Asset Prices in Segmented and Integrated Markets

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New Challenges in Energy Markets
Data Analytics, Modelling and Numerics
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Outline

- Motivation:
  Market Integration, Financialization of Commodities
- Model:
  Equilibrium in Segmentation and Integration
- Results:
  Asset Prices, Interest Rates, and Welfare in Segmentation and Integration.
- Exogenous vs. Endogenous Integration.

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Financialization of Commodities

- Participation of institutional investors to commodity futures since 2004. (Buyuksahin et al., 2008), (Irwin and Sanders, 2011).
- Before 2004, commodity futures uncorrelated with equities and each other. (Bodie and Rosansky, 1980), (Gorton and Rouwenhorst, 2006).
- After, highly correlated with equities and each other: "Financialization". Larger effect on index components (Tang and Xiong, 2012).
- Correlations now low again (Bhardwaj, Gorton, and Rouwenhorst, 2015). Commodity investors negligible for prices? (Hamilton and Wu, 2015)
- Not much theory. Financialization from benchmarking (Basak and Pavlova, 2016). Iterative schemes (Chan, Sircar, and Stein, 2015)

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Market Integration and Orchards

- Asset pricing with multiple cash flows.
  Menzly et al. (2004), Santos and Veronesi (2006).
- International integration.
  Pavlova and Rigobon (2007), Bhamra, Coeurdacier, Guibaud (2014).
- Multiple Lucas trees.
  Cochrane, Longstaff, Santa-Clara (2007), Martin (2012).
- Volatility-stabilized models.
  Karatzas et al. (2005, 2008), Pal (2011), Cuchiero (2017)

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Islands and Trees

- Two islands.
- Two trees, one for each island.
- Each tree feeds its island. People on both islands are similar.
- Crops fluctuate independently, but have similar long-term growth.
- Perishable crops. Must be consumed immediately.
- Trees are the only property on the island.
- What is the price of each tree?
- What if a bridge is built?
- Find a model that is as simple as possible, but not simpler.

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Simplest – and simpler

- Natural attempt.
- Dividend streams as linear, independent Brownian motions:

$$D_t^{(1)} = D_0^{(1)} + \mu_1 t + \sigma_1 B_t^{(1)}$$
$$D_t^{(2)} = D_0^{(2)} + \mu_2 t + \sigma_2 B_t^{(2)}.$$

  Total dividend also linear Brownian motion.
- Exponential utility $U(x) = -e^{-\alpha x}$.
- Both in segmentation and integration, equilibrium prices of the form

$$P_t^{(1)} = a_1 + b_1 D_t^{(1)} \qquad P_t^{(2)} = a_2 + b_2 D_t^{(2)}$$

- Uncorrelated before, uncorrelated after. Nothing to see.
- Exponential utility does not see uncorrelated endowments.
- Model too simple to capture markets' interactions.

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One Tree

- Continuous-time version of Lucas' tree.
- One asset paying dividend stream D_t

$$dD_t = \mu D_t dt + \sigma D_t dB_t$$

- Representative agent with risk aversion γ and impatience β.
- Asset price and safe rate:

$$\frac{P_t}{D_t} = \frac{1}{r_0 - \mu + \gamma\sigma^2}$$

$$r_0 = \beta + \gamma\mu - \gamma(\gamma+1)\frac{\sigma^2}{2}$$

- Constant rate and price-dividend ratio.
- Price equal to expected, risk-adjusted discounted dividends.
- Problem with multiple trees:
  Dividends grow geometrically, consumption aggregation is additive.
- How to make it tractable?

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Sum and Share

- Geometric Brownian motion for total dividend.
  Jacobi process for dividend share of first region.

$$dD_t = \mu D_t dt + \sigma D_t dB_t^D$$
$$dX_t = \kappa(w - X_t)dt + \sigma\sqrt{X_t(1 - X_t)}dB_t^X$$

- $\mu, \sigma > 0$, $w \in (0, 1)$.
- B^D, B^X independent Brownian motions.
- To ensure $X_t \in (0, 1)$ a.s. for all t, assume

$$\frac{\sigma^2}{2\kappa} < w < 1 - \frac{\sigma^2}{2\kappa}$$

  Easy to satisfy for typical parameters.
- Note same parameter σ in both equations. Why?

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Dividends for Regions

- Implied dividend streams $D_t^{(1)} = D_t X_t$ and $D_t^{(2)} = D_t(1 - X_t)$

$$dD_t^{(1)} = ((\mu - \kappa w_2)D_t^{(1)} + \kappa w_1 D_t^{(2)})dt + \sigma\sqrt{D_t^{(1)}(D_t^{(1)} + D_t^{(2)})}dB_t^{(1)}$$
$$dD_t^{(2)} = (\kappa w_2 D_t^{(1)} + (\mu - \kappa w_1)D_t^{(2)})dt + \sigma\sqrt{D_t^{(2)}(D_t^{(1)} + D_t^{(2)})}dB_t^{(2)}$$

where $w_1 := w, w_2 := 1 - w$.

- Brownian motions $B^{(1)}, B^{(2)}$ are **independent**.
  Dividend shocks to different regions uncorrelated.
  Reason to use the same σ in both previous equations.
- For $\kappa = \mu$, volatility-stabilized process.
- Used here for dividends rather than prices.
- Regions symmetric for $w = 1/2$. w controls relative long-term weight.
- Drifts and volatilities higher for smaller region, e.g.,

$$\frac{dD_t^{(1)}}{D_t^{(1)}} = \left(\mu - \kappa(1 - w) + \kappa w \frac{D_t^{(2)}}{D_t^{(1)}}\right) dt + \sigma\sqrt{1 + \frac{D_t^{(2)}}{D_t^{(1)}}}dB_t^{(1)}$$

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Equilibria in Segmentation and Integration

- **Segmentation** equilibrium for region $i = 1, 2$: pair of processes $(r_t^{(i)}, P_t^{(i)})_{t\geq 0}$ such that solution to optimal consumption-investment problem

$$\max_{c\in\mathcal{C},\pi\in\mathcal{P}} \mathbb{E}\left[\int_0^\infty e^{-\beta s}\frac{c_s^{1-\gamma}}{1-\gamma}ds\right]$$

with interest rate r^i and asset price $P^{(i)}$, hence with wealth $(X_t)_{t\geq 0}$ satisfying budget equation

$$dX_t = r_t^{(i)}(X_t - \varphi_t P_t^{(i)})dt + \varphi_t dP_t^{(i)} - c_t dt$$

is well-posed and has solution $c_t = D_t^i$ and $\varphi_t = 1$.

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Present Value Relation

Proposition

Under the well-posedness assumption

$$\theta := \beta - (1-\gamma)\mu + \gamma(1-\gamma)\frac{\sigma^2}{2} > 0$$

the unique equilibrium asset prices are:

$$P_t^{(i)} = E\left[\int_t^\infty \frac{M_s^{(i)}}{M_t^{(i)}} D_s^{(i)} ds\right] \qquad M_t^{(i)} = e^{-\beta t}(D_t^{(i)})^{-\gamma} \qquad \text{(Segmentation)}$$

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Equilibrium interest rates $r_t^{(1)}, r_t^{(2)}, \bar{r}_t$ *are identified by the conditions that* $M_t^{(1)} e^{\int_0^t r_s^{(1)} ds}$, $M_t^{(2)} e^{\int_0^t r_s^{(2)} ds}$, $\bar{M}_t e^{\int_0^t \bar{r}_s ds}$ *are local martingales.*

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Theorem (Segmentation)

- *Let* $\gamma < 1 + \frac{2\kappa}{\sigma^2}\min(w, 1-w)$. *Segmentation prices and rates* $(P_t^{(i)}, r_t^{(i)})_{i=1,2}$ *are*

$$P_t^{(1)} = D_t^{(1)} X_t^{\gamma-1} f^{(1)}(X_t), \qquad r_t^{(1)} = \beta + \frac{1}{X_t}\left(\gamma\mu w - \frac{\gamma(\gamma+1)\sigma^2}{2}\right),$$

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$$W_t^{(i)} = \mathbb{E}_t\left[\int_t^\infty e^{-\beta(s-t)} \frac{\left(D_s^{(i)}\right)^{1-\gamma}}{1-\gamma} ds\right] = \frac{D_t^{1-\gamma}}{1-\gamma} f^{(i)}(X_t), \qquad i = 1, 2.$$

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- Find $f^{(1)}(x) = \mathbb{E}_{X_0=x}\left[\int_0^\infty e^{-\theta s} X_s^{1-\gamma} ds\right]$ in terms of resolvent of X_t.

$$\mathbb{E}\left[\int_0^\infty e^{-\theta s} X_s^{1-\gamma} ds \middle| X_0 = x\right] = \int_0^\infty e^{-\theta s} \left(\int_0^1 y^{1-\gamma} p(s;x,y) m(y) dy\right) ds$$
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$$G(x,y) = \begin{cases} \frac{1}{\omega^1} F_1^1(x)\varphi^{(1)}(y), & x \le y, \\ \frac{1}{\omega^1} F_1^1(y)\varphi^{(1)}(x), & x \ge y, \end{cases}$$

- F_1^1, $\varphi^{(1)}$ fundamental solutions of ODE

$$x(1-x)g''(x) + \frac{2\kappa}{\sigma^2}(w-x)g'(x) = \frac{2\theta}{\sigma^2} g(x).$$

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Theorem (Integration)

Integration prices, rate, and welfare are:

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Questions

- Imagine a shift from segmentation to integration.
- Do prices go up or down?
- What is price correlation before and after integration?
- Does welfare increase?
  For both regions, only one, or none?
- Would regions agree to integration if given the choice?
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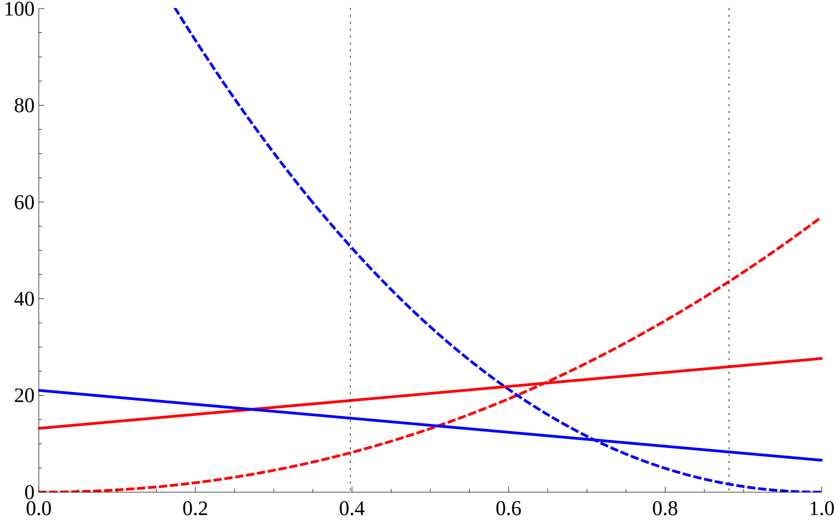
Prices/(Total Consumption)



Prices, as multiples of $D_t = D_t^{(1)} + D_t^{(2)}$, vs. dividend share X_t.
Red: first. Blue: second. Dashed: segmentation. Solid: integration.

Price Levels

- Cyclical prices: increasing with an asset's dividend share.
  More cyclical in segmentation and for smaller region (steeper slope).
- Neither up nor down for sure. But most of the time, down.
- Share unusually low: inflows higher than outflows push price up.
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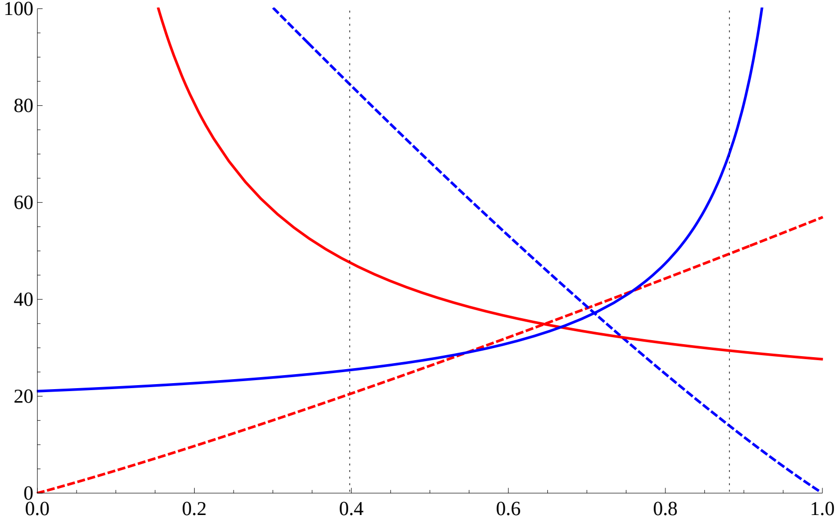
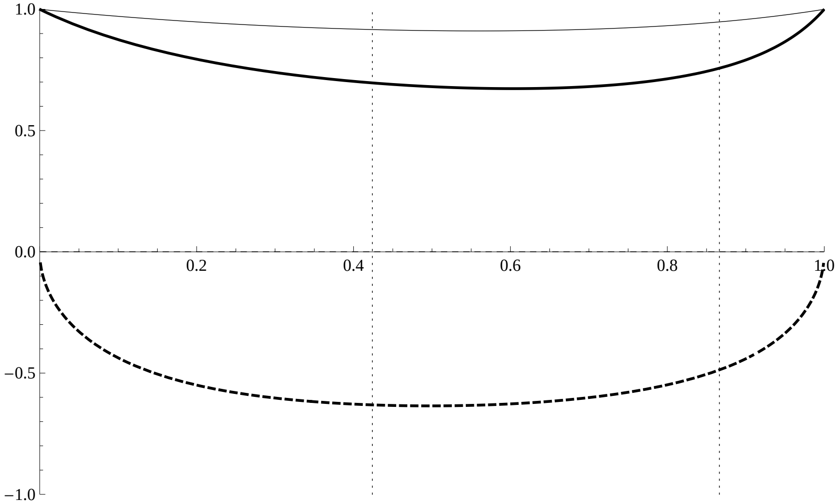
Price-Dividend Ratio



Price-dividend ratios vs. dividend share *w*.
Red: first. Blue: second. Dashed: segmentation. Solid: integration.

Correlation



Return correlation in segmentation (dashed) and integration (solid).

Portfolio

- Segmentation:
  Negative return correlation: negative price-dividend correlation prevails.
  Cross-interaction negligible.
- Integration:
  Negative price-dividend correlation deepens.
  But is overwhelmed by portfolio pressure.
- Though cash-flows are uncorrelated, prices are highly correlated.
  "Excess correlation" makes sense.
- Change in one tilts portfolio. Agent wants to rebalance.
  But supply of assets fixed, whence price increase.
- Like communicating vessels.

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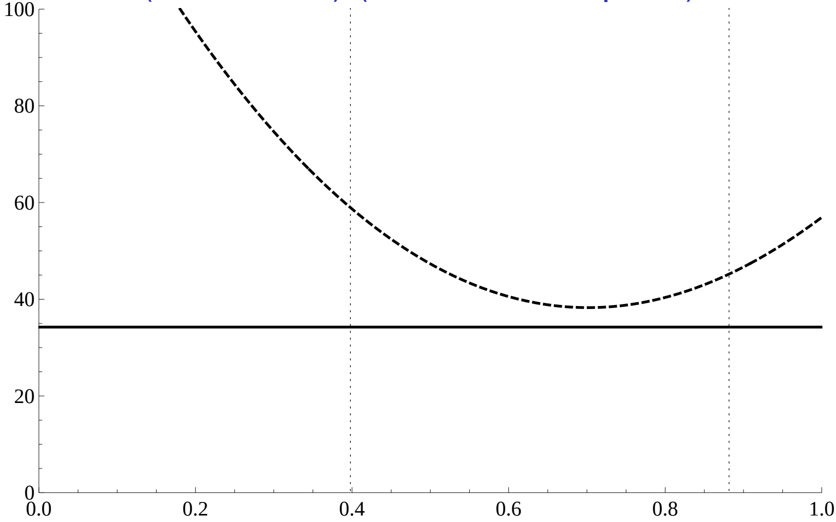

Market value in segmentation (dashed) and integration (solid) vs. share X_t.

- Integration always reduces market value!
- More when one region is much bigger than the other.

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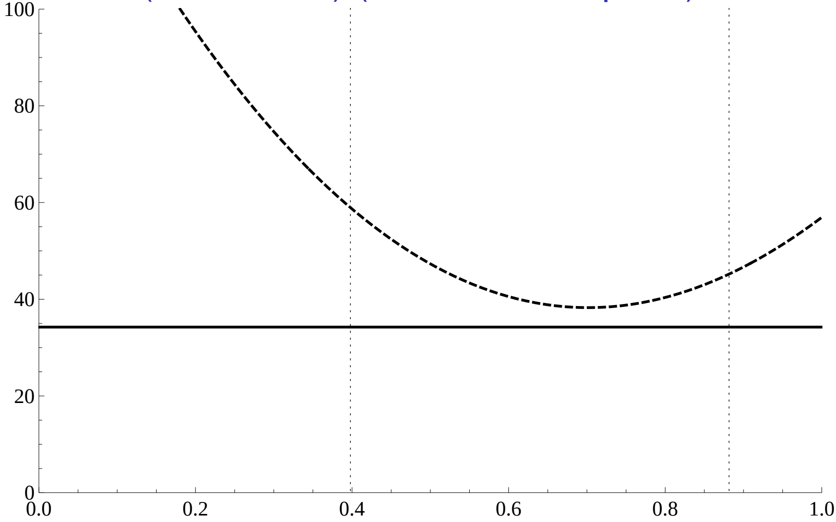

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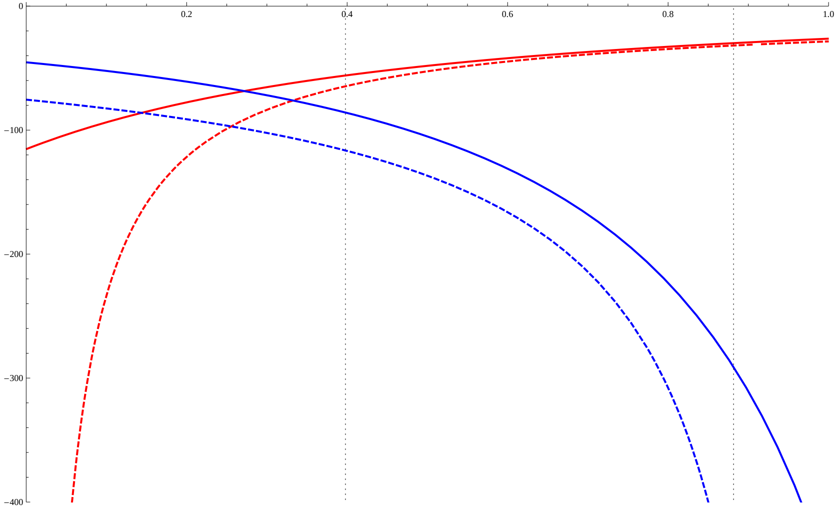
Sometimes Poorer. Always Happier.



Expected utility vs. dividend share.

Red: first. Blue: second. Dashed: segmentation. Solid: integration.

Wealth vs. Welfare

- Integration typically lowers prices.
- But it always increases welfare. For both regions.
- "Loss" in wealth is offset by access to smoother dividend stream.
  Ratio of dividend streams stationary. Neither grows faster than the other.
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  Which makes consumption more valuable.
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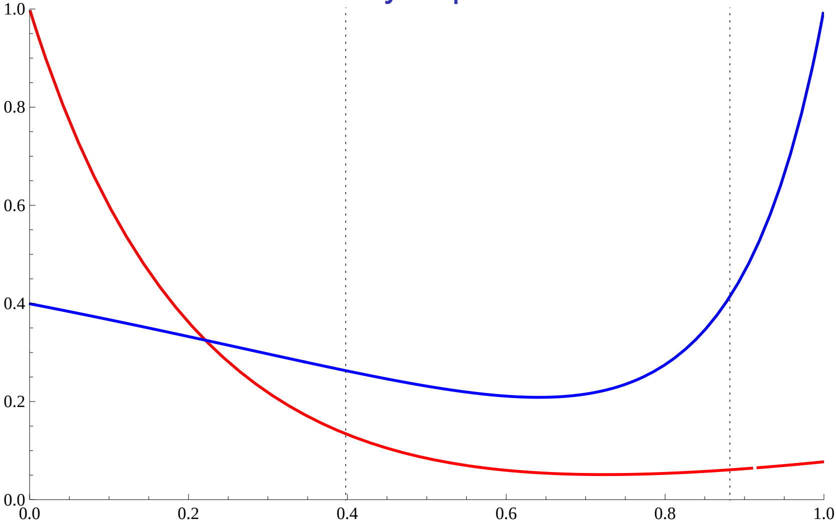
Certainty Equivalent



Fractional reduction in wealth accepted in exchange of integration.
Red: first. Blue: second.

- Integration more important for smaller (blue) region.

Endogenous Integration

- Integration make both regions better off.
- In principle, both agree to integrate.
- But they may negotiate on shares of wealth after post-integration.
- Integration bounds?
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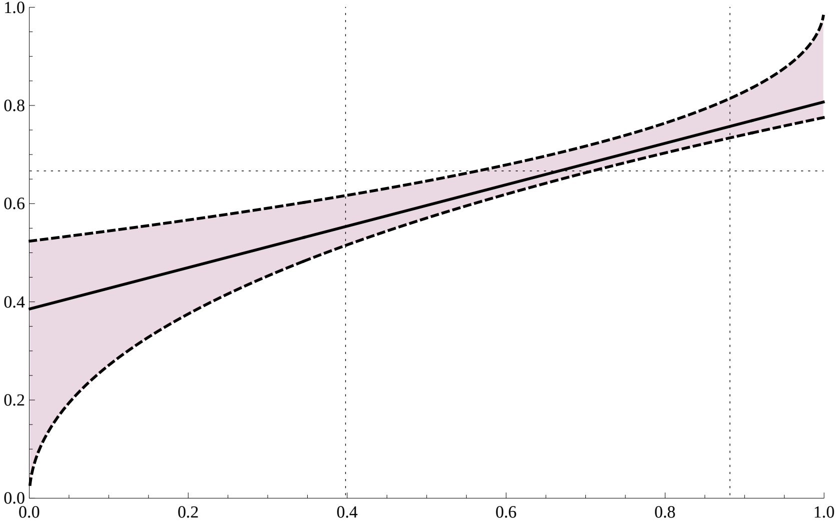
Integration Bounds



Range of wealth shares under which both regions agree to integration.

Conclusion

- Two economies, each with one agent and one asset. Growing together.
- Segmentation vs. Integration. Prices and rates.
- Prices up or down. Mostly down.
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Thank You!

Questions?

https://papers.ssrn.com/abstract=3140433