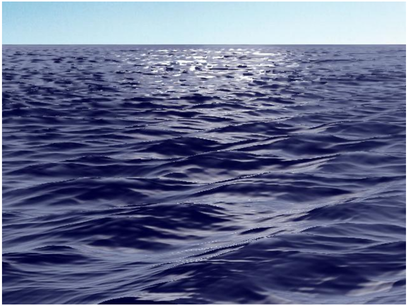
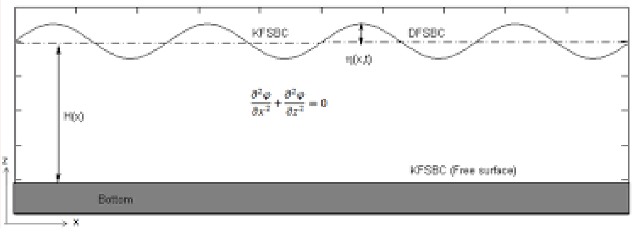
Long time Dynamics of Water Waves

Massimiliano Berti, SISSA,

Advances in Dispersive Equations: Challenges and Perspectives
Banff, July 2019



Time evolution of space periodic water waves in Trieste gulf:



In section it is described by a bidimensional fluid, periodic in x

Water Waves: Euler equations for an irrotational, incompressible fluid in $S_\eta(t) = \{-h < y < \eta(t,x)\}$ under gravity and capillarity

$$\begin{cases} \partial_t \Phi + \frac{1}{2}|\nabla \Phi|^2 + g\eta = \kappa \partial_x \left(\frac{\eta_x}{\sqrt{1+\eta_x^2}} \right) & \text{at } y = \eta(t,x) \\ \Delta \Phi = 0 & \text{in } -h < y < \eta(t,x) \\ \partial_y \Phi = 0 & \text{at } y = -h \\ \partial_t \eta = \partial_y \Phi - \partial_x \eta \cdot \partial_x \Phi & \text{at } \ y = \eta(t,x) \end{cases}$$

$u = \nabla \Phi =$ velocity field, $\mathrm{rot}\, u = 0$ (irrotational),
$\mathrm{div}\, u = \Delta \Phi = 0$ (uncompressible)
$g =$ gravity, $\kappa =$ surface tension coefficient

$$\text{Mean curvature} = \partial_x \left(\frac{\eta_x}{\sqrt{1+\eta_x^2}} \right)$$

Unknowns:

free surface $y = \eta(t,x)$ and the velocity potential $\Phi(t,x,y)$

Zakharov formulation '68

Infinite dimensional Hamiltonian system:

$$\partial_t u = J \nabla_u H(u) , \quad u := \begin{pmatrix} \eta \\ \psi \end{pmatrix} , \quad J := \begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix} ,$$

canonical Darboux coordinates:

$\eta(x)$ and $\psi(x) = \Phi(x, \eta(x))$ trace of velocity potential at $y = \eta(x)$

(η, ψ) uniquely determines Φ in the whole $\{-h < y < \eta(x)\}$ solving the elliptic problem:

Φ is harmonic

$\Delta \Phi = 0$ in $\{-h < y < \eta(x)\}$, $\Phi|_{y=\eta} = \psi$, $\partial_y \Phi = 0$ at $y = -h$

Hamiltonian: total energy on $S_\eta = \mathbb{T} \times \{-h < y < \eta(x)\}$

$$H := \frac{1}{2} \int_{S_\eta} |\nabla \Phi|^2 dxdy + \int_{S_\eta} gy \, dxdy + \kappa \int_{\mathbb{T}} \sqrt{1 + \eta_x^2} \, dx$$

kinetic energy + potential energy + capillary energy

Hamiltonian expressed in terms of (η, ψ)

$$H(\eta, \psi) = \tfrac{1}{2} \int_{\mathbb{T}} \psi(x) \, G(\eta)\psi(x) \, dx + \tfrac{1}{2} \int_{\mathbb{T}} g\eta^2 \, dx + \kappa \int_{\mathbb{T}} \sqrt{1 + \eta_x^2} \, dx$$

Dirichlet–Neumann operator (Craig-Sulem '93)

$$G(\eta)\psi(x) := \sqrt{1 + \eta_x^2} \, \partial_n \Phi|_{y=\eta(x)} = (\Phi_y - \eta_x \Phi_x)(x, \eta(x))$$

Zakharov-Craig-Sulem formulation

$$\begin{cases} \partial_t \eta = G(\eta)\psi = \nabla_\psi^{L^2} H(\eta,\psi) \\ \partial_t \psi = -g\eta - \dfrac{\psi_x^2}{2} + \dfrac{(G(\eta)\psi + \eta_x\psi_x)^2}{2(1+\eta_x^2)} + \dfrac{\kappa\eta_{xx}}{(1+\eta_x^2)^{3/2}} = -\nabla_\eta^{L^2} H(\eta,\psi) \end{cases}$$

Dirichlet–Neumann operator

$$G(\eta)\psi(x) := \sqrt{1+\eta_x^2}\, \partial_n \Phi|_{y=\eta(x)}$$

1. $G(\eta)$ is linear in ψ, non-local,
2. self-adjoint with respect to $L^2(\mathbb{T}_x)$
3. $G(\eta) \geq 0$, $G(1) = 0$
4. $\eta \mapsto G(\eta)$ nonlinear, smooth,
5. $G(\eta)$ is pseudo-differential, $G(\eta) = D_x \tanh(hD_x) + OPS^{-\infty}$

Calderon, Craig, Lannes, Metivier, Alazard, Burq, Zuily, Delort...

Symmetries

Momentum

$$M(\eta, \psi) = \int_{\mathbb{T}} \eta_x(x)\, \psi(x)\, dx$$

x-translation invariance

Invariant subspace: functions even in x. Standing waves

$$\eta(-x) = \eta(x)\,, \quad \psi(-x) = \psi(x)$$

Prime integral: mass

$$\int_{\mathbb{T}} \eta(x)dx$$

Phase space

$$\eta \in H_0^s(\mathbb{T}) := \{\eta \in H^s(\mathbb{T}) \, : \, \int_{\mathbb{T}} \eta(x)dx = 0\}$$

$$u \in H^s(\mathbb{T}) \ \Leftrightarrow \ u(x) = \sum_{k\in\mathbb{Z}} u_k e^{\mathrm{i}kx} \, , \ \ \sum_{k\in\mathbb{Z}} |u_k|^2 \langle k \rangle^{2s} =: \|u\|_{H^s}^2 < +\infty$$

The variable ψ is defined modulo constants: only the velocity field $\nabla_{x,y}\Phi$ has physical meaning.

$$\psi \in \dot{H}^s(\mathbb{T}) = H^s(\mathbb{T})/\sim$$

$$u(x) \sim v(x) \quad \Longleftrightarrow \quad u(x) - v(x) = c$$

Linear water waves theory

Linearized system at $(\eta, \psi) = (0, 0)$

$$\begin{cases} \partial_t \eta = G(0)\psi, \\ \partial_t \psi = -g\eta + \kappa \eta_{xx} \end{cases}$$

Dirichlet-Neumann operator at the flat surface $\eta = 0$ is

$$G(0) = D \tanh(hD)\,, \quad D = \frac{\partial_x}{\mathrm{i}} = \mathrm{Op}(\xi)_{\xi \in \mathbb{R}}$$

Fourier multiplier notation: given $m : \mathbb{Z} \to \mathbb{C}$

$$m(D)h = \sum_{j \in \mathbb{Z}} m(j) h_j e^{\mathrm{i}jx}\,, \quad h(x) = \sum_{j \in \mathbb{Z}} h_j e^{\mathrm{i}jx}$$

Linear water waves system

$$\partial_t \begin{bmatrix} \eta \\ \psi \end{bmatrix} = \begin{bmatrix} 0 & G(0) \\ -g+\kappa\partial_{xx} & 0 \end{bmatrix} \begin{bmatrix} \eta \\ \psi \end{bmatrix}$$

Complex variable

$$u = \Lambda(D)\eta + \mathrm{i}\Lambda^{-1}(D)\psi\,, \quad \Lambda(D) = \left(\frac{g+\kappa D^2}{D\tanh(hD)}\right)^{1/4}$$

Linear Water Waves

$$u_t + i\omega(D)u = 0, \quad \omega(D) = \sqrt{D\tanh(hD)(g+\kappa D^2)}$$

Dispersion relation

$$\omega(\xi) = \sqrt{\xi\tanh(h\xi)(g+\kappa\xi^2)}$$

∞-decoupled harmonic oscillators

$$u(t,x) = \sum_{j\in\mathbb{Z}} e^{-\mathrm{i}\omega(j)t} u_j(0) e^{\mathrm{i}jx}$$

Linear frequencies of oscillations

$$\omega(j) = \sqrt{j\tanh(hj)(g+\kappa j^2)}\,, \quad j\in\mathbb{Z}\,,$$

All solutions are periodic, quasi-periodic, almost periodic in time according to the irrationality properties of $(\omega_j(h,g,\kappa))_{j\in\mathbb{Z}}$

The Sobolev norm is constant

$$\|u(t,\cdot)\|_{H^s} = \|u(0,\cdot)\|_{H^s}$$

Nonlinear water waves

Main questions

1. For which time interval $(-T_{\max}, T_{\max})$ solutions of the nonlinear water waves equations exist?

2. Are there periodic, quasi-periodic, almost periodic solutions (thus global in time) of the nonlinear water waves equations?

Major difficulties:

Gravity-Capillary WW are quasi-linear PDEs

$$u_t + \mathrm{i}\omega(D)u = N(u, \bar{u}) \,, \quad \omega(D) \sim |D|^{3/2}$$

$N =$ quadratic nonlinearity with derivatives of order $N(|D|^{3/2}u)$

Gravity WW are fully nonlinear PDEs

$$u_t + \mathrm{i}\omega(D)u = N(u, \bar{u}) \,, \quad \omega(D) \sim |D|^{1/2}$$

$N =$ quadratic nonlinearity with derivatives of order $N(\partial_x u)$

Singular perturbation of the linear vector field $\mathrm{i}\omega(D)u$

Periodic boundary conditions $x \in \mathbb{T}$

NO dispersive effects of the linear PDE as for $x \in \mathbb{R}^2$, $x \in \mathbb{R}$ and data decaying at infinity:

Global well-posedness: S.Wu, Germain-Masmoudi-Shatah, Ionescu-Pusateri, Alazard-Delort, Ifrim-Tataru, Alazard-Burq-Zuily

Nonlinear water waves, main results:

- **Long time existence/Birkhoff normal form:**
  For *any* small initial data of size ε the solution is defined for long times T_ε
  1. Gravity-capillary:
     M. Berti- J-M. Delort, '17, for any h, most (g, κ), $T_\varepsilon \geq c\varepsilon^{-N}$
  2. M. Berti, R. Feola, L. Franzoi, '19,
     for *all* $g, \kappa, h > 0$ then $T_\varepsilon \geq c\varepsilon^{-2}$
  3. Gravity: M. Berti, R. Feola, F. Pusateri, '18,
     $h = +\infty$, any g, then $T_\varepsilon \geq c\varepsilon^{-3}$
- **KAM:** Existence of quasi-periodic solutions for
  1. Gravity-capillary: Berti-Montalto, '16,
  2. Gravity: Baldi-Berti-Haus-Montalto, '17,

  solutions defined for all times, for "*most*" initial conditions

Focus on long time existence/Birkhoff normal result 3

Proof of a conjecture of Zakharov-Dyanchenko '94

Theorem (M. Berti, R. Feola, F. Pusateri, '18)

INFORMAL STATEMENT:

1. *The gravity water waves equations in $h = +\infty$ are an integrable system up to quartic terms $O(u^4)$*
2. *The solutions with an initial datum $u_0 = O(\varepsilon)$ in Sobolev spaces are defined for times $T_\varepsilon \geq c\varepsilon^{-3}$*

Resonances: dispersion relation $\omega(n) = \sqrt{|n|}$

1) There are no 3-waves resonances:

$$\begin{cases} n_1 \pm n_2 \pm n_3 = 0 \\ \sqrt{|n_1|} \pm \sqrt{|n_2|} \pm \sqrt{|n_3|} = 0 \end{cases}$$

2) There are **non-trivial** 4 **waves resonances**:

$$\begin{cases} n_1 - n_2 + n_3 - n_4 = 0 \\ \sqrt{|n_1|} - \sqrt{|n_2|} + \sqrt{|n_3|} - \sqrt{|n_4|} = 0 \end{cases}$$

has many integer solutions, in addition to the trivial solutions (k, k, j, j): the Benjamin-Feir resonances

$$n_1 = -qm^2, n_2 = q(m+1)^2, n_3 = q(m^2+m+1)^2, n_4 = q(m+1)^2 m^2$$

$$q \in \mathbb{Z} \setminus \{0\}, m \in \mathbb{N}$$

Formal integrability at order 4: resonant system

There is a *formal* symplectic change of variables which transforms the water-waves Hamiltonian into

$$H^{(4)}_{BNF} = \sum_{j \in \mathbb{Z}\setminus\{0\}} \sqrt{|j|}|z_j|^2 + H^{(4)}_{ZD} + \cdots$$

where

$$H^{(4)}_{ZD} = \sum_{\substack{\sigma_1 j_1 + \sigma_2 j_2 + \sigma_3 j_3 + \sigma_4 j_4 = 0,\, \sigma_i = \pm 1\,, \\ \sigma_1 \omega(j_1) + \sigma_2 \omega(j_2) + \sigma_3 \omega(j_3) + \sigma_4 \omega(j_4) = 0}} H^{\sigma_1,\sigma_2,\sigma_3,\sigma_4}_{j_1,j_2,j_3,j_4} z^{\sigma_1}_{j_1} z^{\sigma_2}_{j_2} z^{\sigma_3}_{j_3} z^{\sigma_4}_{j_4}$$

$z^+ = z$, $z^- = \bar{z}$ **There is a null condition**

Theorem (Zakharov-Dyanchenko '94)

The Hamiltonian $\sum_{j\in\mathbb{Z}\setminus\{0\}} \sqrt{|j|}|z_j|^2 + H^{(4)}_{ZD}$ is integrable, possesses the actions $|z_j|^2$, $j \in \mathbb{Z}\setminus\{0\}$, as prime integrals, and, in particular, its flow preserves all Sobolev norms.

Theorem (Birkhoff normal form for gravity WW, B-F-P, '18)

There exists a bounded change of variables in H^s which transforms the gravity water-waves equations with $h = +\infty$ into

$$\partial_t z = \mathrm{i}|D|^{\frac{1}{2}} z + \mathrm{i}\partial_{\bar z} H^{(4)}_{ZD} + \mathcal{X}_{\geq 4}(z)$$

where $H^{(4)}_{BNF}$ is the Zakharov-Dyanchenko Hamiltonian

$$H^{(4)}_{ZD}(z,\bar z) := \frac{1}{4\pi}\sum_{k\in\mathbb{Z}} |k|^3\big(|z_k|^4 - 2|z_k|^2|z_{-k}|^2\big)$$

$$+\frac{1}{\pi}\sum_{\substack{k_1,k_2\in\mathbb{Z},\ \mathrm{sgn}\,(k_1)=\mathrm{sgn}\,(k_2)\\ |k_2|<|k_1|}} |k_1||k_2|^2\big(-|z_{-k_1}|^2|z_{k_2}|^2 + |z_{k_1}|^2|z_{k_2}|^2\big)$$

which preserves all Sobolev norms, and $\mathcal{X}_{\geq 4}(z)$ has energy estimates in H^s:

$$\mathrm{Re}\int_{\mathbb{T}} |D|^s \mathcal{X}_{\geq 4}\cdot \overline{|D|^s z}\, dx \leq C\|z\|^5_{H^s}\,.$$

Corollary: solutions with $\varepsilon u(0) \in H^s$ exist in H^s up to $T_\varepsilon \geq c\varepsilon^{-3}$

Energy estimates of $z_t = \mathcal{X}_{\geq 4}(z)$

How Sobolev norms evolve?

$$\|z\|_s^2 = \sum_{n \in \mathbb{Z} \setminus 0} |n|^{2s} |z_n|^2 = (|D|^s z, |D|^s z)_{L^2}$$

$$\begin{aligned} \frac{d}{dt}\|z\|_s^2 &= (|D|^s z_t, |D|^s z)_{L^2} + (|D|^s z, |D|^s z_t)_{L^2} \\ &= 2\mathrm{Re}(|D|^s \mathcal{X}_{\geq 4}(z), |D|^s z)_{L^2} \\ &\lesssim_s \|z\|_s^5 \end{aligned}$$

not obvious because $\mathcal{X}_{\geq 4}(z)$ is unbounded (order 1). If $\|z(0)\|_s = \varepsilon$ then $\Longrightarrow$

The Sobolev norm $\|z(t)\|_s = O(\varepsilon)$ for a time interval $O(\varepsilon^{-3})$

$\Longrightarrow$ This rigorously justifies the formal Birkhoff normal form expansions used successfully by physicists!

In same spirit that Lindsted formal series in celestial mechanics were rigorously justified a-posteriori by KAM theorem (Moser)

Time of existence

1. $T_\varepsilon \geq c\varepsilon^{-1}$, local existence theory, S. Wu., Lindblad, Coutand-Shkroller, Alazard-Burq-Zuily, ...
2. $T_\varepsilon \geq c\varepsilon^{-2}$, S. Wu, Ifrim-Tataru, if $h = +\infty$ there are no "triple wave interactions" + quasi-linear modified energy

No solutions $k_1, k_2, k_3 \in \mathbb{Z} \setminus 0$ of

$$\begin{cases} |k_1|^{\frac{1}{2}} \pm |k_2|^{\frac{1}{2}} \pm |k_3|^{\frac{1}{2}} = 0 \\ k_1 \pm k_2 \pm k_3 = 0 \end{cases}$$

3. Gravity-capillary waves $T_\varepsilon \geq c\varepsilon^{-N}$, $\forall N$, Berti-Delort '17, we erase parameters (g, κ) to avoid multiple wave interactions

Berti-Feola-Franzoi, '19:
For any value of g = gravity, κ = capillarity, h = depth, the solutions of gravity-capillary water waves exist for

$$T_\varepsilon \geq c\varepsilon^{-2}$$

For general values of (g, h, κ), $\omega_j = \sqrt{j \tanh(hj)(g + \kappa j^2)}$

There are 3-waves resonances (Wilton-ripples)

$$\begin{cases} \omega_{j_1} \pm \omega_{j_2} \pm \omega_{j_3} = 0 \\ j_1 \pm j_2 \pm j_3 = 0 \end{cases} \qquad j_1, j_2, j_3 \in \mathbb{Z} \setminus 0\,,$$

But finitely many. Hamiltonian Birkhoff normal form.

Global existence?

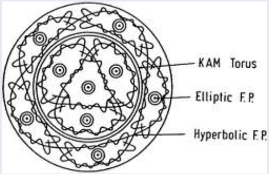
Question: **Do these solutions exist for all times?**

We do not know. Maybe not

Craig-Workfolk: for $\kappa = 0$, $h = +\infty$ the water-waves PDEs are not integrable at the fifth order Birkhoff normal form

(could be Chaotic but with well defined flow)

Expected scenario for nearly-integrable Hamiltonian systems



1. **KAM results**: There are many solutions defined for all times: **selection of "initial conditions" giving rise to global solutions**
2. **Long time existence:** $|t| \leq c\varepsilon^{-N}$. For longer times?
3. **Arnold diffusion:** *What about a solution which does not start on a KAM torus for times* $|t| > c\varepsilon^{-N}$*?* Chaos? Growth of Sobolev norms?

Quasi-periodic solution with n frequencies of $u_t = X(u)$

Definition

$$u(t,x) = U(\omega t, x) \text{ where } U(\varphi, x) : \mathbb{T}^n \times \mathbb{T} \to \mathbb{R},$$

$\omega \in \mathbb{R}^n$(= *frequency vector*) is irrational $\omega \cdot k \neq 0\,, \ \forall k \in \mathbb{Z}^n \setminus \{0\}$

$\Longrightarrow$ the linear flow $\{\omega t\}_{t \in \mathbb{R}}$ is DENSE on $\mathbb{T}^n$

- Global in time
- If $n = 1$ then $U(\omega t, x)$ is time-periodic with period $T = 2\pi/\omega$

Periodic solutions: $n = 1$

- **Plotnikov-Toland**: '01
  Gravity Water Waves with Finite depth
- **Iooss-Plotnikov-Toland** '04, **Iooss-Plotnikov** '05-'09
  Gravity Water Waves with Infinite depth
  Completely resonant, infinite dimensional bifurcation equation
- **Alazard-Baldi** '15,
  Capillary-gravity water waves with infinite depth

Quasi-Periodic solutions: $n \geq 2$

- **Berti-Montalto** '16,
  Gravity-Capillary Water Waves
- **Baldi-Berti-Haus-Montalto**
  Gravity Water Waves '17

Theorem (Baldi, Berti, Haus, Montalto, Inventiones Math. 2018)

For every choice of finitely many tangential sites $\mathbb{S} \subset \mathbb{N} \setminus \{0\}$, *there exists* $\bar{s} > \frac{|\mathbb{S}|+1}{2}$, $\varepsilon_0 \in (0,1)$ *such that: for all* $\xi_j \in (0, \varepsilon_0^2)$, $j \in \mathbb{S}$, $\exists$ *a Cantor like set* $\mathcal{G}_\xi \subset [h_1, h_2]$ *with asymptotically full measure as* $\xi \to 0$, *i.e.* $\lim_{\xi \to 0} |\mathcal{G}_\xi| = h_2 - h_1$, *such that, for any depth* $h \in \mathcal{G}_\xi$, *the* GRAVITY WATER WAVES EQUATION *has a quasi-periodic standing wave solution* $(\eta, \psi) \in H^{\bar{s}}$ *of the form*

$$\eta(\tilde{\omega}_j t, x) = \sum_{j \in \mathbb{S}} \sqrt{\xi_j} \cos(\tilde{\omega}_j t) \cos(jx) + o(\sqrt{|\xi|})$$

$$\psi(\tilde{\omega}_j t, x) = -\sum_{j \in \mathbb{S}} \sqrt{\xi_j} \omega_j^{-1} \sin(\tilde{\omega}_j t) \cos(jx) + o(\sqrt{|\xi|})$$

with frequencies $\tilde{\omega}_j$ *satisfying* $\tilde{\omega}_j - \omega_j(h) \to 0$ *as* $\xi \to 0$.
The solutions are **linearly stable**.

Ideas of proof. Step 1) Poincaré-Birkhoff normal form

With a **bounded** and **invertible** (non symplectic) change of variables we transform water-waves with $h = +\infty$ into

Proposition (Poincaré-Birkhoff Normal Form for gravity WW)

$$\partial_t z = -\mathrm{i}|D|^{\frac{1}{2}} z - \zeta(z)\partial_x z + r_{-\frac{1}{2}}(z; D)z + R^{res}(z) + \mathcal{X}_{\geq 4}(z) \quad (\star)$$

1. $\zeta(z) = \sum_{n\neq 0} n|n||z_n|^2 \in \mathbb{R}$, **constant in** x *and* **integrable**
2. $r_{-\frac{1}{2}}(z;\xi) = \sum_{n\neq 0} r_n(\xi)|z_n|^2$ *symbol of order* $-\frac{1}{2}$ **constant in** x *and* **integrable**
3. $\|R^{res}(z)\|_{\dot{H}^{2s}} \lesssim_s \|z\|_{s_0}\|z\|^2_{\dot{H}^s}$, $s \geq s_0$ **regularizing**
4. $\mathcal{X}_{\geq 4}(z) = O(z^4)$ *has order* 1, *it has energy estimates in* H^s
5. $(\star)$ *is in Poincaré-Birkhoff normal form up to* $O(z^4)$

$(\star)$ is in Poincaré-Birkhoff normal form

The cubic vector field

$$\zeta(z)\partial_x z + r_{-\frac{1}{2}}(z; D)z + R^{res}(z)$$

commutes with $\mathrm{i}|D|^{\frac{1}{2}}$, i.e. in Fourier coordinates $(\star)$ is

$$\dot{z}_n = \mathrm{i}\sqrt{|n|}z_n + \sum_{\substack{\sigma_1 n_1+\sigma_2 n_2+\sigma_3 n_3=n, \sigma_j=\pm, \\ \sigma_1\sqrt{|n_1|}+\sigma_2\sqrt{|n_2|}+\sigma_3\sqrt{|n_3|}=\sqrt{|n|}}} a_{n_1,n_2,n_3}^{\sigma_1,\sigma_2,\sigma_3} z_{n_1}^{\sigma_1} z_{n_2}^{\sigma_2} z_{n_3}^{\sigma_3}, \ \forall n \in \mathbb{Z}\setminus\{0\}$$

- **Notation:** $z^+ = z$, $z^- = \bar{z}$

Rem. 1) $\zeta(z)\partial_x z$ and $r_{-\frac{1}{2}}(z; D)z$ are in Birkhoff normal form: formed by cubic monomial vector fields $z_k \bar{z}_k z_n \partial_{z_n}$

Rem. 2) Benjamin-Feir $z_{-m^2}\overline{z_{(m+1)^2}}z_{(m^2+m+1)^2}\partial_{z_{(m+1)^2 m^2}}$, $m \in \mathbb{N}$

Step 1-I) Birkhoff normal form up to regularizing terms

Performing paradifferential changes of variables, and thanks to algebraic properties of WW, we transform WW in Birkhoff normal form up to smoothing remainders:

$$\partial_t z = \zeta(z)\partial_x z + \mathrm{i}|D|^{\frac{1}{2}} z + r_{-\frac{1}{2}}(z;D)z + R(z) + \mathcal{X}_{\geq 4}(z)$$

where

1. $\zeta(z) = \sum_{n\neq 0} n|n||z_n|^2 \in \mathbb{R}$, **constant in** x and **integrable**
2. $r_{-\frac{1}{2}}(z;\xi) = \sum_{n\neq 0} r_n(\xi)|z_n|^2$ symbol of order $-\frac{1}{2}$ **constant in** x and **integrable**
3. $R(z)$ is a smoothing vector field, $\|R(z)\|_{2s} \lesssim_s \|z\|_s^2$
4. $\mathcal{X}_{\geq 4}(z)$ admits energy estimates since it has a purely imaginary symbol and it is of degree $O(z^4)$

Step 1-II) Poincaré-Birkhoff normal form on $R(z)$

Eliminate all the *quadratic* and *cubic* terms in $R(z)$ which are Birkhoff non-resonant

The loss of derivatives induced by the four-wave-interactions

$$|\omega_{n_1} + \omega_{n_2} - \omega_{n_3} - \omega_n| \geq \frac{1}{\max\left(|n_1|, |n_2|, |n_3|, |n|\right)^\tau}$$

when the left hand side is not zero, are compensated by the fact that $R(z)$ is smoothing

Step 2) Identification of the normal form

A purely algebraic unicity argument proves that

$$-\zeta(z)\partial_x z + r_{-\frac{1}{2}}(z; D)z + R^{res}(z) = \mathrm{i}\partial_{\bar{z}} H^{(4)}_{ZD}$$

where $H^{(4)}_{ZD}$ is the fourth order formal Birkhoff normal form Hamiltonian computed in Zakharov-Dyanchenko and Craig-Workfolk

Remark 1. we do not make symplectic transformations but the third order Birkhoff normal form is a-posteriori Hamiltonian

Idea of proof

$$X_H = X_{H_2} + X_{H_3} + X_{H_4} + \dots$$

We did several transformations which admit a Taylor expansion in u

Regard it as the formal time 1-flow generated by the vector field

$$S = S_2 + \theta S_3 + \dots$$

Transformed vector field

$$X_{H^2} + \underbrace{X_{H_3} + [S_2, X_{H_2}]}_{quadratic}$$

$$+ \underbrace{X_{H_4} + [S_2, X_{H_3}] + \frac{1}{2}[S_2, [S_2, X_{H_2}]] + \frac{1}{2}[S_3, X_{H_2}]}_{cubic} + \cdots$$

$$\Longrightarrow X_{H_3} + [S_2, X_{H_2}] = 0\,,$$

$$\mathcal{X}_3 := X_{H_4} + [S_2, X_{H_3}] + \frac{1}{2}[S_2, [S_2, X_{H_2}]] + \frac{1}{2}[S_3, X_{H_2}]$$

$$= -\zeta(z)\partial_x z + r_{-\frac{1}{2}}(z; D)z + R^{res}(z)$$

Since the adjoint operator $[\,\cdot\,, X_{H_2}]$ acting on quadratic monomial vector fields satisfying momentum conservation is bijective

$$S_2 = X_{F_3}\,, \quad H_3 + \{F_3, H_2\} = 0.$$

$$\Pi_{\rm ker}\Big(u_{j_1}^{\sigma_1} u_{j_2}^{\sigma_2} u_{j_3}^{\sigma_3} \partial_{u_j^\sigma}\Big) :=$$

$$\begin{cases} u_{j_1}^{\sigma_1} u_{j_2}^{\sigma_2} u_{j_3}^{\sigma_3} \partial_{u_j^\sigma} & \text{if } \; -\sigma\omega(j) + \sigma_1\omega(j_1) + \sigma_2\omega(j_2) + \sigma_3\omega(j_3) = 0 \\ 0 & \text{otherwise}\,, \end{cases}$$

$$\begin{aligned} \mathcal{X}_3 = \Pi_{\rm ker}(\mathcal{X}_3) &= \Pi_{\rm ker}\Big(X_{H_4} + [X_{F_3}, X_{H_3}] + \frac{1}{2}[X_{F_3}, [X_{F_3}, X_{H_2}]]\Big) \\ &= \Pi_{\rm ker} X_{H_4 + \{F_3, H_3\} + \frac{1}{2}\{F_3, \{F_3, H_2\}\}} \end{aligned}$$

because $\Pi_{\rm Ker}[S_3, X_{H_2}] = 0$. This is the usual Hamiltonian normal form of ZD.

Thanks for your attention!

Theorem (Berti-Feola-Franzoi, '19)

For any $\kappa, g, h > 0$ there is $s_0 > 0$ and, for any $s \geq s_0$., there are $\varepsilon_0 > 0$, $c > 0$, $C > 0$ such that, for any $\varepsilon \in]0, \varepsilon_0[$, any (η_0, ψ_0) in $H_0^{s+\frac{1}{4}}(\mathbb{T}, \mathbb{R}) \times \dot{H}^{s-\frac{1}{4}}(\mathbb{T}, \mathbb{R})$ with

$$\|\eta_0\|_{H_0^{s+\frac{1}{4}}} + \|\psi_0\|_{\dot{H}^{s-\frac{1}{4}}} < \varepsilon$$

the gravity-capillary water waves equations have a unique classical solution

$$(\eta, \psi) \in C^0(] - T_\varepsilon, T_\varepsilon[, H_0^{s+\frac{1}{4}}(\mathbb{T}, \mathbb{R}) \times \dot{H}^{s-\frac{1}{4}}(\mathbb{T}, \mathbb{R}))$$

with

$$T_\varepsilon \geq c\varepsilon^{-2}$$

satisfying the initial condition $(\eta, \psi)_{|t=0} = (\eta_0, \psi_0)$.

Theorem (Berti-Feola-Franzoi, '19)

For all $\kappa, g, h > 0$, there exists a bounded change of variables in H^s which transforms the gravity-capillary water-waves equations into

$$\partial_t z = \mathrm{i}|D|^{\frac{1}{2}} z + \mathrm{i}\partial_{\bar z} H^{(3)}_{BNF} + \mathcal{X}_{\geq 3}(z)$$

where

$$H^{(3)}_{BNF} = \sum_{\substack{\sigma_1 j_1 + \sigma_2 j_2 + \sigma_2 j_3 = 0 \\ \sigma_1\omega(j_1) + \sigma_2\omega(j_2) + \sigma_3\omega(j_3) = 0}} H^{\sigma_1,\sigma_2,\sigma_3}_{j_1,j_2,j_3} z^{\sigma_1}_{j_1} z^{\sigma_2}_{j_2} z^{\sigma_3}_{j_3}$$

and $\mathcal{X}_{\geq 3}(z)$ has energy estimates in H^s:

$$\mathrm{Re} \int_{\mathbb{T}} |D|^s \mathcal{X}_{\geq 3} \cdot \overline{|D|^s z}\, dx \leq C \|z\|^4_{\dot H^s} .$$

Wilton-ripples: integer solutions $k_1, k_2, k_3 \in \mathbb{Z} \setminus 0$ of

$$\begin{cases} \omega_{k_1} \pm \omega_{k_2} \pm \omega_{k_3} = 0 \\ k_1 \pm k_2 \pm k_3 = 0 \end{cases} \quad \text{are finitely many, } \max_{|k_1|,|k_2|,|k_3|} \leq C$$

Setting $z_L := \sum_{|j|\leq C} z_j e^{\mathrm{i}jx}$, $z_H := \sum_{|j|>C} z_j e^{\mathrm{i}jx}$

$$\begin{cases} \dot z_L = \mathrm{i}\omega(D) z_L + \mathrm{i}\partial_{\bar z} H^{(3)}_{BNF}(z_L) + \Pi_L(\mathcal{X}_{\geq 3}) \\ \dot z_H = \mathrm{i}\omega(D) z_H \qquad\qquad\qquad\qquad + \Pi_H(\mathcal{X}_{\geq 3}) \,. \end{cases}$$

$$\{H^{(3)}_{BNF}, H^{(2)}\} = 0\,, \quad H_2(z) = \sum_{j\neq 0} \omega_j |z_j|^2\,, \quad H_2(z_L(t)) = H_2(z_L(0))$$

Then

$$\|z(t)\|^2_{\dot H^s} \leq C(s)\|z(0)\|^2_{\dot H^s} + C(s)\int_0^t \|z(\tau)\|^4_{\dot H^s}\, d\tau\,, \quad \forall t \in [0, T]\,.$$