Noncommutative Polynomials Describing Convex Sets

Igor Klep U Ljubljana

Geometry of Real Polynomials, Convexity and Optimization

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Based on joint work with

Bill Helton	UC San Diego
Scott McCullough	U Florida
Jurij Volčič	Texas A&M

Outline

Noncommutative Polynomials Describing Convex Sets

1. Semialgebraic sets defined by noncommutative polynomials
2. Polynomials with convex semialgebraic sets
3. Examples and counterexamples
4. Algorithm for testing convexity and producing LMI representations

NC polynomials and linear pencils

Let $x = (x_1, \ldots, x_g)$ and $x^* = (x_1^*, \ldots, x_g^*)$ be freely noncommuting variables. Elements of the free algebra $\mathbb{C}<x, x^*>$ are noncommutative polynomials, e.g.

$$x_1 x_2^* x_2 + 2x_2^* x_1 x_1^* - 3.$$

Given $X = (X_1, \ldots, X_g) \in \mathrm{M}_n(\mathbb{C})^g$ and $f \in \mathrm{M}_d(\mathbb{C}<x, x^*>)$ we have $f(X, X^*) \in \mathrm{M}_{dn}(\mathbb{C})$. If $f^* = f$, then f is hermitian.

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If $A_1, \ldots, A_g, B_1, \ldots, B_g \in \mathrm{M}_d(\mathbb{C})$, then

$$L = I + A_1 x_1 + \cdots + A_g x_g + B_1 x_1^* + \cdots + B_g x_g^*$$

is a (monic) linear pencil of size d.
If $B_j = A_j^*$, then L is a hermitian linear pencil.

Motivation

for noncommutative polynomial inequalities

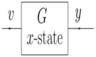
Linear systems engineering

Quantum information theory

Relaxing LMI problems, e.g. LMI domination problem $\mathcal{D}_L \subseteq \mathcal{D}_{\tilde{L}}$

WHY DO

Noncommutative Polynomial Inequalities
=
Noncommutative Real Algebraic Geometry



$$\frac{dx(t)}{dt} = Ax(t) + Bv(t)$$

$$y(t) = Cx(t) + Dv(t)$$

A, B, C, D are matrices

x, v, y are vectors

Asymptotically stable: $\mathrm{Re}(\mathrm{eigvals}(A)) \prec 0 \iff$

$$A^T\mathbf{E} + \mathbf{E}A \prec 0 \quad \mathbf{E} \succ 0$$

Energy dissipating

$G : L^2 \to L^2$

$$\int_0^T |v|^2 dt \geq \int_0^T |Gv|^2 dt$$

$x(0) = 0$

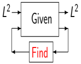
$$\exists \quad \mathbf{E} = \mathbf{E}^T \succeq 0$$

$$H := A^T\mathbf{E} + \mathbf{E}A + $$

$$+\mathbf{E}BB^T\mathbf{E} + C^TC \preceq 0$$

$\mathbf{E}$ is called a storage function

Linear Systems Problems → Matrix Inequalities



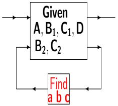
Many such problems, e.g. H^∞ control

The problem is Dimension free: since it is given only by signal flow diagrams and L^2 signals.

A Dimension Free System Problem
is Equivalent to
Noncommutative Polynomial Inequalities

Example:

Get Algebra



$$D = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

DYNAMICS of "closed loop" system: BLOCK matrices

$$\mathcal{A} \quad \mathcal{B} \quad \mathcal{C} \quad \mathcal{D}$$

ENERGY DISSIPATION:

$$H := \mathcal{A}^T\mathbf{E} + \mathbf{E}\mathcal{A} + \mathbf{E}\mathcal{B}\mathcal{B}^T\mathbf{E} + \mathcal{C}^T\mathcal{C} \preceq 0$$

$$\mathbf{E} = \begin{pmatrix} \mathbf{E}_{11} & \mathbf{E}_{12} \\ \mathbf{E}_{21} & \mathbf{E}_{22} \end{pmatrix} \qquad \mathbf{E}_{12} = \mathbf{E}_{21}{}^T$$

$$H = \begin{pmatrix} H_{xx} & H_{xz} \\ H_{zx} & H_{zz} \end{pmatrix} \qquad H_{xz} = H_{zx}^T$$

H^∞ Control

ALGEBRA PROBLEM:

Given the polynomials:

$$H_{xx} = \mathbf{E}_{11}A + A^T\mathbf{E}_{11} + C_1^TC_1 + \mathbf{E}_{12}{}^T\mathbf{b}C_2 + C_2^T\mathbf{b}^T\mathbf{E}_{12}{}^T + \mathbf{E}_{11}B_1\mathbf{b}^T\mathbf{E}_{12}{}^T + \mathbf{E}_{11}B_1B_1^T\mathbf{E}_{11} + \mathbf{E}_{12}\mathbf{b}\mathbf{b}^T\mathbf{E}_{12}{}^T + \mathbf{E}_{12}\mathbf{b}B_1^T\mathbf{E}_{11}$$

$$H_{xz} = \mathbf{E}_{21}A + \frac{a^T(\mathbf{E}_{21}+\mathbf{E}_{12}{}^T)}{2} + \mathbf{c}^TC_1 + \mathbf{E}_{22}\mathbf{b}C_2 + \mathbf{c}^TB_2^T\mathbf{E}_{11}{}^T + \frac{\mathbf{E}_{21}B_1\mathbf{b}^T(\mathbf{E}_{21}+\mathbf{E}_{12}{}^T)}{2} + \mathbf{E}_{21}B_1B_1^T\mathbf{E}_{11}{}^T + \frac{\mathbf{E}_{22}\mathbf{b}\mathbf{b}^T(\mathbf{E}_{21}+\mathbf{E}_{12}{}^T)}{2} + \mathbf{E}_{22}\mathbf{b}B_1^T\mathbf{E}_{11}{}^T$$

$$H_{zx} = A^T\mathbf{E}_{21}{}^T + C_1^T\mathbf{c} + \frac{(\mathbf{E}_{12}+\mathbf{E}_{21}{}^T)\mathbf{a}}{2} + \mathbf{E}_{11}B_2\mathbf{c} + C_2^T\mathbf{b}^T\mathbf{E}_{22}{}^T + \mathbf{E}_{11}B_1\mathbf{b}^T\mathbf{E}_{22}{}^T + \mathbf{E}_{11}B_1B_1^T\mathbf{E}_{21}{}^T + \frac{(\mathbf{E}_{12}+\mathbf{E}_{21}{}^T)\mathbf{b}\mathbf{b}^T\mathbf{E}_{22}{}^T}{2} + \frac{(\mathbf{E}_{12}+\mathbf{E}_{21}{}^T)\mathbf{b}B_1^T\mathbf{E}_{21}{}^T}{2}$$

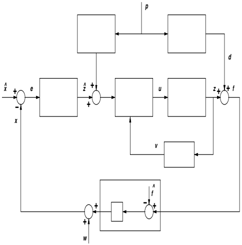
$$H_{zz} = \mathbf{E}_{22}\mathbf{a} + \mathbf{a}^T\mathbf{E}_{22}{}^T + \mathbf{c}^T\mathbf{c} + \mathbf{E}_{21}B_2\mathbf{c} + \mathbf{c}^TB_2^T\mathbf{E}_{21}{}^T + \mathbf{E}_{21}B_1\mathbf{b}^T\mathbf{E}_{22}{}^T + \mathbf{E}_{21}B_1B_1^T\mathbf{E}_{21}{}^T + \mathbf{E}_{22}\mathbf{b}\mathbf{b}^T\mathbf{E}_{22}{}^T + \mathbf{E}_{22}\mathbf{b}B_1^T\mathbf{E}_{21}{}^T$$

(PROB) A, B_1, B_2, C_1, C_2 are knowns.

Solve the inequality $\begin{pmatrix} H_{xx} & H_{xz} \\ H_{zx} & H_{zz} \end{pmatrix} \preceq 0$ for unknowns

a, **b**, **c** and for $\mathbf{E}_{11}$, $\mathbf{E}_{12}$, $\mathbf{E}_{21}$ and $\mathbf{E}_{22}$

More complicated systems give fancier nc polynomials



NC polynomials and linear pencils

Why do?

- Matrix multiplication is not commutative
- The functions we study are typically noncommutative (nc) polynomials

- Engineering problems defined entirely by signal flow diagrams and L^2 performance specs are equivalent to Polynomial Matrix Inequalities
- A system connection law amounts to an algebraic operation on NC quantities, while L^2 performance constraints, through use of quadratic "storage functions", convert to matrix inequalities $M \succ 0$

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- Engineering problems defined entirely by signal flow diagrams and L^2 performance specs are equivalent to Polynomial Matrix Inequalities
- A system connection law amounts to an algebraic operation on NC quantities, while L^2 performance constraints, through use of quadratic "storage functions", convert to matrix inequalities $M \succ 0$
- Convexity is needed for reliable designs and numerics. Often linear systems problems are solved by converting M via ad hoc changes of variables into convex problems or linear matrix inequalities (LMIs)

NC polynomials and linear pencils

Why do?

- Matrix multiplication is not commutative
- The functions we study are typically noncommutative (nc) polynomials
- (Matrix) convexity and LMIs are good for you.
- Matrix convexity ↔ operator systems ↔ quantum information theory
- NC function theory boom (Agler, Ball, McCarthy, Pascoe, Popescu, Shamovich, Vinnikov, Helton-K-McCullough-Volčič, etc.)

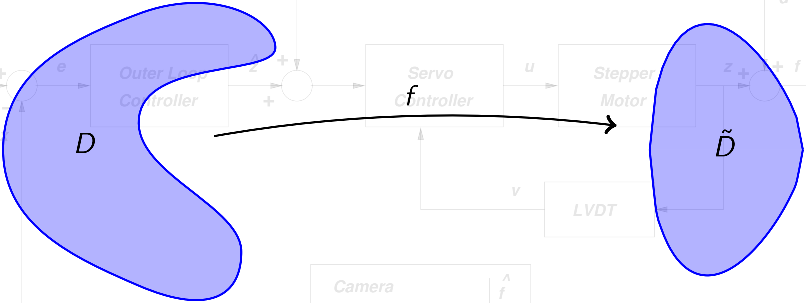
See also Jaka's talk on Thursday

- Convex optimization, polynomial optimization, moment problems (Blekherman, Brändén, Henrion, Infusino, Kummer, Kuhlmann, Lasserre, Naldi, Nie, Plaumann, Putinar, Renegar, Saunderson, Scheiderer, Sinn, Sturmfels, Tunçel, Vinzant, etc.)

What this talk is not about

Change variables to get convexity

- Which sets map bianalytically onto convex sets = spectrahedra?



- If such a map exists parameterize them all.
  Are there many?

Free LMIs

If L is a hermitian linear pencil, then let

$$\mathcal{D}_L = \bigcup_{n\in\mathbb{N}} \mathcal{D}_L(n), \qquad \mathcal{D}_L(n) = \{X \in \mathrm{M}_n(\mathbb{C})^g \colon L(X, X^*) \succeq 0\}$$

be its free spectrahedron or free LMI domain.

Free LMIs and semialgebraic sets

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More generally, if $f \in \mathrm{M}_d(\mathbb{C}{<}x, x^*{>})$ is hermitian and $f(0) \succ 0$, then its free semialgebraic set is $\mathcal{D}_f = \bigcup_n \mathcal{D}_f(n)$, where $\mathcal{D}_f(n)$ is the closure of the connected component of

$$\{X \in \mathrm{M}_n(\mathbb{C})^g \colon f(X, X^*) \succ 0\}$$

containing 0.

Convex free semialgebraic sets are given by LMIs

Theorem (Helton–McCullough (Ann. Math. 2012); Kriel 2018)

Every convex free semialgebraic set is a free spectrahedron.

That is, if for some hermitian $f \in \mathrm{M}_d(\mathbb{C}<x, x^*>)$, $\mathcal{D}_f(n)$ is convex for all n, then $\mathcal{D}_f = \mathcal{D}_L$ for some hermitian linear pencil L.

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Outline for the rest of the talk

(1) For what polynomials f is $\mathcal{D}_f$ convex?

(2) How to check if $\mathcal{D}_f$ is convex?

(3) If $\mathcal{D}_f$ is convex, how to find a hermitian linear pencil L with

$$\mathcal{D}_f = \mathcal{D}_L?$$

Scalar NC polynomials describing convex sets

Theorem (Helton, K, McCullough, Volčič)

Let $f \in \mathbb{C}<x, x^*>$ *be hermitian and irreducible, with* $f(0) > 0$*.*
If $\mathcal{D}_f$ *is a free spectrahedron, then* $\deg f \leqslant 2$ *and* f *is concave:*

$$f = \ell_0 - \sum_k \ell_k \ell_k^*$$

for some affine linear polynomials $\ell_k \in \mathbb{C}<x, x^*>$*.*

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f is irreducible if it does not factor as a product of two non-constants.

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- Theorem is not true for matrix-valued $f \in \mathrm{M}_d(\mathbb{C}\langle x, x^* \rangle)$
- Statement also fails for factorizable $f \in \mathbb{C}\langle x, x^* \rangle$

Linearization (realization) theory

NC rational functions $\mathbb{r}$ admit FM realizations

$$\begin{aligned} \mathbb{r} &= d + c^*\big(A_0 + A_1x_1 + \cdots + A_gx_g + B_1x_1^* + \cdots + B_gx_g^*\big)^{-1}\mathbf{b} \\ &= d + c^*L^{-1}\mathbf{b}, \end{aligned}$$

where $A_j, B_j \in M_d(\mathbb{C})$, $d \in \mathbb{C}$, $\mathbf{b} := \sum_j b_jx_j + b_{g+j}x_j^*$, and $b_i, c \in \mathbb{C}^d$.

Linearization (realization) theory

NC rational functions $\mathbb{r}$ admit FM realizations $\mathbb{r} = 1 + c^* L^{-1}\mathbf{b}$, where L is a $d \times d$ linear pencil, $\mathbf{b} := \sum_j b_j x_j + b_{g+j} x_j^*$ and $b_i, c \in \mathbb{C}^d$.

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- A realization has minimal size iff it is observable and controllable
- Minimal realizations are unique (up to basis change)
  (Ball-Groenewald-Malakorn)
- $\mathrm{dom}(\mathbb{r}) = \mathrm{dom}(L^{-1})$ (Kaliuzhnyi-Verbovetskyi-Vinnikov, Volčič)
- $\mathrm{dom}(\mathbb{r}) =$ all iff$_*$ the coefficients of L are jointly nilpotent iff $\mathbb{r}$ is a polynomial (K-Volčič, K-Pascoe-Volčič)
- $\mathbb{r}^{-1} = 1 - c^*\big(L + \mathbf{b}c^*\big)^{-1}\mathbf{b}$ (Ball-Groenewald-Malakorn)

Linearization theory and reciprocals of polynomials

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- $\mathbb{r}^{-1} = 1 - c^*\big(L + \mathbf{b}c^*\big)^{-1}\mathbf{b}$ (Ball-Groenewald-Malakorn)

The key technique: apply FM realizations to f^{-1} for a polynomial f.

A pencil of the form $L + \mathbf{b}c^*$ for L with jointly nilpotent coefficients is called flip poly.

Irreducible scalar NC polynomials describing convex sets

are of degree $\leqslant 2$. Note to self!! SKIP this??

Proof.

Assume $\mathcal{D}_f = \mathcal{D}_L$ for some (minimal) $L = I + \sum_j A_j x_j + \sum_j A_j^* x_j^*$.

(1) Consider the minimal FM realization $f^{-1} = 1 + c^* \widetilde{L}^{-1} b$.

(2) f is irreducible iff $\widetilde{L}$ is indecomposable (coefficients generate the full matrix algebra).
Helton-K-Volčič, Adv. Math. 2018

(3) $\mathcal{D}_f = \mathcal{D}_L$ & irreducibility imply $\mathcal{Z}_{\widetilde{L}} = \mathcal{Z}_L$, whence L, $\widetilde{L}$ are similar (K-Volčič, CMH 2017).

$$\mathcal{Z}_L := \bigcup_n \{X \in \mathrm{M}_n(\mathbb{C})^g : \det L(X, X^*) = 0\}.$$

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(4) L is hermitian and flip-poly (A_j = nilpotent + rank one)

$$\implies \quad A_j = \begin{pmatrix} \alpha_j & v_j^* \\ u_j & 0 \end{pmatrix}. \qquad \square$$

Example

of a high degree NC polynomial describing a convex set

$$f = \underbrace{\Big(1 + x + x^* - 2xx^* - (x + x^*)xx^*\Big)}_{f_1} \underbrace{\Big(1 + \frac{1}{2}(x + x^*)\Big)}_{s_1}$$

$$L = \begin{pmatrix} 1 + x + x^* & 0 & x \\ 0 & 1 & x \\ x^* & x^* & 1 \end{pmatrix}.$$

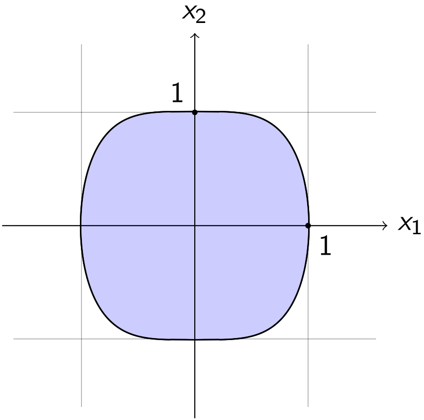
- f is hermitian of degree 4;
- $\mathcal{D}_f = \mathcal{D}_L$ is a free spectrahedron.

Non-example

TV screen

Consider $p(x, y) = 1 - x_1^4 - x_2^2$.

The semialgebraic set $\mathcal{D}_p$ is called the bent TV screen.



TV screen $\mathcal{D}_p(1) = \left\{(x_1, x_2) \in \mathbb{R}^2 \mid 1 - x_1^4 - x_2^2 \geqslant 0\right\}$.

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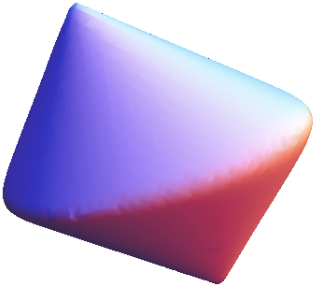

A 3-dimensional slice of $\mathcal{D}_p(2) = \{(X_1, X_2) \in M_2(\mathbb{R})^2_{\mathrm{sym}} \mid I_2 - X_1^4 - X_2^2 \succeq 0\}$.

Non-example

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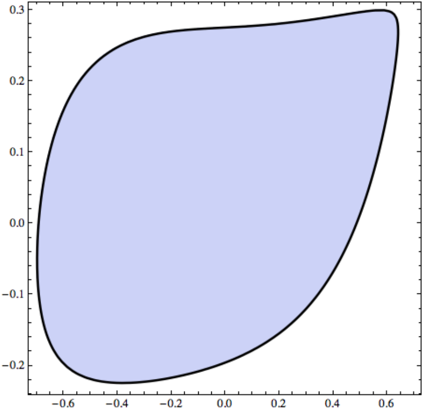

A non-convex 2-dimensional slice of $\mathcal{D}_p(2)$.

Non-example (cont'd)

Convexifying the TV screen $I - X_1^4 - X_2^2 \succeq 0$

Define $$L^1(x,y) = 1 - y_{1111} - y_{22}$$

$$L^2(x,y) = \begin{pmatrix} 1 & x_1 & x_2 & y_{11} \\ x_1 & y_{11} & y_{12} & y_{111} \\ x_2 & y_{21} & y_{22} & y_{211} \\ y_{11} & y_{111} & y_{112} & y_{1111} \end{pmatrix}$$

Set

$$\mathcal{C} := \left\{(X,Y) \mid L^1(X,Y) \succeq 0,\ L^2(X,Y) \succeq 0\right\}.$$

Its projection onto the x coordinates is the spectrahedrop:

$$\hat{\mathcal{C}} := \{X : \ \exists Y \quad (X,Y) \in \mathcal{C}\}.$$

Non-example (cont'd)

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Its projection onto the x coordinates is the spectrahedrop:

$$\hat{\mathcal{C}} := \{X : \ \exists Y \ \ (X, Y) \in \mathcal{C}\} \supsetneq \operatorname{co} \mathcal{D}_p.$$

Open problem:

Does the convex hull of the TV screen have an SDP representation?

Invertibility sets of (non-hermitian) NC polynomials

For a general $f \in \mathrm{M}_d(\mathbb{C}{<}x, x^*{>})$ with $\det f(0) \neq 0$ let

$$\mathcal{K}_f(n) = \text{prime component of } \{X \in \mathrm{M}_n(\mathbb{C})^g : \det f(X, X^*) \neq 0\},$$

$$\mathcal{K}_f = \bigcup_n \mathcal{K}_f(n).$$

Invertibility sets of (non-hermitian) NC polynomials

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- For $d = 1$ there exist irreducible polynomials $f \in \mathbb{C}<x, x^*>$ of arbitrarily high degree with $\mathcal{K}_f$ a free spectrahedron, e.g.

$$f = 1+4(x+x^*)+2(x^2+(x^*)^2)-xx^*-7xx^*(x+x^*)-4x^*x(x+x^*) \\ - xx^*(x^2 + (x^*)^2) + 2xx^*(xx^* + x^*x)(x + x^*).$$

Convexity of $\mathcal{K}_f$

Theorem (Helton, K, McCullough, Volčič)

Let $f \in \mathrm{M}_d(\mathbb{C}\!<\!x, x^*\!>)$ with $f(0) = I$, and write its minimal FM realization $f^{-1} = I + c^* L^{-1}\mathbf{b}$ with

$$L = \begin{pmatrix} L^1 & \star & \star \\ & \ddots & \star \\ & & L^\ell \end{pmatrix},$$

where each L^i is either indecomposable or I.

Let $\widehat{L}$ be the direct sum of those indecomposable blocks L^i that are similar to a hermitian pencil, and let $\check{L}$ be the direct sum of the remaining L^j. The following are equivalent:

(i) *$\mathcal{K}_f$ is a free spectrahedron;*

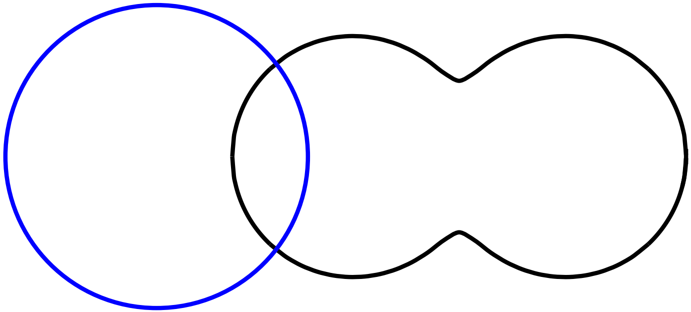
(ii) *$\mathcal{K}_f = \mathcal{K}_{\widehat{L}}$;*

(iii) *$\check{L}$ is invertible on $\operatorname{int} \mathcal{K}_{\widehat{L}}$.*

Convexity of $\bigcap \mathcal{K}_{f_i}$ for irreducible f_i

Corollary

Assume f_i are irreducible. Then $\bigcap \mathcal{K}_{f_i}$ is convex iff each $\mathcal{K}_{f_i}$ is convex.



Convexity of $\bigcap \mathcal{K}_{f_i}$ for irreducible f_i

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Assume f_i are irreducible. Then $\bigcap \mathcal{K}_{f_i}$ is convex iff each $\mathcal{K}_{f_i}$ is convex.

Proof.

- $(f_1 \cdots f_t)^{-1} = I + c^* L^{-1}\mathbf{b}$ is a minimal FM realization[1] with

$$L = \begin{pmatrix} L^1 & \star & \star \\ & \ddots & \star \\ & & L^\ell \end{pmatrix},$$

  where each L^i is either indecomposable or I.
- For every i there exists j_i such that $\mathcal{K}_{L^i} = \mathcal{K}_{f_{j_i}}$.
- If one of the L^i was not similar to a hermitian pencil, then it is redundant by convexity and the Theorem. □

[1] Coded in NCAlgebra, see notebook of Volčič.

All polynomials f with convex $\mathcal{K}_f$

$$f = s_0 \; f_1 \; s_1 \; f_2 \; \cdots \; f_r \; s_r,$$

- f_i irreducible;
- $\mathcal{K}_{f_i}$ convex;
- $\mathcal{K}_{s_i}$ redundant.

An algorithm to determine if $\mathcal{K}_f$ is convex

Check if a rectangular $\check{L}$ is of full rank on $\operatorname{int}\mathcal{D}_{\hat{L}}$

Let $\widehat{L}$ be $d \times d$ hermitian and let $\check{L}$ be a $\delta \times \varepsilon$ affine linear pencil.

Step 1. Solve the following feasibility SDP for $D \in \mathbb{C}^{\delta \times d}$:

$$\operatorname{tr}(\operatorname{Re}(D\check{L})(0)) = 1$$

$$\operatorname{Re}(D\check{L}) = P_0 + \sum_k C_k^* \widehat{L} C_k \quad \text{for some } C_k, P_0, \text{ with } P_0 \succeq 0.$$

Step 2. If infeasible, then $\check{L}(X, X^*)$ is not full rank for some $X \in \operatorname{int}\mathcal{D}_{\hat{L}}$.

Step 3. Otherwise we have a solution D with $V := \ker P_0 \cap \bigcap_k \ker C_k$.

Step 3.1 If $V = (0)$, then $\check{L}$ is full rank on $\operatorname{int}\mathcal{D}_{\hat{L}}$.

Step 3.2. If $\varepsilon' = \dim V > 0$, then let $\check{L}'$ be the $\delta \times \varepsilon'$ pencil whose coefficients are the restrictions of $\check{L}$ to V. Then $\check{L}$ is full rank on $\operatorname{int}\mathcal{D}_{\hat{L}}$ if and only if $\check{L}'$ is full rank on $\operatorname{int}\mathcal{D}_{\hat{L}}$. Now we apply Step 1 to $\check{L}'$.

An algorithm for finding an L with $\mathcal{K}_f = \mathcal{D}_L$

(a) Compute the minimal realization $f^{-1} = I + c^* L^{-1} \mathbf{b}$.

(b) Next find the Burnside decomposition of L into

$$L = \begin{pmatrix} L^1 & \star & \star \\ & \ddots & \star \\ & & L^\ell \end{pmatrix},$$

where each L^i is either indecomposable or I.

(c) Pick one pencil from each similarity class among the L^i.

(d) Find all those L^i that are similar to a hermitian pencil: SDP

$$Q \succeq I, \qquad Q(L^i)^* = L^i Q$$

leads to a hermitian pencil $\widetilde{L}^i = Q^{-\frac{1}{2}} L^i Q^{\frac{1}{2}}$.

(e) The direct sum $\widetilde{L}$ of the hermitian pencils $\widetilde{L}^i$ obtained in (d) satisfies

$$\mathcal{D}_{\widetilde{L}} = \mathcal{K}_f.$$

Conclusions

Take home messages

- Free convex semialgebraic sets are given by LMIs.
- An irreducible polynomial f with convex $\mathcal{D}_f$ must be concave of degree $\leqslant 2$, f is a Schur complement of a hermitian linear pencil.
- An intersection of free convex semialgebraic sets is convex iff all of them are.
- There is an effective algorithm for testing whether $\mathcal{D}_f$ is convex.
- There is an effective algorithm for computing an LMI representation of a convex $\mathcal{D}_f$.