.$$