Blowup dynamics via energy concentration and their stability properties for nonlinear wave equations

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Pure power NLW

- Consider the focussing/defocussing nonlinear wave equation(NLW)

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- Interesting because we we have an energy sub-critical ($p < 5$), energy critical ($p = 5$), and energy super-critical($p > 5$) regime.
- Problem extensively studied in the **defocussing energy critical and subcritical cases** and completely understood there as far as regularity is concerned. However, the situation is much more delicate in the *focussing case* as well as for *both focussing and defocussing in the supercritical case*.

Pure power NLW ; supercritical defocussing case

- In defocussing case, the conserved energy is positive definite :

$$E = \int_{\mathbf{R}^3} \left[\frac{1}{2}|\nabla_{t,x} u|^2 + \frac{1}{p+1}|u|^{p+1}\right] dx$$

Hence singularities would have to be by a 'concentration scenario'.

- It is well-known folklore, backed by a number of examples, that a promising way to build blow ups is by a *self-similar ansatz*

$$u(t,x) = t^{-\frac{2}{p-1}} Q\left(\frac{x}{t}\right),$$

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- For example, when $p = 7$, we get

$$(a^2 - 1)Q''(a) + \left(\frac{8}{3}a - \frac{2}{a}\right)Q'(a) + \frac{4}{9}Q(a) + Q(a)^7 = 0, \; a = \frac{|x|}{t}.$$

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- Imposing boundary conditions $Q(0) = q_0 > 0$, $Q'(0) = 0$, one finds a smooth solution on $[0, 1)$, which behaves like

$$q_1(1-a)^{\frac{2}{3}} + q_2 + o((1-a)^{\frac{2}{3}})$$

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$$\dot{H}^{\frac{7}{6}}$$
- The problem $\Box u = u^7$ on $\mathbf{R}^{3+1}$ is strongly locally well-posed in $H^{\frac{7}{6}}$ and ill-posed below this regularity. Thus simple minded self-similar solutions live just outside the natural well-posedness class.

Pure power NLW ; supercritical focussing case

- The *focussing case* for $p = 7$ behaves fundamentally different. Similar equation for Q :

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- Work by Bizon-Maison-Wasserman('07) shows that there is a *countable family* of q_0 for which there is a *smooth solution across the light cone.* The first of these is the constant q_* such that

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- For the remaining values of q one again encounters the $(1-a)^{\frac{2}{3}}$ singularity across the light cone. Hence *either* C^∞ *or not even minimal regularity*. The infinite family of self-similar solutions is supercritical phenomenon !

Energy critical nonlinear waves

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$$\Box u = -u(-|u_t|^2 + |\nabla_x u|^2) \tag{1}$$

- One can try to find self-similar solution $u(t,x) = W(\frac{x}{t})$. In fact

$$W(a) = Q\left(\frac{a}{1+\sqrt{1-a^2}}\right)$$

works where $Q : \mathbf{R}^2 \longrightarrow S^2$ is a harmonic map. This is of infinite energy, and hence just fails to be in the largest space for which the problem has a meaningful local well-posedness theory.

Singularities for energy critical Wave Maps I

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$$-U_{tt} + U_{rr} + \frac{1}{r}U_r = \frac{\sin(2U)}{2r^2}. \quad (2)$$

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- After some numerical works, it was proved about 10 years ago that (2) admits finite time singularities of the form $U(t, r) = Q(\lambda(t)r) + \epsilon(t, r)$, $Q(r) = 2\arctan r$ the ground state harmonic map.

Singularities for energy critical Wave Maps II

- Two approaches : *Raphael-Rodnianski approach* with C^∞ data giving stable within co-rotational class finite time blow up with essentially uniquely determined $\lambda(t) \sim t^{-1}e^{\sqrt{|\log t|}}$. *K.-Schlag-Tataru approach* with $H^{1+\nu}$, $\nu > \frac{1}{2}$ regularity data with $\lambda(t) = t^{-1-\nu}$ prescribed. Here $\nu > \frac{1}{2}$ can be varied freely ! No stability assertion. Construction later generalised to all $\nu > 0$ (Gao-K. '13).

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- For small $\nu > 0$, the stability of the K.-Schlag-Tataru solutions was established recently by K.-Miao('18) within the co-rotational class. Hence stability per se does not select a unique blow up speed.
- Both methods of construction have been applied in numerous other contexts, on which more later.
- Both methods rely on the construction of an approximate solution via suitable elliptic auxiliary problems, which then get completed to an exact solution via an iterative scheme using

Singularities for energy critical Wave Maps : Raphael-Rodnianski

- The idea is to depart from an ansatz $u(t,r) \sim Q_b(\lambda(t)r)$, where $b = b(t) \sim \frac{1}{\lambda(t)\cdot t}$ is a new time dependent parameter whose precise evolution will be determined uniquely. The choice $b = 1$ in fact corresponds to the self-similar infinite energy profile, and Q_b will be chosen to be an approximate solution. For fixed b, the equation is

$$-\triangle Q_b + b^2 D(\Lambda Q_b) + \frac{\sin(2Q_b)}{2r^2},\ \Lambda = r\partial_r,\ D = 1 + \Lambda$$

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- Instead of solving this exactly, one tries a finite expansion

$$Q_b = \sum_{l=0}^{j} b^{2l} T_l = \sum_{l=0}^{j} \frac{1}{(\lambda t)^{2l}} T_l,\ T_0 = Q$$

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- This is then completed to an exact solution $Q_b + \epsilon$.

Singularities for energy critical Wave Maps : K.-Schlag-Tataru

- Here one also builds an approximate solution $u_{app}(t,r) = u_0 + \sum_{j=1}^{N} v_j$, where $u_0 = Q(\lambda(t)r)$.

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- However, the v_j are chosen very differently. To begin with, one fixes $\lambda(t) = t^{-1-\nu}$. Then one seeks functions v_j of a special form. Specifically, one sets essentially.

$$v_{2j}(t,r) = \frac{1}{(\lambda \cdot t)^j} q_j(a),\ a = \frac{r}{t},$$

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  the idea being that the v_{2j} model the behaviour near the light cone.
- One gets an ode for q_j which is singular both at $a = 0$ and $a = 1$:

$$(1-a^2)\partial_a^2 q_j + [a^{-1} - 2(2+j\nu)a]\partial_a q_j + c_j q_j = h_j$$

Singularities for energy critical Wave Maps : comparison of the solutions

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- A fundamental difference between the approximations is that for the KST solutions the v_j are chosen to vanish to *third order* at the origin $R = 0$. On the other hand, the correction T_1 in the Raphael-Rodniansi ansatz only vanishes the first order at $R = 0$.

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- This is of course related to the fact that the blow up rate in the Raphael-Rodnianski approach is basically determined, while the rate in the KST approach is variable.

How to characterise the blow up solutions I

- It is natural to suspect that imposing C^∞ *smoothness* or also lesser (but more than H^{1+}) smoothness on the data as well as stability of blow up (within the co-rotational class) will essentially select the Raph.-Rod. solutions :
  Conjecture : *If* $u(t,r) = Q(\lambda(t)r) + \epsilon(t,r)$ *is a finite time blow up (at* $t = 0$*) with* C^∞ *initial data, then we have*
  $$\lambda(t) = t^{-j} \cdot f(|\log t|),\ j \in \mathbf{N}$$
  where $f(|logt|)$ *stands for a logarithmic correction. Only the case* $j = 1$ *leads to stable (within the co-rotational class) blow ups.*

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  where $f(|logt|)$ stands for a logarithmic correction. Only the case $j = 1$ leads to stable (within the co-rotational class) blow ups.
- Characterizing the non-smooth KST blow ups may involve the *regularity of the outgoing radiation* past the blow up point. While the KST solutions are constructed within the backward light cone at the singularity, it is natural to expect that ϵ may be continued in the $H^{1+\nu-}$-topology up to $t = 0$ (blow up).

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- **Conjecture** : *Assume that the finite time blow up (at* $t = 0$*) solution* $u(t,r) = Q(\lambda(t)r) + \epsilon(t,r)$ *is such that* $\lim_{t \to 0} \epsilon(t,r) = \epsilon_* \in H^{1+\nu}$, $\lim_{t \to 0} \partial_t \epsilon(t,r) = \epsilon_{**} \in H^{\nu}$, $\nu > 0$. *Then we have*

$$\lambda(t) \geq t^{-1-\nu}$$

for t *sufficiently small.*

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- **Remark** : Observe that due to work by Duyckaerts-Jia-Kenig-Merle one has the *soliton resolution* for finite time blow up solutions which are sufficiently close to Q. This, however, is only in the energy topology, while here a finer topology is being imposed.

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- Results in this direction have been proved for the energy critical focussing NLW $\Box u = -u^{\frac{n+2}{n-2}}$ and type II blow ups in dimensions $n = 3, 4, 5$ by J. Jendrej, but these results are still far from the precision in the above conjecture.

How to characterise the blow up solutions III

- In $n = 3$, the energy critical focussing NLW $\Box u = -u^5$ admits a static solution $W(x) = \left(1 + \frac{|x|^2}{3}\right)^{-\frac{1}{2}}$, which is analogous to Q for critical Wave Maps.

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- K.-Schlag-Tataru('09) constructed finite time blow ups of the form

$$u(t,x) = \lambda^{\frac{1}{2}}(t)W(\lambda(t)x) + \epsilon(t,x),\ \lambda(t) = t^{-1-\nu},$$

with $\epsilon \in H^{1+\frac{\nu}{2}-}$. Similar construction as for critical WM.

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Theorem(Jendrej '15) *If the radiation* $(\epsilon_*, \epsilon_{**}) \in H^2 \times H^1$, *then* $\lambda(t) \geq t^{-\frac{4}{3}}$.
- The KST solutions require $\lambda(t) \geq t^{-3}$ to get H^2 radiation part at blow up time.

Comparison to the harmonic map heat flow

- The parabolic analogue of the co-rotational Wave Maps is the following parabolic problem :

$$u_t = u_{rr} + \frac{1}{r}u_r - \frac{\sin 2u}{2r^2}.$$

admits a quantised set of blow up scales $\lambda(t) \sim \frac{t^k}{|\log t|^{c_k}}$, discovered heuristically by VandenBerg-Hulshof-King, and constructed rigorously by Raphael-Schweyer('13).

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- The continuum of blow up rates for the Wave Maps model is clearly impossible here, due to parabolic smoothing. The quantised rates for the parabolic rates correspond roughly to the (conjectured) quantised set of $\lambda(t)$ corresponding to C^∞ smooth data resulting in finite time blow up for Wave Maps.

Stability of blow up solutions

- An important issue already alluded to is the *stability* of these blow up dynamics . For Wave Maps, given that in the energy critical case thus far all blow ups have been obtained in a suitable symmetry reduction (co-rotational or higher equivariance class), there are **two** questions.

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- Here the the second question is significantly harder than the first since it deals with a system, while the co-rotational problem is scalar. Moreover, the co-rotational problem is truly semilinear (in the sense of no derivatives in nonlinearity), while the second problem does have derivatives.

Stability of blow up solutions

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- **(1)** Symmetry under co-rotational perturbations. **(2)** Symmetry under general perturbations.
- Here the the second question is significantly harder than the first since it deals with a system, while the co-rotational problem is scalar. Moreover, the co-rotational problem is truly semilinear (in the sense of no derivatives in nonlinearity), while the second problem does have derivatives.
- Stability *within co-rotational class* of KST solutions for small enough $\nu > 0$ (recall $\lambda(t) = t^{-1-\nu}$) is a recent result of K.-Miao ('18). It is based on Fourier methods.

Stability of KST solutions I

- Let $u_\nu = Q(\lambda(t)r) + \epsilon_\nu(t,r)$ be one of the co-rotational Wave Maps blow ups constructed by KST with $\lambda(t) = t^{-1-\nu}$, $\nu > 0$ on some internal interval $(0, t_0]$, $t_0 > 0$.

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- **Theorem**(K.-Miao '18) Let $\nu > 0$ be small enough. Let $(\epsilon_0(r), \epsilon_1(r))$ be such that the maps $(r,\theta) \longrightarrow \epsilon_j(r)e^{i\theta}$, interpreted as maps from $\mathbf{R}^2$ into itself have sufficiently small $H^4_{\mathbf{R}^2}$, resp. $H^3_{\mathbf{R}^2}$-norm. In particular, we have $\epsilon_j(0) = 0$, $j = 0, 1$. Then the perturbed data

$$\big(u_\nu(t_0) + \epsilon_0,\, \partial_t u_\nu(t_0) + \epsilon_1\big)$$

lead to another finite time blow up solution of the form

$$u(t,r) = Q(\lambda(t)r) + \epsilon_1(t,r).$$

Thus the KST blow up is stable (under suitable co-rotational perturbations) for $\nu > 0$ small enough.

Stability of KST solutions, remarks

- A puzzling feature here is the fact that the perturbed solutions *blow up in the same space time location*, which may sound paradoxical at first. This is due to the shock the solution experiences across the light cone, and which imparts a certain rigidity to these solutions ; the functions $\epsilon_{0,1}$ are smoother than the solutions u_ν being perturbed, thus not affecting the shock.

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- A similar result, but co-dimension one stability, holds for the KST solutions with $\nu > 0$ small enough in the context of $\Box u = -u^5$ (Burzio-K. '17).
- There are special features in the context of critical Wave Maps, first observed in work by Gustafson-Kang-Tsai, which lead to a more conceptual and elegant method there.

Stability of KST solutions, proof of stability

- Let $u_\nu(t,r) = Q(\lambda(t)r) + \eta(t,r)$ the given blow up solution to be perturbed, and let $u_\nu + \epsilon(t,r)$ be the perturbation. One gets the equation

$$\big(-\partial_{tt} + \partial_{rr} + \frac{1}{r}\partial_r\big)\epsilon - \frac{\cos(2Q(\lambda(t)r))}{r^2}\epsilon = N(\epsilon),$$

with

$$N(\epsilon) = \frac{\cos(2u_\nu) - \cos(2Q(\lambda(t)r))}{r^2} + \frac{\sin(2u_\nu)}{2r^2}(\cos(2\epsilon) - 1) + \frac{\cos(2u_\nu)}{2r^2}(\sin(2\epsilon) - 2\epsilon)$$

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- Passage to the new coordinates $\tau = \int_t^\infty \lambda(s)\,ds$, $R = \lambda(t)r$ leads to time independent potential term.

First setup of perturbation problem

- Write $\tilde{\epsilon}(\tau, R) = R^{\frac{1}{2}}\epsilon(t(\tau), r(\tau, R))$. Obtain perturbation problem

$$\Big(-(\partial_\tau + \frac{\lambda_\tau}{\lambda} R\partial_R)^2 + \frac{1}{4}(\frac{\lambda_\tau}{\lambda})^2 + \frac{1}{2}\partial_\tau(\frac{\lambda_\tau}{\lambda})\Big)\tilde{\epsilon} - \mathcal{L}\tilde{\epsilon}$$
$$= \lambda^{-2}(\tau) R^{\frac{1}{2}} N(\epsilon)$$

where $\mathcal{L}$ is a singular operator

$$\mathcal{L} = -\partial_R^2 + \frac{3}{4R^2} - \frac{8}{(1+R^2)^2}$$

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- $Spec(\mathcal{L}) = [0, \infty)$, purely absolutely continuous. There is a resonance at zero

$$R^{\frac{1}{2}}\partial_\lambda Q(\lambda r)|_{\lambda=1} = \phi_0(R) = \frac{R^{\frac{3}{2}}}{1+R^2}$$

First setup of perturbation problem

- One can associate a Fourier base to $\mathcal{L}$, of the form

$$\phi(R,\xi) = \phi_0(R)(1+O(R^2\xi)),\ R^2\xi \lesssim 1,\ \phi(R,\xi) \sim \sum_{\pm} a_{\pm}(\xi)\frac{e^{\pm iR\xi^{\frac{1}{2}}}}{R^{\frac{1}{2}}\xi^{\frac{3}{4}}},$$

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- Then one gets a associated Fourier representation

$$f(R) = \int_0^\infty \phi(R,\xi)x(\xi)\rho(\xi)\,d\xi,\ x(\xi) = \langle f, \phi(R,\xi)\rangle_{L^2_{dR}}$$

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- This causes complications when controlling the low frequency contributions.

Some remarkable identities

- From now on, we shall use the notation $\phi_0(R) = \frac{R}{1+R^2}$, i. e. we divide by $R^{\frac{1}{2}}$ to go back to the $2-d$ setting.

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- Introduce the operator

$$\mathcal{D} = \partial_R + \frac{1}{R}\frac{R^2-1}{R^2+1},$$

as well as its adjoint (with respect to $\langle \cdot, \cdot \rangle_{L^2_{R\,dR}}$)

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- Then one has the relations

$$\mathcal{D}\phi_0(R) = 0,\ \mathcal{L} = \mathcal{D}^*\mathcal{D}.$$

Introduction of better variables

- Formally one has an inverse of $\mathcal{D}$ given by

$$\phi(g) = \phi_0(R) \int_0^R (\phi_0(s))^{-1} g(s)\, ds$$

One then infers a representation of $\epsilon(\tau, R)$ as

$$\epsilon(\tau, R) = \phi(\mathcal{D}\epsilon) + c(\tau)\phi_0(R).$$

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- The function $\mathcal{D}\epsilon$ solves a wave equation with an elliptic operator which has a much improved spectral representation associated with it.
- On the other hand, the function $c(\tau)$ satisfies a fairly simple ODE.

The system for $\mathcal{D}\epsilon$, $c(\tau)$

- Applying $\mathcal{D}$ to the equation for ϵ and computing a bunch of commutators, one finds

$$
\begin{aligned}
&-\left((\partial_\tau + \frac{\lambda_\tau}{\lambda} R\partial_R)^2 + 3\frac{\lambda_\tau}{\lambda}(\partial_\tau + \frac{\lambda_\tau}{\lambda} R\partial_R)\right)\mathcal{D}\epsilon - \tilde{\mathcal{L}}\mathcal{D}\epsilon \\
&= \lambda^{-2}\mathcal{D}\left(N(\epsilon)\right) - \frac{4R}{(R^2+1)^2}\left(2\left(\frac{\lambda'}{\lambda}\right)^2 + \left(\frac{\lambda'}{\lambda}\right)'\right)\epsilon \\
&- \frac{\lambda'}{\lambda}\frac{4R}{(R^2+1)^2}\left(\partial_\tau + \frac{\lambda'}{\lambda}R\partial_R\right)\epsilon - \frac{\lambda'}{\lambda}\left(\partial_\tau + \frac{\lambda'}{\lambda}R\partial_R\right)\left(\frac{4R}{(R^2+1)}\right. \\
&+ \left(2\left(\frac{\lambda'}{\lambda}\right)^2 + \left(\frac{\lambda'}{\lambda}\right)'\right)\mathcal{D}\epsilon \\
&=: \lambda^{-2}\mathcal{D}\left(N(\epsilon)\right) + \mathcal{R}(\epsilon, \mathcal{D}\epsilon) + \left(2\left(\frac{\lambda'}{\lambda}\right)^2 + \left(\frac{\lambda'}{\lambda}\right)'\right)\mathcal{D}\epsilon.
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- Here $\tilde{\mathcal{L}} = \mathcal{D}\mathcal{D}^*$ has a much better spectral representation.

Spectral theory associated with $\tilde{\mathcal{L}}$

- The generalised eigenfunctions $\phi(R,\xi)$ associated with $\tilde{\mathcal{L}}$ can be easily computed from those used previously for $\mathcal{L}$. Calling the latter $\phi_{KST}(R,\xi)$, we can set

$$\phi(R,\xi) = \xi^{-1}\mathcal{D}\big(R^{-\frac{1}{2}}\phi_{KST}\big).$$

Also, define $\tilde{\rho}(\xi) := \xi\rho(\xi)$, so we now have the asymptotics

$$\tilde{\rho}(\xi) \sim \log^{-2}(\xi),\ \xi \ll 1.$$

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- Then we get the Fourier representation (due first to Bejenaru-K.-Tataru'13)

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- We shall represent the function $\mathcal{D}\epsilon$ in terms of the basis $\phi(R,\xi)$, and translate things to the Fourier side.

Translation of equation to Fourier side

- The issue comes up that the operator $R\partial_R$ occurring in the equation for $\mathcal{D}\epsilon$ does not translate to $-2\xi\partial_\xi$. Instead we have the relation

$$\mathcal{F}\big(R\partial_R f\big) + 2\xi\partial_\xi\mathcal{F}(f) = (-2+\mathcal{K}_0)\mathcal{F}(f),$$

where the operator $\mathcal{K}_0$ is again a Calderon-Zygmund operator of the form

$$\mathcal{K}_0 g(\xi) = \int_0^\infty \frac{F(\xi,\eta)\tilde{\rho}(\eta)}{\xi-\eta} g(\eta)\,d\eta,$$

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with F satisfying certain decay bounds.

- Write now $\mathcal{D}\epsilon(\tau, R) = \int_0^\infty \phi(R,\xi)x(\tau,\xi)\tilde{\rho}(\xi)\, d\xi$.
- Introduce the operator $\mathcal{D}_\tau = \partial_\tau - 2\frac{\lambda_\tau}{\lambda}\xi\partial_\xi - \frac{\lambda_\tau}{\lambda}$.

The equation on Fourier side I

- The equation for x now becomes

$$\left(\mathcal{D}_\tau^2 + \frac{\lambda_\tau}{\lambda}\mathcal{D}_\tau + \xi\right)x(\tau,\xi) = \mathcal{F}\left(\lambda^{-2}(\tau)\mathcal{D}N(\epsilon)\right) + \mathcal{R}(x,\,\tau), \quad (3)$$

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$$(\frac{\lambda_\tau}{\lambda})^2\mathcal{K}_0 x,\ \frac{\lambda_\tau}{\lambda}\mathcal{K}_0\mathcal{D}_\tau x,$$

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  as well as similar ones morally equivalent to these,
- The transport operator on the left in (3) again admits a rather simple explicit propagator. Translated to the physical side, it will not lead to any growth.

The equation on Fourier side I

- $\left(\mathcal{D}_\tau^2 + \frac{\lambda_\tau}{\lambda}\mathcal{D}_\tau + \xi\right)x(\tau,\xi) = 0,\ \left(x(\tau_0,\cdot), \mathcal{D}_\tau x(\tau_0,\cdot)\right) = (x_0, x_1)$ solved by

$$x(\tau,\xi) = \frac{\lambda(\tau)}{\lambda(\tau_0)} \cos\left(\lambda(\tau)\xi^{\frac{1}{2}} \int_{\tau_0}^{\tau} \lambda(u)^{-1} du\right) x_0\left(\frac{\lambda(\tau)^2}{\lambda(\tau_0)^2}\xi\right)$$
$$+ \xi^{-\frac{1}{2}} \sin\left(\lambda(\tau)\xi^{\frac{1}{2}} \int_{\tau_0}^{\tau} \lambda(u)^{-1} du\right) x_1\left(\frac{\lambda(\tau)^2}{\lambda(\tau_0)^2}\xi\right)$$

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- This leads to good weighted energy type bounds. Let

$$\left\|x(\xi)\right\|_{S_0} := \left\|\langle\xi\rangle^{2+\kappa}\xi^{\frac{1}{2}}\langle\log\xi\rangle^{-1-\kappa}x(\xi)\right\|_{L^2_{d\xi}}$$

Then the following energy type bound applies :

Bounds for the parametrix

- We have the bound

$$\sup_{\tau\geq\tau_0}\frac{\lambda(\tau)}{\lambda(\tau_0)}\langle\log(\frac{\lambda(\tau)}{\lambda(\tau_0)})\rangle^{1+\kappa}[\big\|x(\tau,\cdot)\big\|_{S_0}+\big\|\xi^{-\frac{1}{2}}\mathcal{D}_\tau x(\tau,\cdot)\big\|_{S_0}]$$
$$\lesssim\big\|x_0\big\|_{S_0}+\big\|\xi^{-\frac{1}{2}}x_1\big\|_{S_0}.$$

Recall that $\lambda(\tau)\sim\tau^{1+\nu^{-1}}$, so we get decay which becomes better when ν shrinks.

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$$\lesssim\big\|x_0\big\|_{S_0}+\big\|\xi^{-\frac{1}{2}}x_1\big\|_{S_0}.$$

  Recall that $\lambda(\tau)\sim\tau^{1+\nu^{-1}}$, so we get decay which becomes better when ν shrinks.

- We also control the amplitude of ϵ via the norm $\|\cdot\|_{S_0}$. Specifically, putting

$$\epsilon(R)=\phi(\mathcal{D}\epsilon),\ \mathcal{D}\epsilon(R)=\int_0^\infty\phi(R,\xi)x(\xi)\tilde{\rho}(\xi)\,d\xi,$$

  we get the bound

$$\Big\|\frac{\epsilon(R)}{\langle\log R\rangle R}\Big\|_{L^\infty_{dR}}\lesssim\big\|x\big\|_{S_0}$$

The equation for the resonant part $c(\tau)$.

- Recall that we write $\epsilon(\tau, R) = \phi(\mathcal{D}\epsilon) + c(\tau)\phi_0(R)$. The first term here behaves like $O(R^3)$ at the origin, while the second one is $O(R)$ only. This suffices to extract the equation for $c(\tau)$ from the equation for ϵ.

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- Equation governing resonant mode :

$$(\partial_\tau + \frac{\lambda_\tau}{\lambda})^2 c(\tau) + \frac{\lambda_\tau}{\lambda}(\partial_\tau + \frac{\lambda_\tau}{\lambda})c(\tau) + h(\tau) + n(\tau) = 0,$$

where

$$h(\tau) = \lim_{R\to 0} R^{-1}\mathcal{D}^*\mathcal{D}\epsilon(\tau, R), \; n(\tau) = \lim_{R\to 0} R^{-1}\lambda^{-2}(\tau)N(\epsilon)(\tau, R).$$

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- The operator $L_c := \left(\partial_\tau + \frac{\lambda_\tau}{\lambda}\right)^2 + \frac{\lambda_\tau}{\lambda}\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}\right)$ has fundamental system

$$\phi_1(\tau) = \tau^{-1-\nu^{-1}},\ \tau^{-1-\frac{2}{\nu}}$$

Setting up the final fixed point

- In the end, one works with Fourier coefficients $x(\tau, \xi)$ for $\mathcal{D}\epsilon$ which admit a decomposition of the form

$$x(\tau, \xi) = \sum_{N \geq k \geq 1, N \geq j \geq 1} \chi_{\xi > 1} a_{kj}^{\pm}(\tau) \frac{e^{\pm i\nu\tau\xi^{\frac{1}{2}}}}{\xi^{2+\frac{k\nu}{2}}} \langle \log \xi \rangle^{j} + \chi_{\xi > 1} \frac{b(\tau, \xi)}{\xi^{\frac{5}{2}+\frac{\nu}{2}-}}$$
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- To simplify the discussion, we shall pretend in the sequel that $x \in S_0$.
- The point is now to solve the system

$$\Big(\mathcal{D}_\tau^2 + \frac{\lambda_\tau}{\lambda}\mathcal{D}_\tau + \xi\Big)x(\tau,\xi) = \mathcal{F}\big(\lambda^{-2}(\tau)\mathcal{D}N(\epsilon)\big) + \mathcal{R}(x,\tau)$$

$$\Big(\partial_\tau + \frac{\lambda_\tau}{\lambda}\Big)^2 c(\tau) + \frac{\lambda_\tau}{\lambda}\Big(\partial_\tau + \frac{\lambda_\tau}{\lambda}\Big)c(\tau) + h(\tau) + n(\tau) = 0.$$

The final fixed point

- **Prop.** Let $\big\|(x_0, x_1)\big\|_{S_0\times S_1} + |c_0| + |c_1| < \delta_0$ sufficiently small and τ_0 sufficiently large. Then the preceding system admits a unique solution
$$\big(x(\tau,\xi), c(\tau)\big)$$
satisfying the bounds
$$\sup_{\tau\geq\tau_0} \frac{\lambda(\tau)}{\lambda(\tau_0)}\langle\log(\frac{\lambda(\tau)}{\lambda(\tau_0)})\rangle^{1+\frac{\kappa}{2}}[\big\|x(\tau,\cdot)\big\|_{S_0} + \big\|\mathcal{D}_\tau x(\tau,\cdot)\big\|_{S_1}]$$
$$+ \sup_{\tau\geq\tau_0} \frac{\lambda(\tau)}{\lambda(\tau_0)}\langle\log(\frac{\lambda(\tau)}{\lambda(\tau_0)})\rangle^{1+\frac{\kappa}{2}}\tau^{-2}[|c(\tau)| + \tau|c'(\tau)|]$$
$$\lesssim \delta_0$$
In turn
$$\epsilon(\tau, R) = \phi(\mathcal{D}\epsilon) + c(\tau)\phi_0(R)$$
gives the desired perturbation of u_ν.

Stability without equivariance ?

- A recent result by Duyckaerts-Jia-Kenig-Merle('16) characterises blow up solutions $u : \mathbf{R}^{2+1} \longrightarrow S^2$ whose data are close in energy to the family of ground states (i. e. Q up to the symmetries) and which blow up at the origin :

$$\vec{u}(t,x) = \mathcal{R}_{h(t)}^{\alpha(t),\beta(t)} \mathcal{L}_v(\vec{Q}_{\lambda(t)}) + \epsilon(t,x)$$

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- No such examples known, there is some debate whether there may be finite time blow up stable under non-corotational perturbations. It seems the conjecture is that the Raphael-Rodnianski blow ups are unstable, due to 'chaotic' behaviour of the angles $\alpha(t)$ etc.

Stability without equivariance ?

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where $\lim_{t\to 0} \alpha(t)$ *etc exist. Thus the bulk part does not 'spin out of control'.*
- *Strategy* : express perturbation in terms of a suitable frame for TS^2, and express the variables as Fourier series in the angular variable θ, resulting in a countable family of variables.

Other examples of similar blow ups I

- A number of unexpected further blow up constructions of similar nature were accomplished by Galina Perelman and collaborators.

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- *co-rotational critical Schroedinger maps* : $u : \mathbf{R}^{2+1} \longrightarrow S^2$, $u_t = u \times \triangle u$, admits solutions of the form

$$Q(\lambda(t)x) + \epsilon(t,x),\ \lambda(t) = t^{-\frac{1}{2}-\nu},\ \nu > 1.$$

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- *Energy critical focussing NLS* (G. Perelman and C. Ortoleva and G. Perelman, '12)
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Other examples of similar blow ups

- This is a *quasilinear equation* :

$$\partial_t\Big(\frac{u_t}{\sqrt{1-u_t^2+|\nabla u|^2}}\Big)-\sum_{j=1}^{n}\partial_{x_j}\Big(\frac{u_{x_j}}{\sqrt{1-u_t^2+|\nabla u|^2}}\Big)$$

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- Fixing spatial dimension to $2n$, one can consider rotationally symmetric surfaces parametrized by

$$(x,\omega)\in\mathbf{R}^n\times S^{n-1}\longrightarrow(x,u(t,x)\omega)$$

Further assume u radial, i. e. $u=u(t,\rho)$, $\rho=|x|$. Resulting equation

$$(1+u_\rho^2)u_{tt}-(1-u_t^2)u_{\rho\rho}-2u_tu_\rho u_{\rho t}+3(1+u_\rho^2-u_t^2)(u^{-1}-\frac{u_\rho}{\rho}).$$

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- $u(t,\rho)=\rho$ is called the *Simons cone*.

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- Ny work of Bombieri-De Giust-De Giorgi('69) it is known that the complement of the Simons cone can be foliated by a family of minimal surfaces $(aM)_{a>0}$, $(a\tilde{M})_{a>0}$, where M is given by

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- Here part of the problem is to identify the characteristic cone where the solution experiences a (higher order) shock, analogous to the construction for Wave Maps.