Sarason's Toeplitz Product Problem

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BIRS, April 2, 2019

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Hankel and Toeplitz operators

- In each of the three cases above, we have an analytic subspace H of an L^2 space, so there is an orthogonal projection $P : L^2 \to H$. If $\varphi \in L^\infty$, we can define a two linear operators T_φ and H_φ on H by $T_\varphi f = P(\varphi f)$ and $H_\varphi f = (I - P)(\varphi f)$. They are called the Toeplitz operator and Hankel operator induced by φ (or with symbol φ).

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  depending on which case we are considering.
- This integral representation of P allows us to extend its domain to L^1, we we can define Ha-plitz operators with unbounded symbol functions.

Sarason's Toeplitz product problem

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- I came across this problem accidentally and did not know the original motivation. The problem is essentially still open for the Hardy space and the Bergman space.
- We will present a very nice solution for the Fock space!

Sarason's Toeplitz product conjecture

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- The conjecture was disproved for the Bergman space by Aleman, Pott, and Reguera in 2013.

Connections to harmonic analysis

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- A similar formulation exists for the Hardy space and for the Fock space.
- However, such two-weight estimates are highly nontrivial and no simple condition exists for the boundedness of the operators above. The counterexamples for Sarason's conjectures (for H^2 and A^2) were constructed with the help of this connection.

Prior partial results

- For the other direction of Sarason's original conjecture, there were some interesting partial results over the years. One very well-known result was that if

$$\widetilde{|f|^{2+\varepsilon}}\widetilde{|g|^{2+\varepsilon}}$$

is bounded for some $\varepsilon > 0$ that is greater than a certain "cut-off" value depending on the context (Hardy or Bergman), then the Toeplitz product $T_f T_{\overline{g}}$ is bounded.

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  is bounded for some $\varepsilon > 0$ that is greater than a certain "cut-off" value depending on the context (Hardy or Bergman), then the Toeplitz product $T_f T_{\overline{g}}$ is bounded.
- Since there are now counterexamples to Sarason's conjecture in both the Hardy and Bergman space settings, the gap above cannot be bridged in those two classical cases.
- Sarason's problem is still open for the Bergman/Hardy spaces (even for the unit disk).

The Toeplitz product problem for the Fock space

The situation is much different for the Fock space. We have

Theorem (2014, Cho-Park-Zhu)

Let f and g be functions in F^2, not identically zero. Then $T_f T_{\overline{g}}$ is bounded on F^2 if and only if $f = e^q$ and $g = ce^{-q}$, where c is a nonzero complex constant and q is a complex linear polynomial.

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Corollary

The Fock space version of Sarason's conjecture is true: $T_f T_{\overline{g}}$ is bounded on F^2 iff

$$\sup_{z\in\mathbb{C}} \widetilde{|f|^2}(z)\widetilde{|g|^2}(z) < \infty.$$

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Two key ingredients in the proof of the theorem above: (1) there are no bounded functions in F^2 other than the constants, (2) the Weyl unitary representation of the Heisenberg group.

The Heisenberg group

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- Recall that the Mobius group of the unit disk is $\mathbb{D} \times \mathbb{T}$, as every Mobius map can be decomposed into a composition of a rotation and a symmetry: $\varphi(z) = \varphi_a(e^{i\theta} z)$.
- In the case of the Fock space, the symmetries are translations $\tau_a(z) = z - a$. They induce unitary operators on F^2 as follows: $W_a f(z) = f(z - a)k_a(z)$, where k_a is the normalized reproducing kernel at a. It is easy to check that each W_a is a unitary operator on F^2. They are called Weyl operators in quantum physics.

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There are other representations of the Heisenberg group on Hilbert spaces. For example, it is possible to represent the Heisenberg group as (unitary) pseudo-differential operators on $L^2(\mathbb{R}, dx)$, which is usually called the Schrodinger representation.

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The Weyl operators are important for our analysis of Toeplitz and Hankel operators on the Fock space.

Sufficiency for the boundedness of $T_f T_{\overline{g}}$

The sufficiency of the Cho-Park-Zhu theorem follows from the following. The necessity is much more involved.

Lemma

Let $a \in \mathbb{C}$, $f(z) = e^{\overline{a}z}$, and $g(z) = e^{-\overline{a}z}$. We have

$$T_f T_{\overline{g}} = e^{\frac{1}{2}|a|^2} W_a.$$

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To prove the result, let $h \in F^2$. Then

$$\begin{aligned} T_{\overline{g}}h(z) &= \int_{\mathbb{C}} \overline{g(w)}\, h(w) K(z,w)\, d\lambda(w) \\ &= \int_{\mathbb{C}} h(w) e^{(z-a)\overline{w}}\, d\lambda(w) \\ &= h(z-a). \end{aligned}$$

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Therefore, the Toeplitz operator $T_{\overline{g}}$ is an operator of translation, and

$$T_f T_{\overline{g}} h(z) = e^{z\overline{a}} h(z-a) = e^{\frac{1}{2}|a|^2} W_a h(z).$$

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- Consequently, the Toeplitz product $T_f T_{\overline{g}}$, where $f, g \in F^2$, can be bounded for unbounded f and g, but it cannot be compact, unless one of the two functions is identically zero.
- Let $f(z) = e^{az^2}$ and $g(z) = e^{-az^2}$, where $|a| < 1/2$. Then the operator $T = T_f T_{\overline{g}}$ is densely defined on F^2, it is unbounded, but its Berezin transform

$$\widetilde{T}(z) = \langle T_{\overline{g}} k_z, T_{\overline{f}} k_z \rangle = f(z)\overline{g(z)}$$

is bounded on $\mathbb{C}$.

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- Sarason's conjecture is true and a version of the Cho-Park-Zhu theorem holds for Fock spaces defined by

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Two companion problems

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- There are some other related problems, but the two above are the most natural extensions of Sarason's Toeplitz product problem.
- Once again, the problems are wide open for H^2 and A^2. We will now present complete solutions for F^2.

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  is bounded on the unit disk (or even on bounded symmetric domains).
- The natural conjecture was that the above condition should be sufficient.

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- There are no counterexamples and there are no meaningful partial results.
- Several people have worked on the problem for many years and no progress has been made.

Hankel products on F^2

- In light of the fact that Sarason's Toeplitz product problem does not have a satisfactory answer for the Hardy and Bergman space but it has a very nice answer for the Fock space, it is natural and temping to hope that the same situation may be true for the Hankel product problem.

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- Then came surprise a few months ago.

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Pan Ma, Fugang Yan (current PhD student jointly supervised by Dechao Zheng and myself), Dechao Zheng, and myself recently realized and proved the following:

- What was easy for the Hardy and Bergman spaces, namely, the fact that the boundedness of $H_{\bar{f}}^* H_{\bar{g}}$ on H^2 and A^2 implies the boundedness of

$$H(z) = \left[\widetilde{|f|^2}(z) - |\widetilde{f}(z)|^2\right] \left[\widetilde{|g|^2}(z) - |\widetilde{g}(z)|^2\right],$$

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- What was difficult for the Hardy and Bergman spaces (the sufficiency of the boundedness of $H(z)$ for the boundedness of $H_{\bar{f}}^* H_{\bar{g}}$ on H^2 or A^2), which remains open as of now, is actually true for the Fock space.

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- Thus the story for the Fock space is completely different. And there are more details that help us understand why.

Sufficiency for Hankel products on F^2

For sufficiency in the Fock space setting, we actually proved the following.

Theorem (MYZZ, JFA 2019)

Let $f, g \in F^2$.Then the function

$$H(z) = \left[\widetilde{|f|^2}(z) - |\widetilde{f}(z)|^2\right]\left[\widetilde{|g|^2}(z) - |\widetilde{g}(z)|^2\right]$$

is bounded on $\mathbb{C}$ if and only if f and g are of the following form: at least one of f and g is constant; or both f and g are linear polynomials; or there are constants a, b, c, A, and B such that

$$f(z) = A + e^{az+b}, \quad g(z) = B + e^{-az+c}.$$

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Thus we know that the boundedness of $H(z)$ on $\mathbb{C}$ is sufficient for the boundedness of the Hankel product $H_{\overline{f}}^* H_{\overline{g}}$ on F^2. Again, this direction remains an open problem for the Hardy and Bergman space settings.

Necessity for Hankel products on F^2

For necessity in the Fock space setting, we actually proved the following, which is in sharp contrast to the Hardy and Bergman space settings.

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The boundedness of the function

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Furthermore, we are able to determine exactly when $H_{\overline{f}}^*H_{\overline{g}}=0$ on F^2. We also discovered that, on the Fock space, it is possible for $H_{\overline{f}}^*H_{\overline{g}}$ to be compact and non-zero.

Mixed Hankel/Toeplitz products

- Recall that the mixed Hankel/Toeplitz product problem asks for a characterization of analytic functions f and g such that $H_{\bar{f}} T_{\overline{g}}$ is bounded on H^2, or A^2, or F^2.

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- The natural conjecture is the following: the mixed product $H_{\bar{f}}T_{\bar{g}}$ is bounded (on H^2, A^2, or F^2) if and only if the function

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- For H^2 and A^2, it has been known that the "only if" part is true.
- Based on the results about Toeplitz and Hankel products on the Fock space, what is the guess for the mixed product problem for the Fock space?

Mixed products on the Fock space

- It turns out the behavior of the mixed product is more like the behaviour of the Toeplitz product than that of the Hankel product.

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- It was proved recently by Pan Ma, Fugang Yan, Dechao Zheng, and KZ that for f and g in the Fock space, $H_{\overline{f}} T_{\overline{g}}$ is bounded on F^2 iff the function
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  is bounded on the complex plane.
- Furthermore, we have a complete description of which functions satisfy the above condition.

Mixed products on the Fock space

Theorem (MYZZ, 2018)

Suppose f and g are functions in the Fock space F^2. Then the following conditions are equivalent:

- $H_{\overline{f}} T_{\overline{g}}$ *is bounded on* F^2.

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- *The function above is constant on* $\mathbb{C}$.
- *At least one of the following holds:*
  (a) f *is constant.*
  (b) g *is identically zero.*
  (c) f *is a linear polynomial and* g *is a nonzero constant.*
  (d) *There are constants* a, b, c, *and* A *such that*

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As a consequence, we see that the mixed product $H_{\overline{f}} T_{\overline{g}}$ cannot be compact unless it is equal to 0.

Thank You!