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Stability of the superselection sectors of two-dimensional quantum lattice models

Bruno Nachtergaele (UC Davis)

Joint work with

Matthew Cha (MSU, East Lansing)
Pieter Naaijkens (UC de Madrid)

To Horng-Tzer Yau, with admiration.

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Things I learned from Yau

Spectral gaps for lattice systems (Martingale Method)

Lu, S.-L., Yau, H.-T.: Spectral gap and logarithmic Sobolev Inequality for Kawasaki and Glauber dynamics. Commun. Math. Phys. **156**, 399-433 (1993)

Nachtergaele, B.: The Spectral Gap for Some Spin Chains with Discrete Symmetry Breaking. Commun. Math. Phys. **175**, 565-606 (1996)

Continuum fermion dynamics (Relative Entropy Method)

Yau, H.-T., Relative entropy and the hydrodynamics of Ginzburg-Landau models, Lett. Math. Phys. **22**, 63–80 (1991)

Nachtergaele, B. and Yau H.-T.: Derivation of the Euler Equations from Quantum Dynamics. Commun. Math. Phys. **243**, 485–540 (2003)

Outline

- Quasi-particles and excitation spectrum
- Kitaev's quantum double models — the toric-code model
- Infinite systems, GNS representation, (in)equivalent representations, superselection sectors
- Stability of superselection sectors

Excitations as particles.

(i) Spin waves in the Heisenberg ferromagnet: a model of quantum spins on the d-dimensional lattice $\mathbb{Z}^d$ with isotropic nearest neighbor interactions:

$$H^{XXX} = \sum_{|x-y|=1} (S^2 \mathbb{1} - \mathbf{S}_x \cdot \mathbf{S}_y).$$

Holstein and Primakoff (1940) observed that the excitations above the ground state (spin waves) can be regarded as (weakly) interacting bosons with a hard-core constraint.

(ii) The quantum XY chain.

$$H^{XY} = -\sum_x \sigma_x^X \sigma_{x+1}^X + \sigma_x^Y \sigma_{x+1}^Y - h \sum_x \sigma_x^Z,$$

was solved by mapping it to a system of free fermions (Lieb-Schultz-Mattis, 1961).

With a particle description of the elementary excitations in hand, whether it be fermions or bosons, non-interacting or (weakly) interacting, we obtain, at least at the heuristic level, a model for the spectrum and dynamics of the many-body system. This is often a starting point for further analysis of what is, generally, a very hard (intractable) problem.

It is important, however that these (quasi-)particle representations are robust to a degree.

In two space dimensions, particle-like states called anyons, obeying a more general form of statistics, can play the same role. We are interested in the stability of their structure.

There is a difference between the bosons of XXX and the fermions in the XY chain (in addition to the different statistics):

XXX: local

$$S_x^+ = \sqrt{2S} b_x^+ \left[\mathbb{1} - \frac{b_x^+ b_x}{2S} \right]_+^{1/2} ;$$

XY: non-local

$$\sigma_x^+ = c_x^+ \prod_{y<x} (2c_y^+ c_y - \mathbb{1}).$$

We aim to prove 'stability' of the quasi-particles, including the non-local excitations such as, e.g., the anyons of the quantum-double models.

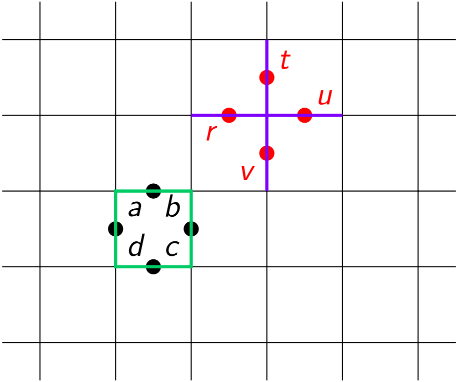
Kitaev's quantum double models (QDM)

(Kitaev, 2003)

• For concreteness, focus on the Toric Code model (TCM). There is a QDM for every finite group G ($G = \mathbb{Z}_2$ for TCM).

• Everything generalizes to arbitrary abelian G and many results also for non-abelian G.

• $\mathcal{H}_e = \mathbb{C}^2$ for all $e \in \mathcal{E}(\mathbb{Z}^2)$, the edges of the square lattice, and we are interested in the infinite-volume model.



$$H = \sum_v (\mathbb{1} - A_v) + \sum_f (\mathbb{1} - B_f)$$

$$A_v = \sigma^1_r \sigma^1_t \sigma^1_u \sigma^1_v$$

$$B_f = \sigma^3_a \sigma^3_b \sigma^3_c \sigma^3_d$$

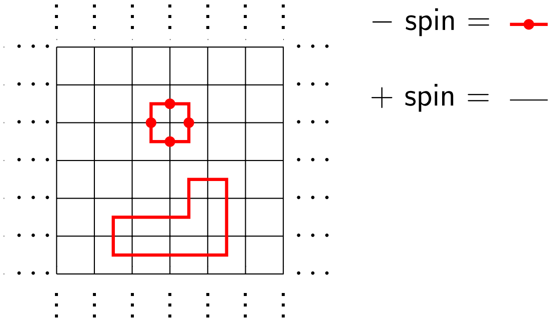
The TCM on the infinite lattice

• On all of $\mathbb{Z}^2$, the model has a unique frustration free (FF) ground state (Alicki-Fannes-Horodecki, 2007): there is a unique state ω_0 on the infinite lattice such that $\omega_0(\mathbb{1} - A_v) = \omega_0(\mathbb{1} - B_f) = 0$ for all vertices v and faces f of the infinite square lattice. (ω_0 is a normalized positive linear functional on the algebra of local observables.)

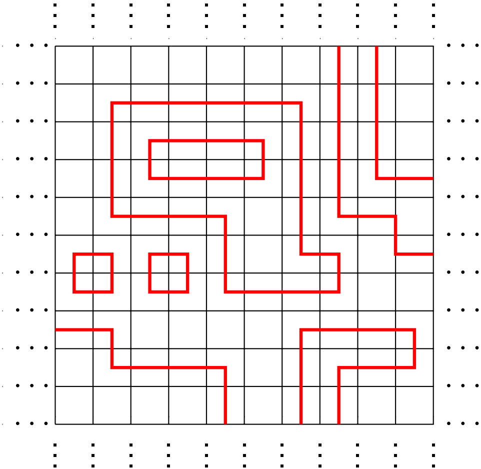
• AFH prove this using the algebra satisfied by the A_v and B_f, by showing that the vanishing of these expectations determines all expectation values.

An alternative description of ω_0 is a gas of loops on (dual) $\mathbb{Z}^2$:

• We represent the spin configurations as a set of paths in the dual lattice:

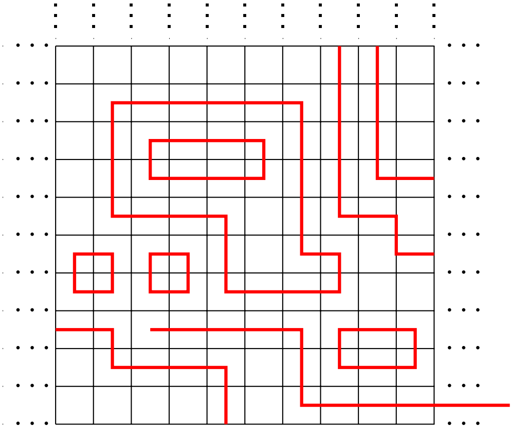


• $(\mathbb{1} - B_f)$ vanishes when the number of − spins is even in all plaquettes, and the terms $(\mathbb{1} - A_v)$ preserve this condition. Note that acting with a product of A_v's on the all + configuration creates closed loops.



The FF ground state is the equal-weigth superposition of all configurations of closed loops.

However, it is also clear that the class of configurations that have one half-infinite dual path ending in f, is also stable under the action of the operators A_v:



The end point can be moved around by local operators but cannot be removed. These are excited states of energy 2.

• The equal-weight superposition of all configurations with fixed end-point is an eigenvector of the Hamiltonian. This state can be obtained from the vacuum Ω by applying a string operator: let ρ be a dual path beginning in p and ending in q, and define the unitary operator

$$F_\rho^\mu = \prod_{x \in \rho} \sigma_x^1,$$

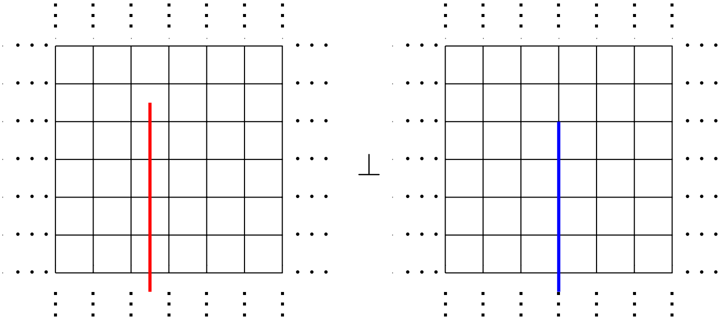
and take $\lim_{q \to \infty}$.

• All configurations correspond to a configuration of dual paths, some open, some closed. Local operators can locally modify them by flipping spins, but parity of # of endpoints is invariant.

• The role of σ^3 and σ^1 can be interchanged if we replace the lattice $\mathbb{Z}^2$ by the dual lattice, again $\mathbb{Z}^2$ (and the same set of spins).

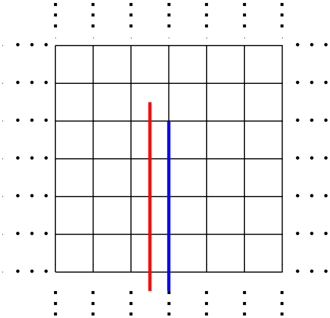
Electric and Magnetic Excitations

Omitting the soup of closed loops from the picture, the 'electric' (ϵ) and 'magnetic' (μ) excited states are associated with the end points of half-infinite paths and dual paths, respectively:



On the infinite lattice these span $\ell^2(\mathbb{Z}^2) \oplus \ell^2(\mathbb{Z}^2)$ worth of energy 2 excitations.

We also introduce ribbon states as the subspace of excitations of energy 4 that a combination of ϵ at vertex v and μ at face f, with $v \in f$:



On the infinite lattice this is a subspace $\cong \ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^4$.

Fixed anyon number representations / spaces

Infinite system: diagonalize Hamiltonian in a GNS representation (or diagonalize Heisenberg dynamics).

- Let $(\mathcal{H}_0, \pi_0, \Omega_0)$ be the GNS triple of ω_0, the unique frustration-free ground state of the Toric Code model on $\mathbb{Z}^2$.
- Let ρ be a dual path beginning in f and ending in f', and consider $\pi_0(F^{\mu}_{\rho})\Omega_0$. This is a 2-anyon state and we are interested in single-anyon excitations.
- Let ρ be a half-infinite dual path starting in f, and consider $\lim_{n\to\infty} \pi_0(F^{\mu}_{\rho_n})\Omega_0$, with ρ_n given by the first n edges in ρ.

But this converges weakly to 0. Nevertheless, we can define the matrix elements of the Hamiltonian:

$$\lim_{\Lambda\uparrow\mathbb{Z}^2} \lim_{n\to\infty} \langle \pi_0(F^{\mu}_{\rho'_n})\Omega_0, \pi_0(H_\Lambda)\pi_0(F^{\mu}_{\rho_n})\Omega_0 \rangle,$$

where ρ' starts at f'.

This works because
$\langle \pi_0(F^{\mu}_{\rho'_n})\Omega_0, \pi_0(H_\Lambda)\pi_0(F^{\mu}_{\rho_n})\Omega_0\rangle = \langle \Omega_0, \pi_0(F^{\mu}_{\rho'_n} H_\Lambda F^{\mu}_{\rho_n})\Omega_0\rangle$, and

$$\tau^{\mu}_{f}(A) = \lim_{n\to\infty} F^{\mu}_{\rho_n} A F^{\mu}_{\rho_n}, \quad A \in \mathcal{A}_{\rm loc} = \cup_{\Lambda\subset\subset\mathbb{Z}^2}\mathcal{A}_\Lambda.$$

converges and defines an automorphism on $\overline{\mathcal{A}_{\rm loc}}$, and

$$\lim_{n\to\infty} F^{\mu}_{\rho'_n} A F^{\mu}_{\rho_n} = F^{\mu}_{f\to f'}\tau^{\mu}_{f}(A).$$

The resulting matrix elements define a bounded s.a. operator on $\mathcal{H}^{\mu} \cong \ell^2(\mathbb{Z}^2)$, so energies are well-defined, but this space cannot be interpreted as a subspace of $\mathcal{H}_0$.
Similarly, define τ^{ϵ}_{v} and $\tau^{\epsilon\mu}_{(v,f)}$ using lattice paths and double paths for the ribbon states that start at vertex v and a pair (v,f) for the ribbon, and define the corresponding anyon Hamiltonians on separate Hilbert spaces $\mathcal{H}^{\epsilon} \cong \ell^2(\mathbb{Z}^2)$ and $\mathcal{H}^{\epsilon\mu} \cong \ell^2(\mathbb{Z}^2)\otimes\mathbb{C}^4$.

Define

$$\omega_f^\mu = \omega_0 \circ \tau_f^\mu; \quad \omega_f^\mu(A) = \lim_{n\to\infty} \omega_0(F_{\rho_n}^\mu A F_{\rho_n}^\mu), \quad A \in \mathcal{A}_{\rm loc}.$$

What is the GNS representation of ω_μ?

$$\omega_f^\mu(A) = \omega_0 \circ \tau_f^\mu(A) = \langle \Omega_0, \pi_0(\tau_f^\mu(A))\Omega_0 \rangle.$$

Therefore, we can take the GNS triple given by $(\mathcal{H}_0, \pi_f^\mu, \Omega_0)$, with $\pi_f^\mu = \pi_0 \circ \tau_f^\mu$.
Since for any face f', $\tau_{f'}^\mu(A) = F_{f\to f'}^\mu \tau_f^\mu(A) F_{f\to f'}^\mu$, π_f^μ and $\pi_{f'}^\mu$ are unitarily equivalent representations and ω_f^μ and $\omega_{f'}^\mu$ are vector states in the same Hilbert space.

Similarly, we have representations π_v^ϵ and $\pi_{v,f}^{\epsilon\mu}$.

We now have four classes of states and representations:

$$K^0 = \{\pi_0\}$$
$$K^\mu = \{\pi_f^\mu \mid \text{ any face } f\}$$
$$K^\epsilon = \{\pi_v^\epsilon \mid \text{ any vertex } v\}$$
$$K^{\epsilon\mu} = \{\pi_{v,f}^{\epsilon\mu} \mid \text{any vertex } v, \text{and face } f\}$$

Within each class the representations are equivalent. Two representations from different classes are inequivalent.

The ϵ and μ anyons behave as hard-core bosons which, however, have mutual statistics: moving one around the other multiplies the state vector by -1. Their combination, the ribbons, are Majorana fermions.

The Stability Question: if the TCM is subjected to (small) perturbations

$$H(\lambda) = H^{\mathrm{TC}} + \lambda \sum_{X} \Phi(X)$$

do we still have a basis for describing the system in terms of these particular anyon types?

A more precise version of the question

We adopt from QFT ('local quantum physics' (Doplicher-Haag-Roberts)) the notion that particle types are given by superselection sectors.

A superselection sector is an equivalence class of representations of the observable algebra generated by composing the vacuum representation π_0 with endopmorphisms τ that satisfy a set of (physically motivated) criteria.

- What are the appropriate criteria for the endomorphisms?
- Is the structure of superselection sectors stable under perturbations?

Superselection criteria

1) Almost-locality in cones: we denote the set of cones in $\mathbb{Z}^2$ with opening angle α by $\mathcal{C}_\alpha$ and require of τ that there is $\alpha \in (0, \pi)$ and $\Lambda \in \mathcal{C}_\alpha$, such that for all $k \geq 0$

$$\lim_{n\to\infty} n^k \sup_{A \in \mathcal{A}_{\Lambda^c_\alpha - n}, \|A\|=1} \|\tau(A) - A\| = 0$$

where '$-n$' denotes translation by n in the direction opposite to the forward direction of the axis of Λ.

2) transportability with respect to the vacuum state: for any two cones $\Lambda, \Lambda' \in \mathcal{C}_\alpha$, and τ (almost) localized in Λ, there is an equivalent τ' (almost) localized in Λ'.

'Almost locality' is the quasi-local version of the 'locality' employed by Doplicher-Haag-Roberts (1971-74) in algebraic QFT and the strict locality in cones used for the TCM by Naaijkens (2011). It is used in Cha's PhD thesis (2017) to treat perturbations of TCM.

Superselection sectors of the TCM

The superselection sectors of the TCM given as the equivalence classes of automorphisms localized in cones (Naaijkens 2011) is given by 4 classes of states equivalent to 4 classes of ground states $K^0, K^\epsilon, K^\mu, K^{\epsilon\mu}$ and can be given the structure of the braided C^* tensor category of the representations of the quantum double $\mathcal{D}(G = \mathbb{Z}_2)$.

• Next, if we add a finite-energy condition, we can show that this structure is stable under uniformly small perturbations of the TCM.

In particular, the same type of anyons describe its low-energy excitations.

Stability of the superselection sectors

- A general class of perturbations of the Hamiltonian:

$$H_\Lambda(s) = H_\Lambda^{\mathrm{TC}} + s \sum_{X \subset \Lambda} \Phi(X).$$

with Φ an interaction such that for some $a > 0$

$$\|\Phi\|_a = \sup_{x,y \in \mathbb{Z}^2} e^{a|x-y|} \sum_{\substack{X \subset \mathbb{Z}^2 \\ x,y \in X}} \|\Phi(X)\| < \infty,$$

For what follows it will be important that H_Λ^{TC} is frustration-free, gapped, and that its ground states satisfy a property called Local Topological Quantum Order.

Theorem (Cha 2017, Cha-Naaijkens-N arXiv:1804.03203)

There exists $s_0 > 0$ such that for $|s| \leq s_0$, there exists a quasi-local automorphism α_s with the following properties:
(i) α_s is the dynamics corresponding to a time-dependent short-range interaction $\Psi(s)$ (Bachmann et al. 2012)
(ii) $\omega_0 \circ \alpha_s$ is a translation invariant infinite volume ground states of the perturbed model, with a positive spectral gap (Bravyi-Hastings-Michalakis, JMP 2010);
(iii) $K^k \circ \alpha_s$, for $k \in \{0, \epsilon, \mu, \epsilon\mu\}$, describe the finite-energy superselection sectors of the perturbed model and are generated by almost localized automorphisms $\tau_s^k = \alpha_s^{-1} \circ \tau^k \circ \alpha_s$;
(iv) The set of superselection sectors of the perturbed model has the same braided (C^*-) fusion tensor category structure as TCM.

Comments and Outlook

- Exploiting quasi-locality is an essential ingredient in many recent results, and can be applied to extended operators.
- Frustration-free models turn out to be a very useful class of examples.
- Stability of the superselection sectors also comes with stability of anyons (fusion and braiding). Anyons exist.
- Thermodynamics and effective equations for many-anyon systems?
- The nature and role 'edges states' for infinite systems with boundary needs mathematical investigation. Bulk-Edge correspondence.
- Interesting examples of stable non-abelian anyons?