From Many Body Problems to Random Matrices

Paul Bourgade (New York University),
Laszlo Erdős (Institute of Science and Technology Austria),
Jeremy Quastel (University of Toronto),
Tai-Peng Tsai (University of British Columbia)

August 4 - 9, 2019

1 Overview of the Field

1.1 Overview of the subject area of the workshop

Universality and integrability. Complex physical systems with many degrees of freedom cannot typically be described deterministically; solving an equation with $N = 10^{23}$ variables is not feasible unless it has a special structure allowing for explicit solutions. Moreover, precise details of the system and the initial conditions are often known only statistically and not exactly. On the crudest level of description, random effects cancel out and effective equations for a few macroscopic physical variables become accurate - a fundamental mechanism that justifies thermodynamics, statistical physics and fluid dynamics. On a finer scale, quite surprisingly, fluctuations around these macroscopic variables tend to exhibit universal behavior.

Universality within and between complex random systems is a striking concept which has played a central role in the direction of research within probability, mathematical physics and statistical mechanics. Complementary to universality, is the exact description of the behaviors that are supposed to be universal as well as the determination which systems are supposed to display them.

The historical example is the Gaussian distribution emerging from the central limit theorem. Since its discovery more than two hundred years ago, it has proved to be an incredibly versatile and robust tool explaining randomness in the physical world, under the assumption of some underlying independence. Systems accurately described in terms of this distribution are said to be in the Gaussian universality class.

Ongoing efforts to understand this universality class are essentially of two types. First, integrability consists in finding possibly new statistics for a few, rigid models, with methods including combinatorics and representation theory. Second, universality means enlarging the range of models with random matrix statistics, through probabilistic methods.

In recent years there has been an immense amount of progress in the rigorous mathematical understanding of certain universal scaling limits, not only on the universality and integrability fronts, but also both in equilibrium and non-equilibrium statistical physical systems. On the random matrix side, microscopic scaling limits are often described in terms of determinantal or pfaffian point processes. On the non-equilibrium side, systems like growth processes are described through the Kardar-Parisi-Zhang universality class. There is reason to believe that these two directions share many (as of yet) unexploited relationships. For instance, the field of quantum integrable systems was developed to study equilibrium systems, but has now found itself center stage in the KPZ universality class. Conversely, methods developed in stochastic PDEs for non-equilibrium systems have begun to make their way into constructive field theory, and the dynamics of Dyson

Brownian motion proved to be essential to understand static spectral universality.

We give below a succinct overview of recent progress for random matrices, with the key technique of dynamics, and the recent progress on the KPZ universality class.

The Wigner-Dyson-Mehta conjecture. As a first test for his paradigm, Wigner conjectured - together with Dyson and Mehta - that the microscopic eigenvalue statistics of large symmetric matrices with independent entries - so-called Wigner matrices - do not depend on the distribution of the matrix elements [57, 46]. This is an analogue of the central limit theorem for eigenvalue statistics at the microscopic scale, but with a non-Gaussian limit. Along an important series of breakthroughs in the past ten years, the Wigner-Dyson-Mehta conjecture has been proved, enlarging the universality class from the integrable Gaussian ensembles to the Wigner matrices: the Gaudin-Mehta and Tracy-Widom distributions appear naturally as a simple transform with input independent, arbitrarily distributed, random variables (see e.g. [28, 56, 30, 29]).

The universality class has subsequently been enlarged to many other models, as was done recently for covariance matrices, β-ensembles, matrices with correlated entries, matrices of free-convolution type, Erdős-Rényi and d-regular random graphs.

Universality for these models may use a variety of tools including the Itzykson Zuber integral, the Lindeberg principle, transport maps and multiscale analysis. However, for all of these models, possibly the most fundamental idea - introduced by Erdős, Schlein and Yau [28] - consists in using dynamics to prove the - static - universality result by interpolation.

Dynamics. The Dyson Brownian motion [26] describes the dynamics of eigenvalues of random matrices, when matrix entries follow standard independent Brownian paths. At the technical level, the analysis of Dyson Brownian motion can be performed either through hydrodynamics or probabilistic coupling. Erdős, Schlein, Yau and Yin proved that this dynamics reaches local equilibrium very rapidly, so that the noise regularization could be effectively removed by density arguments.

From a more general perspective, non-equilibrium dynamics describes systems which are in motion. Examples include fluid dynamics, weather, and models in the biological sciences. Although nonequilibrium dynamics are ubiquitous in the real world, it is one of the most challenging domains of mathematics and physics. Unfortunately we still have only limited mathematical tools to analyze general dynamics, but at least the Dyson Brownian motion is now well understood even for arbitrary initial conditions.

The following two particular models of stochastic dynamics became better understood in recent years and are of particular interest: the Kardar- Parisi-Zhang (KPZ) equation for its own sake, and Dyson Brownian motion for its applications, as previously mentioned. It is a remarkable fact that the fluctuations of the KPZ solutions are very closely connected to random matrix theory, through the Tracy Widom distribution(s).

Kardar-Parisi-Zhang. The KPZ equation is a stochastic equation in one dimension which has elements of stochastic integrability. It is also related to asymptotic representation theory. A substantial part of the program was centered around the KPZ equation. Originally, the equation was proposed by Kardar, Parisi, and Zhang (1986) [42] to describe the motion of growing fronts. Large scale mathematical activities started in the seminal work of Baik, Deift and Johansson (2000) [4]. The wide interest in the KPZ equation stems from its role in connecting seemingly different mathematical worlds, in particular, Dysons Brownian motion, quantum Toda chain and related integrable models, statistical mechanics of line ensembles, directed polymers in a random medium, tilings, stochastic lattice gases, and stochastic conservation laws in one dimension. During recent years, a wide spectrum of advances have been achieved. We list only a few examples:

(i) Hairer's rigorous and robust regularity theory structures gave firm grounds to the KPZ equation [36, 37].

(ii) Universality of the KPZ equation. Based on heuristic arguments, one expects the KPZ equation to be universal in the sense that microscopic models of surface growth presenting either weak noise or a weak asymmetry is expected to converge to the KPZ equation when viewed at a suitable large scale. Some results were recently established rigorously for a class of continuous models. (Hairer, Quastel [39]).

(iii) More recently, a clear understanding of the KPZ fixed point for any initial data was provided in a key result of Quastel, Matetski and Remenik [45].

These recent progresses raise questions about universality of discrete models supposed to converge to the KPZ fixed point. For example, one natural question is whether ASEP models exhibit the whole range of possible limiting behaviors discovered in (iii).

Delocalization and universality beyond mean field. In Wigner's original theory, the eigenvectors play no role. However, their statistics are essential in view of a famous dichotomy of spectral behaviors, widely studied since Andersons tight binding model [3] for conductor-insulator transition: random matrix eigenvalue distributions should coincide with delocalization of eigenstates, while Poisson and Gaussian universality classes for the spectrum occur together with localization. The localized phase is well understood since the early work of Fröhlich and Spencer [33], but delocalization and random matrix universality have still not been proved for any operator relevant in physics. Indeed Wigner matrices are mean field models - all matrix entries are random - which strongly limits their physical relevance. An outstanding problem consists in proving the Anderson transition phenomenon for at least one model.

One well-known such model is given by band matrices (see e.g. [55]): each point in space randomly interacts only with a finite-range of neighbours. These matrices are believed to exhibit the insulator-conducting transition, as famously conjectured twenty five years ago by Fyodorov and Mirlin [34]: for example, in dimension 1, if the random band has width greater than $N^{1/2}$, with N the matrix dimension, the system is supposedly delocalized, otherwise it is localized. Very little is known about the delocalized phase. Recently, a mean field reduction technique was introduced to identify the eigenvalue statistics for some non mean field band matrices [9, 10]. Interestingly, this reduction technique means that universal spectral statistics follow from quantum unique ergodicity of the eigenvectors, a notion of delocalization introduced by Rudnick and Sarnak in the context of the Laplacian on manifolds [50].

Another very intriguing idea is the supersymmetric (SUSY) approach that explicitly computes local statistics of quite general disordered models, including the Anderson model, but it uses mathematically ill-defined saddle point analysis on Grassmann integrals [27]. Currently there is no idea how to remedy this unfortunate situation where physicists have found an obviously powerful tool but mathematicians failed to create its rigorous foundations. Nevertheless, some aspects of SUSY analysis can be made rigorous and have been used to analyse band matrices [24, 51, 52, 53, 54].

1.2 Statement of the objectives of the workshop

The focus of this workshop was to enlarge the basin of attraction of integrable models, by developing new analytic tools to advance our understanding of universality, both for random matrices and stochastic growth models. It brought together leading figures that are working on questions related to this field and place them in contact with young researchers. The chief interests did not only include the exposition of recent work, but also a collaborative effort to understand the relationships between the seemingly disparate results and techniques and foster new collaborations.

Organisational aspects. The workshop had 19 speakers and 35 participants in a very active and fundamental area at the intersection of probability, analysis and mathematical physics. The organizers have invited leading researchers in this field, including women mathematicians, and made sure promising postdocs attend the meeting for their scientific development. The organizing committee also made sure that the meeting contribute to the enhancement and improvement of scientific and educational activities through the dissemination of the results. Most speakers provided slides which have become immediately available through the workshop webpage. The video facilities at BIRS have allowed wide range dissemination of this very active research field.

Why now? A special year was organized on random matrix theory and the KPZ equation at the Institute for Advanced Study in 2013-2014. The organizing committee felt that now is a good time to join again these research groups, after five years of intensive progress.

Probability as a field has seen tremendous advances in recent years with breakthroughs in random matrix theory, stochastic PDEs and the Kardar-Parisi-Zhang universality class. For instance, random matrix theory has found some new uses in the study of random 2d geometries, as have stochastic PDEs; stochastic PDEs and random matrix theory have found new uses in the study of random growth. In light of all of these new techniques and advances, it also seems quite appropriate to try to reintroduce many of the fundamental challenges related to localization, delocalization, dynamics and perhaps refocus people's attention (and new methods) on some of these questions.

2 Some major open problems

This conference brought together international experts working on the broad theme of dynamics and universality of complex disordered systems, who are at the forefront of current progress in these fields. These areas provided a vast source of open questions and interesting phenomena for the past 50 years. Despite the long investigation and remarkable progress, several fundamental problems remain unsolved:

(i) Random Matrix Universality beyond mean field models, in particular delocalization

(ii) A robust characterization for convergence of discrete models to the KPZ fixed point

(iii) Universality of the Tracy Widom distribution in first passage percolation

(iv) Random matrix statistics in semiclassical analysis, i.e. the Bohigas Giannoni Schmidt conjecture [8]

Problems number (iii) and (iv) seem out of reach with current techniques, but there is some starting point for problems (i) and (ii) and some progress was mentioned during the workshop. This progress, and some more specific open problems, are detailed in the next section.

3 Presentation Highlights

The meeting From Many Body Problems to Random Matrices was held Aug 4-9, 2019. There were many very high level talks surveying the latest developments in the general field spanning mathematical physics and probability, mostly by junior researchers, with many interesting comments and discussions with the more senior attendants. Some key progress and ideas of the talks are mentioned in te next paragraph. Some more details (and possibly some open problems) for each talk are given just after.

Monday morning was devoted to several talks on the recent proofs of the Lee-Huang-Yang asymptotics for the ground state energy of the Bose-Einstein condensate, a problem that has been worked on intensively in the math-physics community for the last several decades. In the afternoon there were a variety of results on hydrodynamic limits without local equilibrium, quantum lattice models, and the dynamical approach to eigenvalue empirical distributions of complex matrix ensembles. As well, there was the talk of Novak on the successful culmination of a decade work on deriving Hurwitz numbers from matrix integrals. Tuesday had a breakthrough result of Ding on the mathematical approach to Anderson localization problems, as well as Tatyana Shcherbyna on the supersymmetric approach to random band matrices. Wednesday morning concentrated on large random graphs, then some hints at new universality classes for statistics in numerical algorithms. The final talks of the meeting contained remarkable new results by a number of young researchers: Bauerschmidt has found a way to extend the Bakry-Emery arguments to non-convex potentials, with an application to obtain the log-Sobolev inequality for dynamical sine-Gordon; Aggarwal has managed to extend the three step program for universality of random matrices to obtain the local statistics in the bulk of lozenge tiling models for general domains; Sosoe and coauthors have extended the Fields medal work of Martin Hairer to certain stochastically forced wave equations; Cipolloni extended the random matrix universality to places where there is a cusp in the eigenvalue density, and to the edge of non Hermitian random matrix models; finally, Landon calculated the fluctuations of the overlap in the spherical spin glass model.

Benjamin Schlein: Excitation spectrum of Bose Einstein condensates [7]. This talk considered systems of N trapped bosons interacting through a repulsive potential with scattering length of the order 1/N, i.e. in the Gross-Pitaevskii regime. Schlein and collaborators determined the low-energy spectrum of the Hamilton operator in the limit of large N, confirming the predictions of Bogoliubov theory.

Jan Phillip Solovej: On the Lee-Huang-Yang universal asymptotics for the ground state energy of a Bose gas in the dilute limit. [12, 13, 31, 32] In 1957 Lee, Huang, and Yang (LHY) predicted a universal expression for a two-term asymptotic formula for the ground state energy of a dilute Bose gas. The formula is universal in the sense that the two terms depend on the interaction potential only through its scattering length. In 2009 Yau and Yin proved an upper bound of the LHY form for a fairly large class of potentials [58]. The speaker discussed recent joint work with Fournais complementing this by a corresponding lower bound, establishing the LHY universality formula.

Stefano Olla: Some problems in hyperbolic hydrodynamic limits: random masses and non-linear wave equation with boundary tension. The speaker illustrated some recent results about hydrodynamic limit in Euler scaling for one dimensional chain of oscillators:

(1) in the harmonic case with random masses, Anderson localization allows to obtain Euler equation in the hyperbolic scaling limit, while temperature profile does not evolve in any time scale.

(2) If the chain is in contact with a Langevin heat bath conserving momentum and volume (isothermal evolution), we prove convergence to L^2-valued weak entropic thermodynamic solutions of the non-linear wave equation, even in presence of boundary tension.

Open problem related to this talk. Uniqueness of the solution to the limiting hydrodynamic equation.

Bruno Nachtergaele: Stability of the superselection sectors of two-dimensional quantum lattice models [15]. Kitaev's quantum double models provide a rich class of examples of two-dimensional lattice systems with topological order in the ground states and a spectrum described by anyonic elementary excitations. The infinite volume ground states of the abelian quantum double models come in a number of equivalence classes called superselection sectors. Nachtergaele and collaborators showed that the superselection structure remains unchanged under uniformly small perturbations of the Hamiltonians.

Open problems related to this talk. Among important remaining questions are the thermodynamics and effective equations for many-anyon systems, and interesting examples of stable non-abelian anyons.

Todd Kemp: Geometric Matrix Brownian Motion and the Lima Bean Law. [25] The non-normality (and lack of explicit symmetry) of the Geometric matrix Brownian motion has made understanding its large-N limit empirical eigenvalue distribution quite challenging. There are two sides to this problem: proving that the empirical law of eigenvalues converges (which amounts to certain tightness conditions on singular values), and computing what it converges to. In the case of the geometric matrix Brownian motion, the speaker exposed the explicit calculation of the conjectured limit empirical eigenvalue distribution. It has an analytic density with a nice polar decomposition, supported on a region that resembles a lima bean for small time, then folds over and becomes a topological annulus for large time.

Open problems related to this talk. The question of convergence is still a work in progress.

Jonathan Novak: A tale of two integrals. The Harish-Chandra/Itzykson-Zuber integral [40] and its additive counterpart, the Brezin-Gross-Witten integral, play an important role in random matrix theory. The author presented his recent work which proves a longstanding conjecture on the large dimension asymptotic behavior of these special functions. Hurwitz and monotone Hurwitz numbers play an important role.

Jian Ding: Localization near the edge for the Anderson Bernoulli model on the two dimensional lattice. [23] The speaker considers a Hamiltonian given by the Laplacian plus a Bernoulli potential on the two dimensional lattice. He explained how, for energies sufficiently close to the edge of the spectrum, the resolvent on a large square is likely to decay exponentially. This implies almost sure Anderson localization for energies sufficiently close to the edge of the spectrum, answering a longstanding open question. The proof follows the program of Bourgain-Kenig [11], using a new unique continuation result inspired by a Liouville theorem of Buhovsky-Logunov-Malinnikova-Sodin [14]. Tha speaker also explained how Li and Zhang [44] proved a similar result in 3d.

Open problems related to this talk. Key questions are localization through the spectrum for $d = 2$ with weak potentials, and localization/delocalization phase transition for $d \geq 3$ with weak potentials.

Alexander Elgart: Localization at the bottom of the spectrum of a disordered XXZ spin chain . Quantum spin chains provide some of the mathematically most accessible examples of quantum many-body systems. However, even these toy models pose considerable analytical and numerical challenges, due to the fact that the number of degrees of freedom involved grows exponentially fast with the system's size. The speaker discussed the recent progress in establishing many body localization at the bottom of the spectrum of a disordered XXZ chain. In particular, he mentioned in progress introducing a new approach to many body localization that works beyond the droplet phase.

Mariya Shcherbyna: Central Limit Theorem for the entanglement entropy of free disordered fermions. [48] The speaker considered the macroscopic disordered system of free lattice fermions with the one-body Hamiltonian, which is the Schrdinger operator with i.i.d. potential in $d > 1$. Assuming that the fractional moment criteria for the Anderson localization is satisfied, she proved a Central Limit Theorem for the large block entanglement entropy.

Tatyana Shcherbyna: Universality for random band matrices. [51, 52, 53, 54] As explained in Section 1.1, random band matrices (RBM) are natural intermediate models to study eigenvalue statistics and quantum propagation in disordered systems, since they interpolate between mean-field type Wigner matrices and random Schrodinger operators. In particular, RBM can be used to model the Anderson metal-insulator phase transition (crossover) even in 1d. The speaker discussed some recent progress in application of the supersymmetric method (SUSY) and transfer matrix approach to the analysis of local spectral characteristics of some specific types of 1d RBM. In particular she explained work in progress to obtain the localization-delocalization transition at the level of 2 point correlation function, for some Hermitian Gaussian random band matrices.

Open problems related to this talk. Important natural questions are higher order correlation functions, other symmetry classes, eigenvector statistics and universality of the transition for the RBM model.

Morris Yau: Convex relaxations for robust statistics. Much of the theory of machine learning is concerned with the optimization of non-convex functions. Convex relaxations and their associated hierarchies (sum-of-squares, Lassere) provide a systematic approach for approximately optimizing non-convex functions. Recent breakthroughs in robust statistics have produced the first polynomial time (efficient) algorithms for computing the robust mean of a high dimensional Gaussian. Building on these developments, the speaker constructed a framework for robust learning via convex relaxations yielding the first polynomial time algorithm for robust regression when the overwhelming majority of the dataset is comprised of outliers.

Amir Dembo: Large deviations of subgraph counts for sparse random graphs. [21] The speaker discussed recent developments in the emerging theory of nonlinear large deviations, focusing on sharp upper tails for counts of a fixed subgraph in large sparse random graphs, such as Erdős-Rényi or uniformly d-regular. He explained his approach via quantitative versions of the regularity and counting lemmas suitable for the study of sparse random graphs in the large deviations regime.

Open problems related to this talk. For example, Sidorenko's conjecture (see (1.31) in [21])

Antti Knowles: Extremal eigenvalues of sparse Erdos-Renyi graphs. [2] The speaker reviewed recent results on the extremal eigenvalues of the adjacency matrix A of the Erdos-Renyi graph G(N,p). If p is large then, after a suitable rescaling, A behaves like a Wigner matrix and its extremal eigenvalues converge to the edges -2, +2 of the asymptotic support of the eigenvalue distribution. If p is small, this is no longer true. The behaviour of the extremal eigenvalues for small p was explained, and in particular the transition around a critical p. The proof is based on a tridiagonal representation of A and on a detailed analysis of the geometry of the neighbourhood of the large degree vertices. An important ingredient is a matrix inequality obtained via the associated nonbacktracking matrix and an Ihara-Bass formula.

Open problems related to this talk. Delocalized or localized behavior of eigenvectors near energy levels $-2, 0, 2$, for small connectivity.

Percy Deift: Universality in numerical computation with random data [22, 49]. The speaker described various universality results in numerical computation with random data. The talk provided an overview of prior results, and also some recent developments.

Roland Bauerschmidt: Log-Sobolev inequality for the continuum Sine-Gordon model [6]. The speaker presented a multiscale generalisation of the Bakry-Emery criterion [5] for a measure to satisfy a Log-Sobolev inequality. It relies on the control of an associated PDE well known in renormalisation theory: the Polchinski equation. His criterion remains effective for measures which are far from log-concave. Indeed, he explained that the massive continuum Sine-Gordon model on R^2 with $\beta < 6\pi$ satisfies asymptotically optimal Log-Sobolev inequalities for Glauber and Kawasaki dynamics.

Open problems related to this talk. Applications of this new LSI criterion to other dynamics with equilibrium φ^4, Ising models, for example.

Amol Aggarwal: Universality for Lozenge Tiling Local Statistics [1]. The speaker considered uniformly chosen random lozenge tilings of essentially arbitrary domains and showed that the local statistics of this model around any point in the liquid region of its limit shape are given by the infinite-volume, translation-invariant, extremal Gibbs measure of the appropriate slope. It confirms a famous prediction of Cohn-Kenyon-Propp from 2001 [20] in the case of lozenge tilings.

Open problems related to this talk. Edge statistics of Tracy-Widom type in such tiling models.

Philippe Sosoe: On the two-dimensional hyperbolic sine-Gordon equation [47]. The speaker considered the two-dimensional stochastic sine-Gordon equation (SSG) in the hyperbolic setting. In particular, by introducing a suitable time-dependent renormalization, Sosoe and collaborators proved local well- posedness of SSG for any value of a parameter $\beta > 0$ in the nonlinearity. This is in contrast to the parabolic case studied by Hairer and Shen [38] and Chandra-Hairer-Shen [16], where the parameter is restricted to the subcritical range $\beta^2 < 8\pi$.

Giorgio Cipolloni: Universality at criticality: Cusp and Circular Edge [17, 18, 19]. As explained in Section 1.1, in the last decade, Wigner-Dyson-Mehta (WDM) conjecture has been proven for very general random matrix ensembles in the bulk and at the edge of the self consistent density of states (scDos). The speaker explained this recent work on universality at the cusp of the scDos. About universality for non-Hermitian matrices much less is known (see [35] for the integrable model). The author explained his proof of universality at the circular edge of any non-Hermitian matrix X with entries i.i.d. real or complex centered random variables.

Open problems related to this talk. Bulk universality in non perturbative setting for non-Hermitian random matrices. For the same model, edge universality beyond two moment matching.

Benjamin Landon: Fluctuations of the overlap of the spherical SK model at low temperature [43]. The speaker considered the fluctuations of the overlap between two replicas in the 2-spin spherical SK model in the low temperature phase. He showed that the fluctuations are of order $N^{-1/3}$ and are given by a simple, explicit function of the eigenvalues of a GOE matrix.

Open problems related to this talk. Fluctuations of overlaps in presence of a magnetic field, and a quenched result.

4 Outcome of the Meeting

The meeting was very well appreciated by all the attendees, who shared and learned some of the most important developments in probability and mathematical physics in the last several years. The younger researchers also had an opportunity to meet, interact with and discuss mathematics with several of the most senior people in the field. Several very recent breakthroughs were presented for the first time. It was an unambiguous success.

References

[1] Amol Aggarwal, Universality for Lozenge Tiling Local Statistics, 2019.

[2] J. Alt, R. Ducatez, A. Knowles Extremal eigenvalues of critical Erds-Rnyi graphs.

[3] Anderson, P., Absences of diffusion in certain random lattices, *Phys. Rev.* **157** (1958), 1492-1505.

[4] Baik, Jinho, Deift, Percy, Johansson, Kurt, On the distribution of the length of the longest increasing subsequence of random permutations, *J. Amer. Math. Soc.* **12** (1999), 1119-1178.

[5] Bakry, Dominique, Emery, Michel, Diffusions hypercontractives, *Séminaire de probabilités, XIX, 1983/84* **12** (1985).

[6] R. Bauerschmidt, T. Bodineau, Log-Sobolev inequality for the continuum Sine-Gordon model, 2019.

[7] C. Boccato, C. Brennecke, S. Cenatiempo, B. Schlein. Bogoliubov excitation spectrum for Bose-Einstein condensates.

[8] O. Bohigas, M.-J. Giannoni, C. Schmidt, Characterization of chaoic quantum spectra and universality of level fluctuation laws, Phys. Rev. Lett. 52 (1984), 1-4.

[9] Bourgade, P., Erdős, L., Yau, H.-T., Yin, J., Universality for a class of random band matrices, *Advances in Theoretical and Mathematical Physics* **21** (2017).

[10] Bourgade, P., Yau, H.-T., Yin, J., Random band matrices in the delocalized phase, I: Quantum unique ergodicity and universality (2018).

[11] Bourgain, Jean ; Kenig, Carlos E. On localization in the continuous Anderson-Bernoulli model in higher dimension. Invent. Math. 161 (2005), no. 2, 389-426.

[12] Birger Brietzke, Jan Philip Solovej, J., The Second Order Correction to the Ground State Energy of the Dilute Bose Gas (2019).

[13] Birger Brietzke, Soren Fournais, Jan Philip Solovej, J., A simple 2nd order lower bound to the energy of dilute Bose gases (2019).

[14] Lev Buhovsky, Alexander Logunov, Eugenia Malinnikova, Mikhail Sodin, A discrete harmonic function bounded on a large portion of Z^2 is constant, 2018.

[15] Matthew Cha, Pieter Naaijkens, Bruno Nachtergaele, On the stability of charges in infinite quantum spin systems, arXiv:1804.03203.

[16] Ajay Chandra, Martin Hairer, Hao Shen, The dynamical sine-Gordon model in the full subcritical regime, 2018.

[17] Giorgio Cipolloni, Laszlo Erdos, Torben Kruger, Dominik Schroder, Cusp Universality for Random Matrices II: The Real Symmetric Case, 2018.

[18] Giorgio Cipolloni, Laszlo Erdos, Dominik Schroder, Edge Universality for non-Hermitian Random Matrices, 2019.

[19] Giorgio Cipolloni, Laszlo Erdos, Dominik Schroder, Optimal Lower Bound on the Least Singular Value of the Shifted Ginibre Ensemble.

[20] Cohn, Henry, Kenyon, Richard, L., Propp, James, A variational principle for domino tilings, *J. Amer. Math. Soc.* **21** (2001).

[21] Nicholas A. Cook, Amir Dembo, Large deviations of subgraph counts for sparse Erdos-Renyi graphs (2018).

[22] Deift, P., Trogdon, T., Universality for the Toda algorithm to compute the eigenvalues of a random matrix, *Comm. Pure Appl. Math., to appear*.

[23] Jian Ding, Charles K Smart, Localization near the edge for the Anderson Bernoulli model on the two dimensional lattice, *Inventiones, to appear*.

[24] Disertori, M., Pinson, L., Spencer, T., Density of states for random band matrices, Commun. Math. Phys., 232, 83-124, 2002.

[25] Bruce K. Driver, Brian C. Hall, Todd Kemp, The Brown measure of the free multiplicative Brownian motion, 2019.

[26] Dyson, Freeman J. A Brownian-motion model for the eigenvalues of a random matrix. J. Mathematical Phys. 3 (1962), 1191–1198.

[27] Efetov, K., Supersymmetry in disorder and chaos, Cambridge University Press, 1997.

[28] Erdös, L., Schlein, B., Yau, H.-T., Universality of random matrices and local relaxation flow, Inv. Math., 187, 2011, 1, 75-119.

[29] Erdős, L., Yau, H.-T., A dynamical approach to random matrix theory, series=Courant Lecture Notes in Mathematics, 28, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI. 2017.

[30] Erdős, L., Yau, H.-T., Yin, J., Bulk universality for generalized Wigner matrices, 2012, Probab. Theory Related Fields, 154, 1-2.

[31] Soren Fournais, Marcin Napiorkowski, Robin Reuvers, Jan Philip Solovej, J., Ground state energy of a dilute two-dimensional Bose gas from the Bogoliubov free energy functional (2019).

[32] Soren Fournais, Jan Philip Solovej, J., The energy of dilute Bose gases (2019).

[33] Fröhlich, J., Spencer, T., Absence of diffusion in the Anderson tight binding model for large disorder or low energy, Commun. Math. Phys., 88, 1983, 151-184.

[34] Fyodorov, Y.V., Mirlin, A.D., Scaling properties of localization in random band matrices: a sigma-model approach, Phys. Rev. Lett., 67, 1991.

[35] Ginibre, J., Statistical ensembles of complex, quaternion, and real matrices, J. Mathematical Phys., 6, 1965, 440-449.

[36] Hairer, M., A theory of regularity structures, Invent. Math., 198, 2014, 2, 269-504.

[37] Hairer, Martin, Solving the KPZ equation, Ann. of Math. (2), 178, 2013, 2, 559-664.

[38] Hairer, Martin, Shen, Hao, The dynamical sine-Gordon model Comm. Math. Phys., 341, 2016, 3, 933-989.

[39] Hairer, Martin, Quastel, Jeremy, A class of growth models rescaling to KPZ, Forum Math. Pi, 6, 2018, 112.

[40] Itzykson, C., Zuber, J. B., The planar approximation. II, J. Math. Phys., 21, 1980, 3, 411-421.

[41] Linjun Li, Lingfu Zhang, Anderson-Bernoulli Localization on the 3D lattice and discrete unique continuation principle, 2019.

[42] Kardar, Mehran; Parisi, Giorgio; Zhang, Dynamic Scaling of Growing Interfaces, Physical Review Letters, 1986.

[43] Benjamin Landon, Philippe Sosoe, Fluctuations of the overlap at low temperature in the 2-spin spherical SK model, 2019.

[44] Linjun Li, Lingfu Zhang, Anderson-Bernoulli Localization on the 3D lattice and discrete unique continuation principle, 2019.

[45] Konstantin Matetski, Jeremy Quastel, Daniel Remenik, The KPZ fixed point, 2016.

[46] Mehta, M. L., Random matrices, Pure and Applied Mathematics (Amsterdam), 142, Elsevier/Academic Press, Amsterdam, 2004, xviii+688.

[47] Tadahiro Oh, Tristan Robert, Philippe Sosoe, Yuzhao Wang, On the two-dimensional hyperbolic stochastic sine-Gordon equation, 2019.

[48] L. Pastur, M. Shcherbina, Szego-type Theorems for One-Dimensional Schrodinger Operator with Random Potential (smooth case)

[49] Pfrang, C., Deift, P., Menon, G., How long does it take to compute the eigenvalues of a random symmetric matrix?, Random matrix theory, interacting particle systems, and integrable systems, Cambridge Univ. Press, New York, 2014, 411-442.

[50] Rudnick, Zeév, Sarnak, Peter, The behaviour of eigenstates of arithmetic hyperbolic manifolds, Comm. Math. Phys., 161, 1994, 1, 195-213.

[51] Shcherbina, T., Universality of the local regime for the block band matrices with a finite number of blocks, J. Stat. Phys., 155, 2014, 3, 466-499.

[52] Shcherbina, T., On the Second Mixed Moment of the Characteristic Polynomials of 1D Band Matrices, Communications in Mathematical Physics, 328, 2014, 45-82.

[53] Shcherbina, T., Universality of the second mixed moment of the characteristic polynomials of the 1D band matrices: Real symmetric case, J. Math. Phys., 56, 2015.

[54] Shcherbina, M., Shcherbina, T., Characteristic polynomials for 1D random band matrices from the localization side, Communications in Mathematical Physics, 351, 2017.

[55] Spencer, T., Random banded and sparse matrices (Chapter 23) in Oxford Handbook of Random Matrix Theory edited by G. Akemann, J. Baik, and P. Di Francesco, 2010.

[56] Tao, T., Vu, V., Random matrices: universality of local eigenvalue statistics, Acta Math., 206, 2011, 1, 127-204.

[57] Wigner, E., Distribution of neutron resonance level spacing, In International conference on the neutron interactions with the nucleus (Columbia University, New York, 1957), Columbia Univ. Rept. CU-175 (TID-7547), 1957, 49-50.

[58] Yau, Horng-Tzer, Yin, Jun, The second order upper bound for the ground energy of a Bose gas, J. Stat. Phys., 136, 2009, 3, 453-503.