Rank-one perturbations and Anderson-type Hamiltonians

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at BIRS in April 2019

Thanks to NSF-DMS.

Idea of Self-adjoint Perturbation Theory

- Throughout consider self-adjoint operators on a separable Hilbert space $\mathcal{H}$.
- The general context of this talk is perturbation theory:

  Given operator A, what can we say about the spectral properties of $A+B$ for $B \in$ Class$X$?

- Classically Class $X = \{$trace cl.$\}$, $\{$Hilb.–Schmidt$\}$, $\{$comp.$\}$, or some other von Neuman–Schatten class $\mathcal{S}_p$.
- Here our goal is to:

  Relate rank one pert. w/ Anderson-type Hamiltonians!

- Rank one pert. are $A_\gamma := A + \gamma\langle\cdot,\varphi\rangle\varphi$ with $\varphi \in \mathcal{H}$, $\gamma \in \mathbb{R}$.
- This is interesting, as $\{\gamma\langle\cdot,\varphi\rangle\varphi\} \subset \mathcal{S}_p$ for all $1 \leq p \leq \infty$, while Anderson-type Hamiltonians have a random perturbation that is almost surely non-compact.

Origins, applications and connections of rank one pert.

- Differential operators with changing boundary conditions:
  - Sturm–Liouville operators (Weyl 1910),
  - Half-line Schrödinger operator $Au = -\frac{d^2}{dx^2}u + Vu$,
  - Maybe soon PDEs.
- Describe all self-adjoint extensions of a symmetric operator with deficiency indices $(1, 1)$.
- Anderson-type Hamiltonian
- Large random matrices, free probability probability
- Decoupling of CMV matrices
- Adding partition vertices to quantum graphs
- Nehari interpolation problem
- Holomorphic composition operators
- Rigid functions
- Functional models (Sz.-Nagy–Foiaș, deBranges–Rovnyak, Nikolski–Vasyunin)
- Two weight problem for Hilbert/Cauchy transform
- Existence of the limit in the Julia–Carathéodory quotient

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Unitary Equivalence and Spectral Decompositions

- $A \sim T$ means unitary equivalence of operators, i.e. $UAU^{-1} = T$ for some unitary U.
- $A \sim T (\mathrm{mod}\, \text{Class } X)$, if $(UAU^{-1} - T) \in \text{Class } X$ for some unitary operator U.
- $T_{\mathrm{ac}} \sim \left(M_z \big|_{\oplus \int \mathcal{H}(z) d\mu_{\mathrm{ac}}(z)} \right)$. Think "$\oplus \int \mathcal{H}(z) d\mu(z) = L^2(\mu)$".
- $\sigma_{\mathrm{ess}}(T) = \sigma(T) \backslash \{\text{isolated point spectrum of finite mult.}\}$.

Perturbation Theory (think "A and $T = A + B$")

Theorem (von Neuman early 1900's)

$A \sim T$(*Mod compact operators*) $\overset{A\ bdd}{\Leftrightarrow}$ $\sigma_{\text{ess}}(A) = \sigma_{\text{ess}}(T)$.

Theorem (Kato–Rosenblum 1950's, Carey–Pincus 1976)

$A \sim T$(*Mod trace class*) $\Leftrightarrow$ $A_{\text{ac}} \sim T_{\text{ac}}$, *conditions.*

Theorem (Aronszajn–Donoghue Theory 1970-80's)

Complete info for eigenvalues and absolutely cont. part of A *and* $T = A_\gamma$. *Singular parts are mutually singular and 'interlacing'.*

A and T are said to be completely non-equivalent, if there are no non-trivial closed inv. subspaces $\mathcal{H}_1, \mathcal{H}_2 \leq \mathcal{H}$ w/ $A|_{\mathcal{H}_1} \sim T|_{\mathcal{H}_2}$.

Theorem (Poltoratski 2000)

Let A *and* T *be cyclic, self-adjoint, completely non-equivalent operators with purely singular spectrum so that* $\sigma(A) = \sigma(T) = K$ *and* $\sigma_{\text{pp}}(A) \cap \partial K = \sigma_{\text{pp}}(T) \cap \partial K = \varnothing$.
Then we have $A \sim T(\text{mod}$ *rank one*$)$.

Anderson-type Hamiltonians

- Self-adjoint operator A on a separable Hilbert space $\mathcal{H}$.
- $\{\varphi_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$ orthonormal basis.
- $\omega = (\omega_1, \omega_2, \ldots)$, where ω_i's are i.i.d. random variables distributed according to an absolutely continuous probability measure $\mathbb{P}$ on $\mathbb{R}$.
- Anderson-type Hamiltonian A_ω on $\mathcal{H}$ is given by

$$A_\omega := A + \sum_n \omega_n \langle \cdot , \varphi_n \rangle \varphi_n.$$

- Perturbation is almost surely a non-compact operator.
- Special case: discrete random Schrödinger operator on $l^2(\mathbb{Z}^d)$

$$Af(x) = -\triangle f(x) = -\sum_{n\in\mathbb{Z}^d, |n|=1} (f(x+n) - f(x)),$$

$$\varphi_n(x) = \delta_n(x) = \begin{cases} 1 & x = n, \\ 0 & \text{else}. \end{cases}$$

Simon–Wolff

Recall $A_\omega = A + \sum_n \omega_n \langle \cdot\,, \varphi_n\rangle \varphi_n$ vs. $A_\gamma = A + \gamma\langle \cdot\,, \varphi\rangle\varphi$, $\gamma \in \mathbb{R}$.

- Simon–Wolff 1986 provided a characterization of when rank-one perturbation problems A_γ are pure point for Lebesgue a.e. $\gamma \in \mathbb{R}$. With this they showed that the one-dimensional discrete random Schrödinger operator exhibits 'Anderson localization'.
- Their idea was to sweep through the parameter domain for the perturbed operators' random coupling constants.
- This technique was the first kind of connection between rank one perturbations and Anderson-type Hamiltonians.
- I believe their success inspired numerous mathematical physicists to work on rank one perturbations.

Theorem (L. 2019 in BJMA)

Consider the Anderson-type Ham. $A_\omega = A + \sum \omega_n \langle \cdot , \varphi_n \rangle \varphi_n$. *together with corresponding scalar-valued spectral measure* μ_ω. *Assume* A *is bounded and of finite multiplicity. For almost all* $(\omega, \eta) \in (\prod_n \mathbb{P} \times \prod_n \mathbb{P})$ *we have:*

1) $(\mu_\omega)_{\text{ac}} \sim (\mu_\eta)_{\text{ac}}$,
2) $\sigma_{\text{ess}}(A_\omega) = \sigma_{\text{ess}}(A_\eta)$ *and*
3) *If* $(A_\omega)_{\text{ess}}$ *is cyclic almost surely and* $|\partial$*ess-supp*$(\mu_\omega)_{\text{ac}}| = 0$, *then* $(A_\omega)_{\text{ess}} \sim (A_\eta)_{\text{ess}} (\text{mod}$ *rank one*$)$.

Proof of "$(\mu_\omega)_{\rm ac} \sim (\mu_\eta)_{\rm ac}$":

Absolutely continuous distributions $\mathbb{P}$ satisfy the Kolmogorov 0-1 law. So properties that are invariant under finite rank pert. of H are enjoyed by H_ω for almost all or almost no ω (deterministic).
Fix $\varepsilon > 0$ and ω. Consider the Borel function $x \mapsto D_\varepsilon \mu_\omega(x)$ where

$$D_\varepsilon \mu_\omega(x) := \frac{\mu_\omega([x-\varepsilon, x+\varepsilon])}{2\varepsilon}.$$

The essential support of the absolutely continuous part is given by

$$\text{ess-supp}(\mu_\omega)_{\rm ac} = \left\{ x \in \mathbb{R} : 0 < \limsup_{\varepsilon \to 0} D_\varepsilon \mu_\omega(x) < \infty \right\}.$$

By the Kato–Rosenblum theorem, the symmetric difference between ess-supp$(\mu_{(0,0,0,\dots)})_{\rm ac} \Delta$ess-supp$(\mu_{(\omega_1,\dots,\omega_n,0,\dots)})_{\rm ac}$ has zero Lebesgue measure. So by the Kolmogorov 0-1 law, Lebesgue almost every point $x \in \mathbb{R}$ lies in ess-supp$(\mu_\omega)_{\rm ac}$ for almost all, or almost no ω.
In fact, we have proved that ess-supp$(\mu_\omega)_{\rm ac}$ is a deterministic set.

Proof idea "$(A_\omega)_{\mathrm{ess}} \sim (A_\eta)_{\mathrm{ess}} (\mathrm{mod}$ rank one)":

- Cauchy transform for non-negative σ:

$$K\sigma(z) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{d\sigma(t)}{t - z}, \qquad z \in \mathbb{C}_+.$$

- Use the Krein–Lifshitz spectral shift function

$$u = -\arg(1 - \pi\gamma K\nu_\gamma),$$

  which drops from π to 0 at isolated points of $\mathrm{supp}(\nu_\gamma)_{\mathrm{s}}$ and lies in $(0, \pi)$ a.e. on the absolutely continuous spectrum.

- $u \overset{1:1}{\longleftrightarrow} \{\nu_\gamma\} \overset{1:1}{\longleftrightarrow} A_\gamma$.
- Define auxiliary singular measures and then modify their Krein–Lifshitz spectral shift function.
- Use Poltoratski's sufficient conditions to ensure that the singular parts correspond to rank one perturbations.

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