Weak containment vs amenability for group actions and groupoids

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Classification Problems in von Neumann Algebras

September 30, 2019

SUMMARY

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WEAK CONTAINMENT PROPERTY (WCP)

- Let $\mathcal{G}$ be a **locally compact groupoid** with Haar system (e.g. $\mathcal{G} = X \rtimes G$ the transformation groupoid for an action of a locally compact group G on a locally compact space X). We say that $\mathcal{G}$ has the **WCP** *if its full C^*-algebra $C^*(\mathcal{G})$ coincides with its reduced C^*-algebra $C^*_r(\mathcal{G})$*. For $\mathcal{G} = X \rtimes G$, this means that $C_0(X) \rtimes G = C_0(X) \rtimes_r G$.

- We say that an **action** $G \curvearrowright A$ **of a locally compact group** G **on a** C^***-algebra** A has the **WCP** if $A \rtimes G = A \rtimes_r G$.

⇝ **Problem :** find "good conditions" for the WCP to be realized.

POSITIVE RESULTS for the WCP

$\leadsto$ Let G be a locally compact group.

- G **has the WCP iff it is amenable** (*Hulanicki 1964, 1966*). Moreover, in this case $C_r^*(G)$ is nuclear (*Guichardet 1969*).
- $C_r^*(G)$ nuclear $\not\Rightarrow$ G amenable in general (*Takesaki 1964, Connes 1976*)
- $C_r^*(G)$ nuclear $\Rightarrow$ G amenable when G is discrete (*Lance 1973*).

$\leadsto$ **If $\mathcal{G}$ is an amenable l.c. groupoid, then $\mathcal{G}$ has the WCP** and $C_r^*(\mathcal{G})$ is nuclear (*Renault 1980*). If $C_r^*(\mathcal{G})$ is nuclear and the isotropy groups are discrete then $\mathcal{G}$ is amenable.

$\leadsto$ $G \curvearrowright A$ **with G amenable has the WCP** (*Takai 1975*). Moreover if A is nuclear then $A \rtimes_r G$ is nuclear (*Rosenberg 1977*)

Characterisation of actions such that $A \rtimes_r G$ is nuclear for G discrete

Theorem (AD 1987) : Let $G \curvearrowright A$ be an action of a **discrete** group on a C^*-algebra A. The following conditions are equivalent :

(i) $A \rtimes G$ is nuclear ;

(ii) $A \rtimes_r G$ is nuclear ;

(iii) the von Neumann algebra $A^{**} \rtimes_{vn} G$ is injective ;

(iv) the action $G \curvearrowright A^{**}$ (by bitransposition) is amenable in the von Neumann sense and A^{**} is injective.

$\leadsto$ What is an amenable action on a von Neumann algebra ?

AMENABLE ACTIONS ON von NEUMANN ALGEBRAS

▶ The notion of amenable action of a l.c. group G on $L^\infty(X,\mu)$ was introduced by *Zimmer 1977* in terms of a fixed point property.

⤳ Later on he proved, **when G is discrete**, that $G \curvearrowright L^\infty(X,\mu)$ **is amenable if there exists an equivariant norm-one projection**

$$L^\infty(G) \otimes L^\infty(X,\mu) \to L^\infty(X,\mu).$$

This was extended to any l.c. group G by *Adams-Elliott-Giordano 1994*.

⤳ *Zimmer 1977* : $L^\infty(X,\mu) \rtimes_{vn} G$ with G discrete is injective iff $G \curvearrowright L^\infty(X,\mu)$ is amenable.

AMENABLE ACTIONS ON von NEUMANN ALGEBRAS

▶ **Definition** (AD 1979) : We say that **a continuous action of a l.c. group G on a von Neumann algebra M is amenable iff there exists an equivariant norm-one projection $L^\infty(G) \otimes M \to M$.**

⤳ We have (AD 1980) :

$G \curvearrowright M$ **is amenable iff** $G \curvearrowright Z(M)$ **is amenable**

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Proposition (AD 1980) : Let $G \curvearrowright M$ where G is a discrete group. TFAE :

- $G \curvearrowright M$ is amenable and M is injective ;
- $M \rtimes_{vn} G$ is injective.

AMENABLE ACTIONS ON von NEUMANN ALGEBRAS, cont'd

▶ **Definition** : Let $\alpha : G \curvearrowright A$. A function $\theta : G \to A$ is **positive type** (w.r.t. the action) if for any $g_1, \cdots, g_n \in G$ the matrix $[\alpha_{g_i}(\theta(g_i^{-1} g_j))] \in M_n(A)$ is non-negative.

Proposition (AD 1987) : $G \curvearrowright M$, with G discrete, is amenable iff there exists a net $\theta_i : G \to Z(M)$ of **finitely supported** positive type functions such that $\theta_i(e) \leq 1$ for each i and $\lim_{i \to \infty} \theta_i(g) = 1$ **ultraweakly** for each $g \in G$.

AMENABLE ACTIONS ON C^*-ALGEBRAS

Recall :

Theorem : Let $G \curvearrowright A$ be an action of a **discrete** group on a C^*-algebra A. The following conditions are equivalent :

(i) $A \rtimes G$ is nuclear ;

(ii) $A \rtimes_r G$ is nuclear ;

(iii) the von Neumann algebra $A^{**} \rtimes_{vn} G$ is injective ;

(iv) the action $G \curvearrowright A^{**}$ (by bitransposition) is amenable in the von Neumann sense and A^{**} is injective.

This motivates the following definition :

► **Definition** : We say that $G \curvearrowright A$, with G **discrete, is amenable, as a** C^***-algebra action** if the action $G \curvearrowright A^{**}$ is **amenable in the von Neumann sense**.

AMENABLE ACTIONS ON von NEUMANN ALGEBRAS, cont'd

WARNING !

$\rightsquigarrow$ $G \curvearrowright \ell^\infty(G)$ is amenable when $\ell^\infty(G)$ is seen as a von Neumann algebra, but is amenable when $\ell^\infty(G)$ is seen as a C^*-algebra iff G is exact.

AMENABLE ACTIONS ON C^*-ALGEBRAS, cont'd

Proposition (AD 1987) : Every amenable action $G \curvearrowright A$ has the WCP.

Recall :

$\rightsquigarrow$ $G \curvearrowright A$, with G discrete, is **amenable** iff there exists a net $(\theta_i : G \to Z(A^{**}))_i$ of finitely supported positive type functions such that $\theta_i(e) \leq 1$ for each i and $\lim_{i\to\infty} \theta_i(g) = 1$ **ultraweakly** for each $g \in G$.

$\rightsquigarrow$ *Buss-Echterhoff-Willett 2019* have introduced a stronger definition of amenable action :

► **Definition** : $G \curvearrowright A$, with G discrete, is **strongly amenable** if there exists a net $(\theta_i : G \to Z(M(A)))_i$ of finitely supported positive type functions such that $\theta_i(e) \leq 1$ for each i and $\lim_{i\to\infty} \theta_i(g) = 1$ **strictly** for each $g \in G$.

AMENABLE ACTIONS ON C^*-ALGEBRAS, cont'd

⤳ **Remark** : Let $G \curvearrowright X$ and so $G \curvearrowright C_b(X)$. Given $\theta : G \to C_b(X)$, set $\tilde{\theta}(x, g) = \theta(g)(x)$. Then θ is positive type iff $\tilde{\theta}$ is positive type on the groupoid $X \rtimes G$, *i.e.* for any $g_1, \cdots, g_n \in G$ and any $x \in X$ the matrix $[\tilde{\theta}(g_i^{-1}x, g_i^{-1}g_j)] \in M_n(\mathbb{C})$ is non-negative.

▶ **Definition** : The transformation groupoid $X \rtimes G$ is amenable if there exists a net $(\tilde{\theta}_i)_i$ of positive type functions on the groupoid $X \rtimes G$, with compact support and and $(\tilde{\theta}_i(e, \cdot) \leq 1$, which converges to 1 uniformly on compact subsets of $X \rtimes G$.

Proposition (AD 1987) : The groupoid $X \rtimes G$ is amenable iff $G \curvearrowright C_0(X)$ is strongly amenable, iff $G \curvearrowright C_0(X)$ is amenable.

AMENABLE ACTIONS ON C^*-ALGEBRAS and the WCP

Let $G \curvearrowright A$, with G discrete.

⟿ Recall :

strongly amenable action $\Rightarrow$ **amenable action** $\Rightarrow$ **WCP**.

⟿ **Question** : What about the converses ?

Yuhei Suzuki (2018) has shown that **every <u>exact</u> discrete group has an action with the WCP on a <u>simple unital nuclear</u> C^*-algebra**.
If G is exact but not amenable the action is not strongly amenable since $Z(M(A)) = Z(A) = \mathbb{C}$.

⟿ However, in this example, $A \rtimes_r G$ is nuclear, and therefore the action is amenable.

Hence we have :

- **amenable action** $\not\Rightarrow$ **strongly amenable action**
- **WCP** $\not\Rightarrow$ **strongly amenable action**.

AMENABLE ACTIONS ON C^*-ALGEBRAS and the WCP, cont'd

Lemma : If $G \curvearrowright A$ is amenable there exists a ucp equivariant map $\Phi : \ell^\infty(G) \to Z(A^{**})$. The converse is true when G is **exact**.

Proof : If there exists an equivariant projection $\ell^\infty(G) \otimes Z(A^{**}) \to Z(A^{**})$ we consider its restriction to $\ell^\infty(G)$. Conversely assume that G is exact. Then $G \curvearrowright \ell^\infty(G)$ is amenable (**in the** C^* **sense**) and there exists a net $(\theta_i : G \to \ell^\infty(G))_i$ of finitely supported positive type functions such that $\theta_i(e) \leq 1$ for each i and $\lim_{i\to\infty} \theta_i(g) = 1$ in norm for each $g \in G$. Then, considering the net $(\Phi \circ \theta_i : G \to Z(A^{**}))_i$ show that $G \curvearrowright A$ is amenable. □

Theorem (*Matsumura 2014*) : Let $G \curvearrowright X$ be an action of a discrete **exact** group. Then the WCP of the action (or of the groupoid $X \rtimes G$) implies the amenability of the action.

Proof : One constructs a representation $\pi : C_0(X) \rtimes_r G \to \mathcal{B}(H)$, extends it to a cp map $\mathcal{B}(\ell^2(G) \otimes H) \to \mathcal{B}(H)$ and take the restriction to $\ell^\infty(G) \otimes A$. □

AMENABILITY vs WCP for GROUP BUNDLES

▶ A **groupoid group bundle** is a l.c. groupoid $\mathcal{G}$ such that $r = s$ and r is an *open* map from $\mathcal{G}$ onto its space of units X. It can be seen as a field of groups $x \in X \mapsto G(x) = \{\gamma \in \mathcal{G} : r(\gamma) = x = s(\gamma)\}$ arranged in a continuous way.

⤳ $C^*(\mathcal{G})$ is a $C_0(X)$-algebra with fibres $C^*(G(x))$ [1] :
if $\Pi_x : C^*(\mathcal{G}) \to C^*(G(x))$ is the quotient map, the map $x \mapsto \|\Pi_x(a)\|$ is **upper** semicontinuous for $a \in C^*(\mathcal{G})$.

⤳ if $\pi_x : C^*_r(\mathcal{G}) \to C^*_r(G(x))$ is the quotient map, the map $x \mapsto \|\pi_x(a)\|$ is **lower** semicontinuous for $a \in C^*_r(\mathcal{G})$.

1. $(\varphi f)(\gamma) = \varphi \circ r(\gamma) f(\gamma)$ for $\varphi \in C_0(X)$ and $f \in C_c(\mathcal{G})$

AMENABILITY vs WCP for GROUP BUNDLES, cont'd

Let $\mathcal{G}$ be a groupoid group bundle over $X = \mathcal{G}^{(0)}$. For $x \in X$ let π_x be the canonical surjective map from $C_r^*(\mathcal{G})$ onto $C_r^*(G(x))$. We set $U_x = X \setminus \{x\}$ and denote by $\mathcal{G}(U_x)$ the subgroupoid of those $\gamma \in \mathcal{G}$ such that $r(\gamma) \in U_x$.

Proposition : Let $\mathcal{G}$ groupoid group bundle and let us consider the following conditions.

(1) $\mathcal{G}$ is amenable ;

(2) for every $a \in C_r^*(\mathcal{G})$ the function $x \mapsto \|\pi_x(a)\|$ is continuous ;

(3) the sequence $0 \to C_r^*(\mathcal{G}(U_x)) \to C_r^*(\mathcal{G}) \to C_r^*(G(x)) \to 0$ is exact for every $x \in X$.

Then we have $(1) \Rightarrow (2) \Leftrightarrow (3)$. Moreover these three conditions are equivalent when $\mathcal{G}$ has the weak containment property.

AMENABILITY vs WCP for GROUP BUNDLES, cont'd

Assume that $\mathcal{G}$ has the weak containment property. For every $x \in X$ the following diagram is commutative

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C^*(\mathcal{G}(U_x)) & \longrightarrow & C^*(\mathcal{G}) & \longrightarrow & C^*(G(x)) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \lambda & & \downarrow \lambda_x & & \\
0 & \longrightarrow & C^*_r(\mathcal{G}(U_x)) & \longrightarrow & C^*_r(\mathcal{G}) & \xrightarrow{\pi_x} & C^*_r(G(x)) & \longrightarrow & 0
\end{array}$$

The first line is exact and λ is an isomorphism. It follows that if the second line is exact, then λ_x is injective, *i.e.* $G(x)$ is amenable for every $x \in X$, and therefore $\mathcal{G}$ is amenable.

AMENABILITY vs WCP for GROUP BUNDLES, cont'd

We have :

⤳ $\mathcal{G}$ is amenable $\Leftrightarrow$ $\mathcal{G}$ has the WCP and for every $x \in X$ the sequence

$$0 \to C_r^*(\mathcal{G}(U_x)) \to C_r^*(\mathcal{G}) \to C_r^*(G(x)) \to 0$$

is exact.

⤳ **Problem** : Does the WCP imply the exactness of this sequence for every $x \in X$?

HLS GROUPOIDS

$\rightsquigarrow$ **Example : Higson-Lafforgue-Skandalis (HLS) groupoids**

Let Γ be a finitely generated **residually finite** group and let $\Gamma \supset N_0 \supset N_1 \cdots \supset N_k \supset \cdots$ be a decreasing sequence of finite index normal subgroups with $\cap_k N_k = \{e\}$. We set $\Gamma_k = \Gamma / N_k$ and $\Gamma_\infty = \Gamma$ and denote by $\rho_k : \Gamma \to \Gamma_k$ the quotient map. Let $\widehat{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ be the Alexandroff compactification of $\mathbb{N}$. Let $\mathcal{G}$ be the quotient of $\widehat{\mathbb{N}} \times \Gamma$ by the equivalence relation

$$(k, s) \sim (l, t) \quad \text{if} \quad k = l \quad \text{and} \quad \rho_k(s) = \rho_k(t).$$

Equipped with the quotient topology, $\mathcal{G}$ is an étale Hausdorff groupoid, a bundle of groups, whose fibre at k is $G(k) = \Gamma_k$.

$\rightsquigarrow$ *HLS 2002* :
the sequence $0 \longrightarrow C_r^*(\mathcal{G}(\mathbb{N})) \longrightarrow C_r^*(\mathcal{G}) \longrightarrow C_r^*(G(\infty)) \longrightarrow 0$ is **not exact** whenever Γ is **infinite and has Kazdhan's property (T)** (it is not even exact in K-theory !)

HLS GROUPOIDS, cont'd

Proposition (AD 2015) : The sequence

$$0 \longrightarrow C_r^*(\mathcal{G}(\mathbb{N})) \longrightarrow C_r^*(\mathcal{G}) \longrightarrow C_r^*(G(\infty)) \longrightarrow 0$$

is **exact iff the group Γ is amenable** (thus iff the HLS groupoid $\mathcal{G}$ is amenable).

$\rightsquigarrow$ *Willett 2015* : **There exist HLS groupoids that have the WCP and are not amenable**. He takes $\Gamma = \mathbb{F}_2$ and constructs a decreasing sequence $\mathbb{F}_2 \supset N_0 \supset N_1 \cdots \supset N_k \supset \cdots$ of finite index normal subgroups of $\mathbb{F}_2$ with $\cap_k N_k = \{e\}$ such the corresponding HLS groupoid has the weak containment property.

EXACT GROUPOIDS

⤳ Knowing *a priori* some exactness property of a groupoid seems useful in order to show that its weak containment property implies its amenability.

⤳ **Questions**

- What is the right definition of exactness for a groupoid ?
- Would WCP, added with a good definition of exactness, implies amenability ?

EXACT GROUPOIDS. For a l.c. **group** G there are three possible definitions of exactness :

(1) **amenability at infinity** meaning the existence of an amenable action of G on a *compact* space X (i.e. the groupoid $X \rtimes G$ is amenable).

(2) KW-**exactness** [a], i.e. for every G-equivariant exact sequence $0 \to I \to A \to B \to 0$ of G-C^*-algebras, the corresponding sequence $0 \to I \rtimes_r G \to A \rtimes_r G \to B \rtimes_r G \to 0$ of reduced crossed products is exact.

(3) C^*-**exactness**, i.e. or every short exact sequence $0 \to J \to B \to B/J \to 0$ of C^*-algebras, the following sequence $0 \to C_r^*(G) \otimes J \to C_r^*(G) \otimes B \to C_r^*(G) \otimes (B/J) \to 0$ is exact, where $\otimes$ denotes the minimal (or spatial) tensor product.

a. for Kirchberg-Wassermann

(1) $\Rightarrow$ (2) $\Rightarrow$ (3) is immediate. That (2) $\Rightarrow$ (1) is due to *Brodski-Cave-Li* (2017) in the non-discrete case. In the discrete case, these 3 conditions are equivalent *(Kirchberg-Wassermann (1999), Ozawa (2000))*.

WEAK INNER AMENABILITY

▶ **Definition :** A locally compact group G is **weakly inner amenable** if there exists a state on $L^\infty(G)$ that is invariant under conjugacy.

⟿ Discrete groups and amenable groups are weakly inner amenable.

Proposition : G is weakly inner amenable iff the following property is satisfied :
for all compact subset $K \subset G$, $\forall \varepsilon > 0$ there exists a *continuous bounded positive type, properly supported* function* f *on the group* $G \times G$ such that $|f(t,t) - 1| \leq \varepsilon$ for all $t \in K$.

If G is weakly inner amenable there exists a net (ξ_i) in $C_c(G)$ with $\|\xi_i\|_2 = 1$ and $\lim_i \langle \xi_i, \lambda_t \rho_t \xi_i \rangle = 1$ uniformly on compact subsets of G.
Then set $f_i(s,t) = \langle \xi_i, \lambda_t \rho_s \xi_i \rangle$.

⟿ Converse proved by *Crann-Tanko (JFA 2017)*

* i.e. $(supp\, f) \cap (K \times G \cup G \times K)$ compact for every compact subset of G

WEAK INNER AMENABILITY, cont'd

Proposition (AD 2000) : Let G be a weakly inner amenable l.c. group. Then the 3 definitions of exactness are equivalent.

- **Definition :** A l.c. groupoid $\mathcal{G}$ is **weakly inner amenable** if for every compact subset K of $\mathcal{G}$ and every $\varepsilon > 0$ there exists a continuous bounded positive type, properly supported function f on the groupoid product $\mathcal{G} \times \mathcal{G}$ such that $|f(\gamma, \gamma) - 1| \leq \varepsilon$ for all $\gamma \in K$.

- **Definition :** A l.c. groupoid $\mathcal{G}$ is **amenable at infinity** if it has an amenable action on a fibre l.c. space (Y, p) over $\mathcal{G}^{(0)}$ such that $p : Y \to \mathcal{G}^{(0)}$ is proper.

EXACT GROUPOIDS

For an étale groupoid we have the same results.

Proposition (AD 2000, 2016) : Let $\mathcal{G}$ be an étale groupoid that is weakly inner amenable. TFEA :

(1) $\mathcal{G}$ is amenable at infinity ;

(2) $\mathcal{G}$ is KW-exact ;

(3) $\mathcal{G}$ is C^*-exact.

⤳ **Question** : Is every étale groupoid weakly inner amenable ?

EXACT GROUPOIDS

▶ **Definition** : A l.c. groupoid $\mathcal{G}$ is **inner exact** if for every invariant open subset U of $X = \mathcal{G}^{(0)}$ (*i.e.*, $r(\gamma) \in U \Leftrightarrow s(\gamma) \in U$) the sequence

$$0 \to C_r^*(\mathcal{G}(U) \to C_r^*(\mathcal{G}) \to C_r^*(\mathcal{G}(X \setminus U) \to 0$$

is exact ($\mathcal{G}(U) = r^{-1}(U)$ and $\mathcal{G}(X \setminus U) = r^{-1}(X \setminus U)$).

⤳ **Examples** : Locally compact groups, minimal l.c. groupoids (*i.e.* without non-trivial invariant open subsets, KW-exact l.c. groupoids are inner amenable.

⤳ **Problem** : Let $\mathcal{G}$ be an inner exact groupoid. Is is true that if $\mathcal{G}$ has the WCP then $\mathcal{G}$ is amenable.
Answer is yes for l.c. groups (*Hulanicki*), for transitive l.c. groupoids (*Buneci 2001*), groupoid group bundles and more generally for l.c. groupoids $\mathcal{G}$ such that the orbit space $\mathcal{G}^{(0)}/\mathcal{G}$ is T_0 (*Bönicke, 18/09/2019*).

OPEN QUESTIONS

- For a l.c. groupoid :

$$WCP + \text{ (inner) exactness } \Rightarrow \text{ amenability ?}$$

- For an action $G \curvearrowright A$

$$WCP + G \text{ exact } \Rightarrow \text{ amenability ?}$$

Roe-Willett 2014 : If G is discrete and $G \curvearrowright \partial G$ has the WCP then G is exact and so the action is amenable.