Fractional revival of threshold graphs under Laplacian dynamics

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Adjacency matrix and Laplacian matrix of a graph

Definition

Let X be a weighted graph on vertices $1, 2, \cdots, n$. Denote the weight of the edge between vertices u and v by $w_{u,v}$. Then the adjacency matrix $A(X)$ of X is an $n \times n$ matrix and is defined via

$$a_{uv} = \begin{cases} w_{u,v} & \text{if } u \text{ and } v \text{ are adjacent} \\ 0 & \text{if } u \text{ and } v \text{ are not adjacent,} \end{cases}$$

and the Laplacian matrix of X is defined as $L(X) = D(X) - A(X)$, where $D(X)$ is a diagonal matrix whose u-th diagonal entry is the degree of vertex u: $deg(u) = 2w_{u,u} + \sum_{v \neq u} w_{u,v}$. If $D(X)$ is a scalar matrix, then X is said to be regular.

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Both $A(X)$ and $L(X)$ are symmetric matrices. If X does not have loops, then $L(X)$ is positive semi-definite with $\mathbf{1} = (1, 1, \cdots, 1)^T$ as an eigenvector associated to eigenvalue 0.

Quantum state transfer on graphs

Fidelity of state transfer

Let M be the Laplacian matrix $L(X)$ (resp. the adjacency matrix $A(X)$) of a (weighted) graph X on n vertices. Let $U(t) = e^{itM}$, then the Laplacian (resp. adjacency) fidelity of transfer on graph X from vertex u to vertex v at time t is given by $p_{uv}(t) = |\left(U(t)\right)_{uv}|^2 = |e_u^T U(t) e_v|^2$.

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The fidelity is a number between 0 and 1.

1. If $p_{uv}(t) = 1$ for vertices u, v and time t
   1. If $u \neq v$, then X is said to admits Laplacian (resp. adjacency) perfect state transfer (PST) between u and v.
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   2. If $u = v$, then X is said to be Laplacian (resp. adjacency) periodic at vertex u at time t.
2. If $p_{uv}(t_m) \to 1$ as $m \to \infty$, then the system admits Laplacian (resp. adjacency) pretty good state transfer (PGST) between u and v.

Fractional revival

Fractional revival in terms of fidelity

If there is a time t, and two distinct vertices u and v such that $p_{uu}(t) + p_{uv}(t) = 1$, with $p_{uv}(t) > 0$, then we say there is Laplacian (resp. adjacency) fractional revival (FR) between u and v.

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Fractional revival in terms of $U(t)$

If there is a time t, and two distinct vertices u and v, such that $U(t)e_u = \alpha e_u + \beta e_v$ for some $\alpha, \beta \in \mathbb{C}$ with $\beta \neq 0$ (since $U(t)$ is unitary, we know that $|\alpha|^2 + |\beta|^2 = 1$), then we say that there is fractional revival (FR) from u to v at time t.

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If $p_{uu} = p_{uv}$, or equivalently, $|\alpha| = |\beta|$, then the revival is said to be balanced.

Generalized fractional revival

If there is a time t, and a proper subset S of $V(X)$, such that $|S| \geq 3$ and that for any vertex $u \in S$, $U(t)_{uv} = 0$ if $v \notin S$, and that the unweighted graph associated to the submatrix $U(t)_{[S,S]}$ is connected, then we say that there is generalized fractional revival between vertices in S.

Calculate the fidelity through diagonalization

The two dynamics

A regular graph admits adjacency PST (PGST, or FR) if and only if it admits Laplacian PST (PGST, or FR).

$e^{itL} = e^{it(D-A)} = e^{it(dI-A)} = e^{itd}e^{-itA} = e^{itd}\overline{e^{itA}}$, where d is the degree of the graph.

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Any real symmetric matrix is diagonalizable by a real orthogonal matrix.

Assume that M is diagonalized by a orthogonal matrix $Q = [q_{uv}]$ to $Q^T MQ = \Lambda = diag(\lambda_1, \dots, \lambda_n)$. Then

$p_{uv}(t) = |e_u^T e^{itM} e_v|^2 = |e_u^T e^{itQ\Lambda Q^T} e_v|^2 = |e_u^T Q e^{it\Lambda} Q^T e_v|^2 =$
$|q_u^T e^{it\Lambda} q_v|^2 = |\sum_{\ell=1}^n e^{it\lambda_\ell} q_{u\ell} q_{v\ell}|^2.$

Spectral decomposition of a Hermitian matrix

Spectral decomposition

Let M be an $n \times n$ Hermitian matrix. Assume that $\lambda_1, \ldots, \lambda_s$ are all the distinct eigenvalues of M, and for each $j = 1, \ldots, s$, let E_j represent the orthogonal projection matrix onto the eigenspace associated to the eigenvalue λ_j. Then the spectral decomposition of M is $M = \sum_{r=1}^{s} \lambda_r E_r$.

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1. $E_j^2 = E_j$ and $E_j E_k = 0$ if $j \neq k$;
2. $\sum_{r=1}^{s} E_r = I_n$;
3. If $f(x)$ is an analytic function which is defined at each eigenvalue of M, then $f(M) = \sum_{r=1}^{s} f(\lambda_r) E_r$.

Therefore $e^{itM} = \sum_{r=1}^{s} e^{it\lambda_r} E_r$, and $p_{uv}(t) = \sum_{r=1}^{s} e^{it\lambda_r} (E_r)_{uv}$.

Cospectral vertices and eigenvalue support of a vertex

Let M be either the adjacency matrix or Laplacian matrix of a (weighted) graph. Assume the spectral decomposition of M is $M = \sum_{r=1}^{s} \lambda_r E_r$.

Two vertices u and v are said to be strongly cospectral with respect to M if for each $r = 1, \ldots, s$, $E_r e_u = \pm E_r e_v$.

Equitable partition of a graph and quotient graph

Definition

Let X be a graph on n vertices. A partition $\pi = (C_1, \ldots, C_k)$ of $V(X)$ is equitable if for any $\ell, j \in \{1, \ldots, k\}$, the number of neighbours in C_ℓ of a vertex in C_j is the same for all vertices in C_j.

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$j \neq \ell$: the bipartite graph on $C_j \cup C_\ell$ formed by the edges between the two cells is semi-regular.

$j = \ell$: the induced subgraph on C_j is regular.

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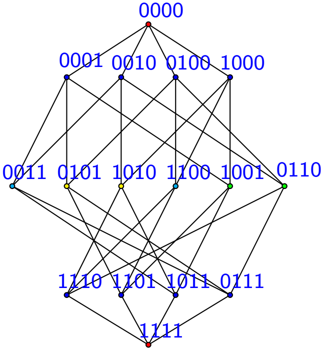
Definition

Assume that $\pi = (C_1, \ldots, C_k)$ is an equitable partition of a graph X. Define the symmetrized quotient graph $\widehat{X/\pi}$ of X with respect to π to be the unweighted graph that has the cells of π as its vertices, and with an edge of weight $\sqrt{c_{j\ell}c_{\ell j}}$ between C_j and C_ℓ whenever $c_{j\ell}c_{\ell j} \neq 0$, where $c_{j\ell}$ is the number of neighbours a vertex in cell C_j has in cell C_ℓ.

$A(\widehat{X/\pi})$ is of size $k \times k$, with (j, ℓ) entry equal to $\sqrt{c_{j\ell}c_{\ell j}}$.

An example of an equitable partition of a graph



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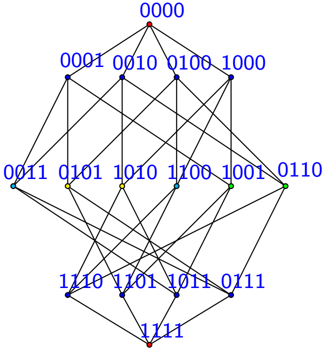

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$c_{21} = 1, c_{32} = 2, c_{43} = 3, c_{54} = 4.$

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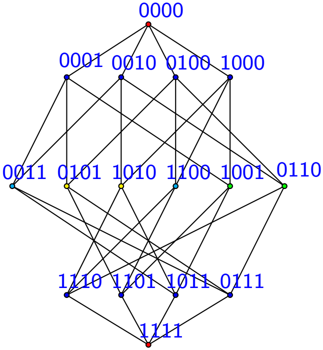

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$$A(\widehat{Q_4/\pi}) =$$

$$\begin{bmatrix} 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & \sqrt{6} & 0 & 0 \\ 0 & \sqrt{6} & 0 & \sqrt{6} & 0 \\ 0 & 0 & \sqrt{6} & 0 & 2 \\ 0 & 0 & 0 & 2 & 0 \end{bmatrix}$$

Graph and its symmetrized quotient graph

Graph and symmetrized quotient graph under XY dynamics (Bachman et al., 2012)

Let X be a graph with an equitable partition π, where vertices u and v are singleton cells. Then for any time t,

$$\left(e^{itA(X)}\right)_{uv} = \left(e^{itA(\widehat{X/\pi})}\right)_{\{u\}\{v\}}.$$

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Weighted path (Christandl–Datta–Ekert–Landahl, 2004)

For any positive integer n, there is a weighted path on n vertices which admits adjacency PST between its end vertices.
One possible weights set is $w_{u,u+1} = \sqrt{u(n-u)}$ for each $u \in \{1, \dots, n-1\}$.

Weighted paths with adjacency fractional revival

Use weighted path with PST (Genest–Vinet–Zhedanov, 2016)

Let R_m denote the antidiagonal matrix of size $m \times m$.
Assume A is the adjacency matrix of a weighted path with loops that admits adjacency PST between the two end vertices.

Let $Q = \begin{bmatrix} \sin(\theta) I_{\frac{n}{2}} & \cos(\theta) R_{\frac{n}{2}} \\ \cos(\theta) R_{\frac{n}{2}} f & -\sin(\theta) I_{\frac{n}{2}} \end{bmatrix}$ if n is even, and

$Q = \begin{bmatrix} \sin(\theta) I_{\frac{n-1}{2}} & 0_{\frac{n-1}{2}} & \cos(\theta) R_{\frac{n-1}{2}} \\ 0^T_{\frac{n-1}{2}} & 1 & 0^T_{\frac{n-1}{2}} \\ \cos(\theta) R_{\frac{n-1}{2}} & 0_{\frac{n-1}{2}} & -\sin(\theta) I_{\frac{n-1}{2}} \end{bmatrix}$ if n is odd.

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Furthermore, the deformation only changes the middle edge weight (also the weights of the loops on the middle two vertices of the path when n is even).

Therefore there is a weighted path with loops of any length that admits adjacency FR between the two end vertices.

Some known graphs

Graphs known with adjacency FR

Weighted paths with loops.

Some weighted cubelike graphs.

Some weighted graphs obtained from hypercubes.

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A weighted path with adjacency FR (Chan–Coutinho–Tamon–Vinet–Zhan, 2018)

There is adjacency FR between vertices 1 and 3 at time $t = \frac{\pi}{\sqrt{w^2+1}}$. When $w \neq 1$, the two vertices are not strongly cospectral.

Laplacian FR

1. Spectral decomposition $L(X) = \sum_{r=1}^{s} \lambda_r E_r \implies U(t) = e^{itL(X)} = \sum_{r=1}^{s} e^{it\lambda_r} E_r \implies E_r U(t) = e^{it\lambda_r} E_r$ for any r

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2. Laplacian FR at t between 1 and 2, $U(t) = \begin{bmatrix} U_1 & 0_{2,n-2} \\ 0_{n-2,2} & U_2 \end{bmatrix}$.

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3. $[E_r e_1, E_r e_2] \begin{bmatrix} u_{11} - e^{it\lambda_r} & u_{12} \\ u_{21} & u_{22} - e^{it\lambda_r} \end{bmatrix} = [E_r e_1, E_r e_2](U_1 - e^{it\lambda_r} I_2) = 0$

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4. For a (complex) symmetric matrix U, any real right eigenvector is a real left eigenvector at the same time: $Ux = \lambda x$, x real, then $x^T U = \lambda x^T$.
5. Real eigenvectors associated to distinct eigenvalues of a symmetric matrix U are orthogonal: $\lambda x^T y = (x^T U)y = x^T(Uy) = \mu x^T y$.

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6. $\mathbf{1}$ is an eigenvector of U_1, U_2 associated to 1, and $\sigma(L(X)) = \{\lambda_1, \dots, \lambda_s\} \implies \sigma(e^{itL(X)}) = \{e^{it\lambda_1}, \dots, e^{it\lambda_s}\}$.

Strong cospectrality of two vertices involved in Laplacian FR

If a connected weighted graph X admits Laplacian FR between two vertices u and v at time t, then the two vertices are strongly cospectral with respect to the Laplacian matrix $L(X)$: for $L(X) = \sum_r \lambda_r E_r$,
either $E_r e_u = E_r e_v$ (if $\frac{t\lambda_r}{2\pi} \in \mathbb{Z}$), or $E_r e_u = -E_r e_v$ (if $\frac{t\lambda_r}{2\pi} \notin \mathbb{Z}$).

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The FR time t satisfies that $\frac{t\lambda_r}{2\pi} \in \mathbb{Z}$ for some $\lambda_r \neq 0$.

Generalized Laplacian FR

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Assume that X is a weighted graph that admits generalized Laplacian FR between vertices in $S = \{1, 2, \ldots, m\} \subset V(X)$ at time t, and that $U_1 = U(t)_{[S,S]} = (e^{itL(X)})_{[S,S]}$ has 1 as a simple eigenvalue. Consider the spectral decomposition $L(X) = \sum_{r=1}^{s} \lambda_r E_r$. For each $r = 1, \ldots, s$, the vectors $E_r e_1, E_r e_2, \cdots, E_r e_m$ are linearly dependent, and either

$$E_r e_1 = E_r e_2 = \cdots = E_r e_m \text{ if } e^{it\lambda_r} = 1, \text{ or} \tag{1}$$

$$E_r e_1 + E_r e_2 + \cdots + E_r e_m = 0 \text{ if } e^{it\lambda_r} \neq 1. \tag{2}$$

A graph with (generalized) Laplacian FR

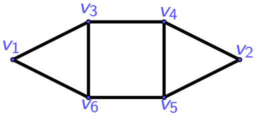

Figure: Graph X, with Laplacian eigenvalues:0, 1, $3^{(2)}$, 4,5

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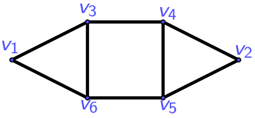

Figure: Graph X, with Laplacian eigenvalues:0, 1, $3^{(2)}$, 4,5

There is Laplacian FR between vertices v_1 and v_2, and generalized FR between vertices $\{v_3, v_4, v_5, v_6\}$ at time $\frac{2\pi}{3}$. (1 is not simple for U_2)

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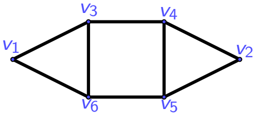

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There is also generalized Laplacian FR between vertices $\{v_1, v_4, v_5\}$, and between vertices $\{v_2, v_3, v_6\}$ at time π. (1 is simple for both U_1 and U_2)

Threshold graphs

Definition

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Characterize a connected threshold graph with join and union

A connected graph X is a threshold graph if and only if one of the two conditions is true:

1) there are indices $m_1, \ldots, m_{2k}$ with $m_1 \geq 2$ such that $X = ((((O_{m_1} \vee K_{m_2}) \cup O_{m_3}) \vee K_{m_4}) \cdots) \vee K_{m_{2k}} \equiv \Gamma(m_1, \ldots, m_{2k})$,

2) there are indices $m_1, \ldots, m_{2k+1}$ with $m_1 \geq 2$ such that $X = ((((K_{m_1} \cup O_{m_2}) \vee K_{m_3}) \cup O_{m_4}) \cdots) \vee K_{m_{2k+1}} \equiv \Gamma(m_1, \ldots, m_{2k+1})$.

Spectral property of threshold graphs

Laplacian eigenvalues of the threshold graph $\Gamma(m_1, m_2, \ldots, m_{2k})$:

1. $\lambda_0 = 0$ (multiplicity 1),
2. $\lambda_j = m_{j+1} + m_{j+3} + \cdots + m_{2k}$ for any odd integer $j \in \{1, \ldots, 2k\}$ (multiplicity $\begin{cases} m_1 - 1, & \text{if } j = 1 \\ m_j & \text{otherwise.} \end{cases}$)
3. $\lambda_j = \sigma_j + m_{j+2} + \cdots + m_{2k}$ for any even integer $j \in \{1, \ldots, 2k\}$ (multiplicity m_j), where $\sigma_j := m_1 + m_2 + \cdots + m_j$ for $j = 1, 2, \ldots, 2k$.

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Orthogonal idempotents: $E_0 = \frac{1}{\sigma_{2k}} J_{\sigma_{2k}}$,

$$E_1 = \begin{bmatrix} I_{m_1} - \frac{1}{m_1} J_{m_1} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

$$E_j = \begin{bmatrix} \frac{m_j}{\sigma_{j-1}\sigma_j} J_{\sigma_{j-1}} & -\frac{1}{\sigma_j} J_{\sigma_{j-1}, m_j} & 0_{\sigma_{j-1}, \sigma_{2k}-\sigma_j} \\ -\frac{1}{\sigma_j} J_{m_j, \sigma_{j-1}} & I_{m_j} - \frac{1}{\sigma_j} J_{m_j} & 0_{m_j, \sigma_{2k}-\sigma_j} \\ 0_{\sigma_{2k}-\sigma_j, \sigma_{j-1}} & 0_{\sigma_{2k}-\sigma_j, m_j} & 0_{\sigma_{2k}-\sigma_j, \sigma_{2k}-\sigma_j} \end{bmatrix}.$$

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Laplacian FR between u and v, $\xrightarrow{\text{strong cospectrality}} \{u, v\} = \{1, 2\}$ and $m_1 = 2$.

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In this case, $(e^{itL(X)})_{1,1} = \frac{1}{2}e^{itm_2} + \frac{1}{2}$, $(e^{itL(G)})_{1,2} = -\frac{1}{2}e^{itm_2} + \frac{1}{2}$.

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Hence, if in addition,

- tm_2 (and therefore $tm_1 = 2t$) is an even integer multiple of π, then the graph G is periodic at vertex 1 (and vertex 2);
- tm_2 (and therefore $tm_1 = 2t$) is an odd integer multiple of π, then the graph G admits Laplacian perfect state transfer between vertex 1 and 2;
- tm_2 (and therefore $tm_1 = 2t$) is not an integer multiple of π, then the graph G admits Laplacian fractional revival between vertex 1 and 2.

Laplacian FR in threshold graphs

Laplacian FR in connected threshold graphs

The threshold graph $X = \Gamma(m_1, \ldots, m_e)$, where $e = 2k$ or $2k+1$, admits Laplacian FR between two vertices u and v at time t iff

1. $\{u, v\} = \{1, 2\}$ and $m_1 = 2$, and
2.
   1. $m_1 \frac{t}{\pi} = 2\frac{t}{\pi} \notin \mathbb{Z}$
   2. $(m_1 + m_2)\frac{t}{2\pi}, m_j \frac{t}{2\pi} \in \mathbb{Z}$ for $j = 3, \ldots, e$.

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Balanced Laplacian FR in connected threshold graphs

Balanced FR between u, v at time t in $X = \Gamma(m_1, \ldots, m_e)$ iff

1. $m_1 = 2$ with $\{u, v\} = \{1, 2\}$,
2. $t = \frac{2\ell+1}{4}\pi$ for some non-negative integer ℓ,
3. $m_2 = \frac{2(2s+1)}{2\ell+1}$, for the same integer ℓ as in 2.) above, and for a non-negative integer s of distinct parity from ℓ such that $(2\ell+1)|(2s+1)$ (in this case $\frac{2s+1}{2\ell+1} \equiv 3 \pmod 4$), and
4. $m_j \equiv 0 \pmod 8$ for $j = 3, \ldots, e$.

Generalized Laplacian FR in threshold graphs

Consider the threshold graph $X = \Gamma(m_1, \ldots, m_e)$, where $e = 2k$ or $2k+1$, and let $C_\ell, \ell = 1, \ldots, e$ denote the cells of the partition π of $V(X)$ according to the parameters $m_\ell, \ell = 1, \ldots, e$. Then X admits generalized Laplacian FR between vertices in $S \subset V(X)$ at some time $t > 0$ iff, for some integer $j < e$,

1. $\frac{tm_e}{2\pi}, \frac{tm_{2k-1}}{2\pi}, \ldots, \frac{tm_{j+2}}{2\pi}, \text{ and } \frac{t\sigma_{j+1}}{2\pi} \in \mathbb{Z}$,
2. $\frac{tm_{j+1}}{2\pi} \notin \mathbb{Z}$.

In this case, $S = C_1 \cup \cdots \cup C_j$, and X is periodic at all vertices in the cells $C_{j+1}, \ldots, C_e$.

Almost equitable partition

Definition

Let X be a graph on n vertices. A partition $\pi = (C_1, \ldots, C_k)$ of $V(X)$ is almost equitable if for any $\ell \neq j \in \{1, \ldots, k\}$, the number of neighbours in C_ℓ of a vertex in C_j is the same for all vertices in C_j.

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Quotient

Assume that $\pi = (C_1, \ldots, C_k)$ is an almost equitable partition of a graph X. Let $B = [b_{j\ell}]$ with $b_{j\ell} = \begin{cases} -\sqrt{c_{j\ell}c_{\ell j}} & \text{if } \ell \neq j \\ \sum_{r \neq j} c_{jr} & \text{if } \ell = j \end{cases}$.

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If u and v are both singleton cells of π. Then for any time t,

$$\left(e^{itL(X)}\right)_{u,v} = \left(e^{itB}\right)_{\{u\}\{v\}}$$

Construct more graphs with Laplacian PST, PGST or FR

More graphs with Laplacian FR

Suppose that the graph $X = (V, E)$ has an almost equitable partition π of V, with vertices u and v being singleton cells. If there is Laplacian PST, PGST or FR between vertices u and v, then for any graph Y obtained from X by adding or deleting any collection of edges within the cells of π, Y also admits Laplacian PST, PGST or FR.

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The partition of $X=\Gamma(m_1,\ldots,m_e)$, according to the indices $m_1,\ldots,m_e$ is an equitable partition, so is any refinement of it.

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$O_2 \vee K_6$ admits Laplacian FR at time $\pi/4$ and Laplacian PST at time $\pi/2$, then so does the complete bipartite graph $K_{2,6}$, since it can obtained from $O_2 \vee K_6$ by removing all the edges inside K_6.

Thank you!