Part 2

Multivariate Modules for (m,n)-Rectangular Combinatorics

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MODULES OF DIAGONAL HARMONIC POLYNOMIALS

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Action of $GL_\infty \times \mathbb{S}_n$ on Polynomials in the variables

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \\ \vdots & \vdots & \ddots & \vdots \\ z_1 & z_2 & \cdots & z_n \\ \vdots & \vdots & \ddots & \vdots \end{pmatrix}$$

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Action of $GL_\infty \times \mathbb{S}_n$ on Polynomials in the Variables

Action of GL_∞

$f(X) \longmapsto f(T \cdot X)$

Action of $\mathbb{S}_n$

$f(X) \longmapsto f(X \cdot \sigma)$

$$\sigma \circlearrowright$$

$$T \circlearrowright \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \\ \vdots & \vdots & \ddots & \vdots \\ z_1 & z_2 & \cdots & z_n \\ \vdots & \vdots & \ddots & \vdots \end{pmatrix}$$

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GL_∞-CHARACTER = MULTIVARIATE HILBERT SERIES

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_m \\ y_1 & y_2 & \cdots & y_m \\ \vdots & \vdots & \ddots & \vdots \\ z_1 & z_2 & \cdots & z_m \\ \vdots & \vdots & \ddots & \vdots \end{pmatrix} \begin{matrix} \to q_1 \\ \to q_2 \\ \vdots \\ \to q_k \\ \vdots \end{matrix}$$

SYMMETRIC IN THE q_i

AND

SCHUR POSITIVE

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The Module $\mathcal{E}_{m,n}$

$\mathcal{M}_{m,n}$ smallest module containing

$$V_{m,n} := \det\left(x_i^a \theta_i^b\right)_{\substack{1\leq i\leq n \\ (a,b)\in\gamma_{m,n}}}$$

closed under

- Partial derivatives
- Polarization

θ_i inert variables (degree 0)

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Polarization $m=2$

$$\left(y_1\frac{\partial}{\partial x_1}+y_2\frac{\partial}{\partial x_2}\right)(x_1^2-x_2^2)=2(y_1x_1-y_2x_2)$$

$$\left(y_1\frac{\partial}{\partial x_1}+y_2\frac{\partial}{\partial x_2}\right)^2(x_1^2-x_2^2)=2(y_1^2-y_2^2)$$

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$$\mathcal{E}_{m,n} := \mathcal{M}_{m,n} / \Lambda^* \mathcal{M}_{m,n}$$

$\Lambda^* \mathcal{M}_{m,n}$ SMALLEST MODULE CONTAINING

$$\mathcal{F} V_{m,n}$$

$\mathcal{F}$ ALL DIAGONAL SYMMETRIC DERIVATION WITHOUT CONSTANT TERM

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EXAMPLES OF $f \in \Lambda^*$

$m=3$

$$\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}$$

$$\frac{\partial^2}{\partial x_1 \partial y_1} + \frac{\partial^2}{\partial x_2 \partial y_2} + \frac{\partial^2}{\partial x_3 \partial y_3}$$

NOTATION

$$\xi_n = \xi_{n,n}$$

$$\xi_{rn+1,n} = \xi_{rn,n}$$

$$\xi_{2,n} = \sum_{d=0}^{\lfloor n/2 \rfloor} s_{\lfloor n/2 \rfloor - d} \circledast s_{(2^d,\, 1^{n-2d})}$$

$$\xi_{2r,2} = s_{r-1} \circledast s_2 + s_r \circledast s_{11}$$

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A TOY EXAMPLE $\mathcal{E}_2$ $\Delta_2 = x_2 - x_1$

GL_∞ - ACTION

$$\mathcal{E}_2 = \underset{\substack{\circlearrowleft\\ s_2}}{\overset{1}{\mathbb{Q}\{1\}}} \oplus \mathbb{Q}\{\underset{\substack{\circlearrowleft\\ s_{11}}}{\overset{q_1}{x_2 - x_1}}, \underset{\substack{\circlearrowleft\\ s_{11}}}{\overset{q_2}{y_2 - y_1}}, \cdots, \underset{\substack{\circlearrowleft\\ s_{11}}}{\overset{q_k}{z_2 - z_1}}, \cdots\}$$

S_n - ACTION

$$\mathcal{E}_2 = 1 \otimes s_2 + (q_1 + q_2 + \cdots + q_k + \cdots) \otimes s_{11}$$

$$= 1 \otimes s_2 + s_1 \otimes s_{11}$$

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$$\xi_{m,n} = \sum_{\mu \vdash n} \sum_{\lambda} r_{\lambda\mu}\, \Delta_\lambda \otimes \Delta_\mu$$

$$\xi_{m,n} = \cdots + \langle \xi_{m,n}, \Delta_\mu \rangle \otimes \Delta_\mu + \cdots$$

$$\langle \xi_{m,n}, \Delta_\mu \rangle = \sum_{\lambda} r_{\lambda\mu}\, \Delta_\lambda$$

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$\xi_{m,n}$

EXPLICITLY CALCULATED FOR $m \leq n \leq 6$ BY P. HUBERT AND N. THIÉRY (NOT EASY)

First Conjecture

$$\xi_{mn,n}(q;z) = q^{c_{mn}-n(\mu)}\, e_{mn}\big(\tilde{H}_\mu\big|_{t^{n(\mu)}};\, z\big)$$

$$c_{mn} := \sum_{(a,b)\in\gamma_{mn}} a \qquad \mu_i := \#\text{ cells on row } i \text{ in } \gamma_{mn}$$

$$\tilde{H}_\mu(q,t;z) = \ldots + t^{n(\mu)}\,\omega Q'_{\mu'}(q;z)$$

Q'_μ dual basis of Hall-Littlewood P_μ

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THE LOCAL HOOK $e_k^\perp$-THEOREM

For all hook shapes, we have

$$e_k^\perp \langle \mathcal{E}_n, s_{1^n} \rangle = \langle \mathcal{E}_n, s_{(k+1,1^{n-k-1})} \rangle$$

Second Conjecture

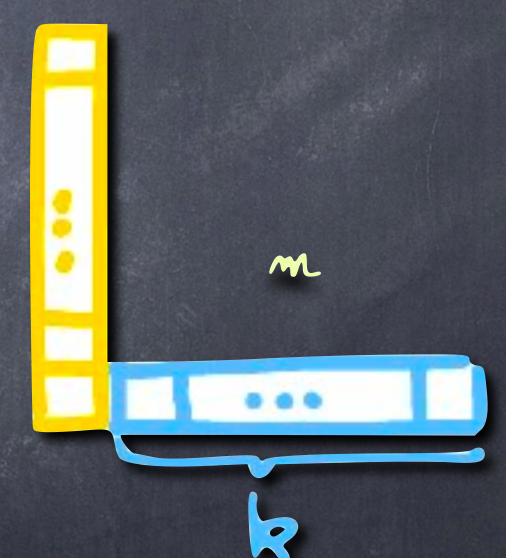
For all hook shapes, we have

$$e_k^{\perp} \langle \xi_{m,n}, s_{1^n} \rangle = \langle \xi_{m,n}, s_{(k+1,1^{n-k-1})} \rangle$$

where

coefficient of s_μ ↓

$$\xi_{m,n} = \cdots + \langle \xi_{m,n}, s_\mu \rangle \otimes s_\mu + \cdots$$



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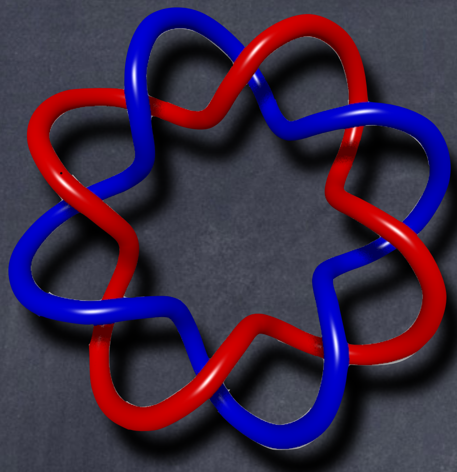

THE SUPERPOLYNOMIAL OF THE (m,n)-TORUS LINK

KHOVANOV-ROZANSKY HOMOLOGY OF (m,n)-TORUS LINKS

$$(1+a)\sum_{k=0}^{n-1} \langle \xi_{m,n}, s_{(k+1,1^{n-k-1})} \rangle\, a^k$$

↑

(m,n)-TORUS LINK

EVALUATE (m, n)-TORUS LINK q, t

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$$\langle \mathcal{E}_{m,n}, s_{(k+1,1^{n-k-1})} \rangle = \langle \mathcal{E}_{n,m}, s_{(k+1,1^{m-k-1})} \rangle$$

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$$
\begin{aligned}
\mathcal{E}_{6,4} = s_2 \otimes s_4 &+ (s_{21} + s_3 + s_{31} + s_4 + s_5) \otimes s_{31} \\
&+ (s_{111} + s_{22} + s_{31} + s_4 + s_{41} + s_6) \otimes s_{22} \\
&+ (s_{211} + s_{31} + s_{32} + 2s_{41} + s_5 + s_{51} + s_6 + s_7) \otimes s_{211} \\
&+ (s_{311} + s_{42} + s_{51} + s_{61} + s_8) \otimes s_{1111}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{E}_{4,6} = s_1 \otimes s_{42} + s_2 \otimes s_{411} + s_2 \otimes s_{33} & \\
&+ (s_{11} + s_{21} + s_2 + 2s_3 + s_4) \otimes s_{321} \\
&+ (s_{21} + s_{31} + s_3 + s_4 + s_5) \otimes s_{3111} \\
&+ (s_{21} + s_{31} + s_3 + s_5) \otimes s_{222} \\
&+ (s_{111} + s_{22} + s_{21} + 2s_{31} + s_{41} + 2s_4 + s_5 + s_6) \otimes s_{2211} \\
&+ (s_{211} + s_{32} + s_{31} + 2s_{41} + s_{51} + s_5 + s_6 + s_7) \otimes s_{21111} \\
&+ (s_{311} + s_{42} + s_{51} + s_{61} + s_8) \otimes s_{111111}
\end{aligned}
$$

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LENGTH COMPONENTS

LENGTH OF AN EXPRESSION

$$\mathcal{G} = \sum_{\mu \vdash n} \sum_{\lambda} a_{\lambda\mu}\, s_\lambda \otimes s_\mu$$

$$\ell(\mathcal{G}) := \max_{a_{\lambda\mu} \neq 0} \ell(\lambda)$$

$$\ell\Big(\sum_{\lambda} \langle \mathcal{G}, s_{\lambda\mu} \rangle\, s_\lambda\Big) := \max_{a_{\lambda\mu} \neq 0} \ell(\lambda)$$

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LENGTH PROPERTIES

$$\ell(\xi_{m,n}) = \min(m-1, n-1)$$

$$\ell(\langle \xi_n, s_\mu \rangle) = n - \mu_1$$

RECONSTRUCTION OF $A_n : \langle \xi_{nn}, \Delta_{11\cdots1} \rangle$

USING THE LOCAL HOOK $e_b^\perp$-THEOREM

$n = 4$

$$A_4 = \Delta_6 + \Delta_{31} + \Delta_{41} + \Delta_{111}$$

$$e_3^\perp A_4 = \Delta_1 + \Delta_2 + \Delta_3$$

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$$\mathcal{G} = \sum_{\mu \vdash n} \sum_{\lambda} a_{\lambda\mu}\, s_\lambda \otimes s_\mu$$

$$\mathcal{G}^{\langle k \rangle} := \sum_{\mu \vdash n} \sum_{\ell(\lambda) \leq k} a_{\lambda\mu}\, s_\lambda \otimes s_\mu$$

$\ell(\lambda)$: # PARTS OF λ

WRITE $\mathcal{G}_1 =_k \mathcal{G}_2$ IFF $\mathcal{G}_1^{\langle k \rangle} = \mathcal{G}_2^{\langle k \rangle}$

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THE GLOBAL $e_k^\perp$ - CONJECTURE

FOR ALL $0 \leq k \leq m-1$ WE HAVE

- $e_k^\perp \xi_{m+n,n} =_2 \Delta'_{e_{(m-1-k)}}(e_{mn})$
- $e_{m-1}^\perp \xi_{m+n,n} = \xi_{m,n}$

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$$\xi_4 = 1 \otimes \Delta_4 + (\Delta_1 + \Delta_2 + \Delta_3) \otimes \Delta_{31}$$

$$+ (\Delta_{21} + \Delta_2 + \Delta_4) \otimes \Delta_{22}$$

$$+ (\Delta_{11} + \Delta_{21} + \Delta_{31} + \Delta_3 + \Delta_4 + \Delta_5) \otimes \Delta_{211}$$

$$+ (\Delta_{111} + \Delta_{31} + \Delta_{41} + \Delta_6) \otimes \Delta_{1111}$$

$$e_1^{\perp} \Big| e_2^{\perp} \quad \downarrow$$

$$\Delta'_{e_2}(e_4) = (\Delta_1 \otimes \Delta_{22} + (1) \otimes \Delta_1 + \Delta_3) \otimes \Delta_{211}$$

$$+ (\Delta_{11} + \Delta_1 + (\Delta_2 + \Delta_3) \otimes \Delta_{22}) \otimes \Delta_{1111}$$

$$+ (\Delta_{11} + \Delta_{21} + \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4) \otimes \Delta_{211}$$

$$+ (\Delta_{11} + \Delta_{21} + \Delta_{31} + \Delta_3 + \Delta_4 + \Delta_5) \otimes \Delta_{1111}$$

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Reconstruction of $\mathcal{E}_n$ using the global $e_k^\perp$-conjecture

$$\Delta'_{e_1}(e_4) = \Delta_1 \otimes \Delta_{22} + (1+\Delta_1+\Delta_2)\otimes \Delta_{211} + (\Delta_1+\Delta_2+\Delta_3)\otimes \Delta_{1111}$$

$e_2^\perp$

$$\mathcal{E}_4 = 1\otimes \Delta_4 + (\Delta_1+\Delta_2+\Delta_3)\otimes \Delta_{31} + (\Delta_{21}+\Delta_2+\Delta_4)\otimes \Delta_{22} + (\Delta_{11}+\Delta_{21}+\Delta_{31}+\Delta_3+\Delta_4+\Delta_5)\otimes \Delta_{211} + (\Delta_{111}+\Delta_{31}+\Delta_{41}+\Delta_6)\otimes \Delta_{1111}$$

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EXPLICIT FORMULAS FOR $c_{\lambda\mu}$ WHEN λ IS A HOOK

BY N. WALLACE

e-Positivity and Local Δ-Conjecture

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e-Positivity Phenomenon

$$\nabla \mathcal{F}_n := \sum_{k\geq 0} h_k^\perp \, \xi_n$$

$$= \sum_{\mu \vdash n} \sum_{\lambda} d_{\lambda\mu} \, \Delta_\lambda \otimes e_\mu$$

$$d_{\lambda\mu} \in \mathbb{N}$$

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$$\mathscr{F}_{13} = 1 \otimes e_3,$$

$$\mathscr{F}_{23} = s_1 \otimes s_3 + 1 \otimes e_{21},$$

$$\mathscr{F}_{33} = (s_{11} + s_3) \otimes e_3 + (2s_1 + s_2) \otimes e_{21} + 1 \otimes e_{111},$$

$$\mathscr{F}_{53} = (s_{21} + s_4) \otimes e_3 + (s_1 + s_{11} + 2s_2 + s_3) \otimes e_{21} + (1 + s_1) \otimes e_{111},$$

$$\mathscr{F}_{63} = (s_{22} + s_{41} + s_6) \otimes e_3 + (2s_2 + 2s_{21} + s_3 + s_{31} + 2s_4 + s_5) \otimes e_{21}$$
$$+(1 + 2s_1 + s_{11} + s_2 + s_3) \otimes e_{111},$$

$$\mathscr{F}_{83} = (s_{32} + s_{51} + s_7) \otimes e_3$$
$$+(s_2 + s_{21} + s_{22} + 2s_3 + 2s_{31} + s_4 + s_{41} + 2s_5 + s_6) \otimes e_{21}$$
$$+(1 + 2s_1 + s_{11} + 2s_2 + s_{21} + s_3 + s_4) \otimes e_{111},$$

$$\mathscr{F}_{14} = 1 \otimes e_4,$$

$$\mathscr{F}_{24} = s_2 \otimes e_4 + s_1 \otimes e_{31} + 1 \otimes e_{22},$$

$$\mathscr{F}_{34} = (s_{11} + s_3) \otimes e_4 + (s_1 + s_2) \otimes e_{31} + s_1 \otimes e_{22} + 1 \otimes e_{211},$$

$$\begin{aligned}\mathscr{F}_{44} = {} & (s_{111} + s_{31} + s_{41} + s_6) \otimes e_4 + (2s_{11} + s_{21} + 2s_3 + s_{31} + s_4 + s_5) \otimes e_{31} \\ & + (s_{11} + s_2 + s_{21} + s_4) \otimes e_{22} + (3s_1 + 2s_2 + s_3) \otimes e_{211} + 1 \otimes e_{1111},\end{aligned}$$

$$\begin{aligned}\mathscr{F}_{64} = {} & (s_{311} + s_{42} + s_{51} + s_{61} + s_8) \otimes e_4 \\ & + (s_{21} + s_{211} + 2s_{31} + s_{32} + s_4 + 2s_{41} + 2s_5 + s_{51} + s_6 + s_7) \otimes e_{31} \\ & + (s_{11} + s_{111} + s_2 + s_{21} + s_{22} + s_{31} + s_4 + s_{41} + s_6) \otimes e_{22} \\ & + (2s_1 + 2s_{11} + 2s_2 + 2s_{21} + 4s_3 + s_{31} + 2s_4 + s_5) \otimes e_{211} \\ & + (1 + s_1 + s_2) \otimes e_{1111},\end{aligned}$$

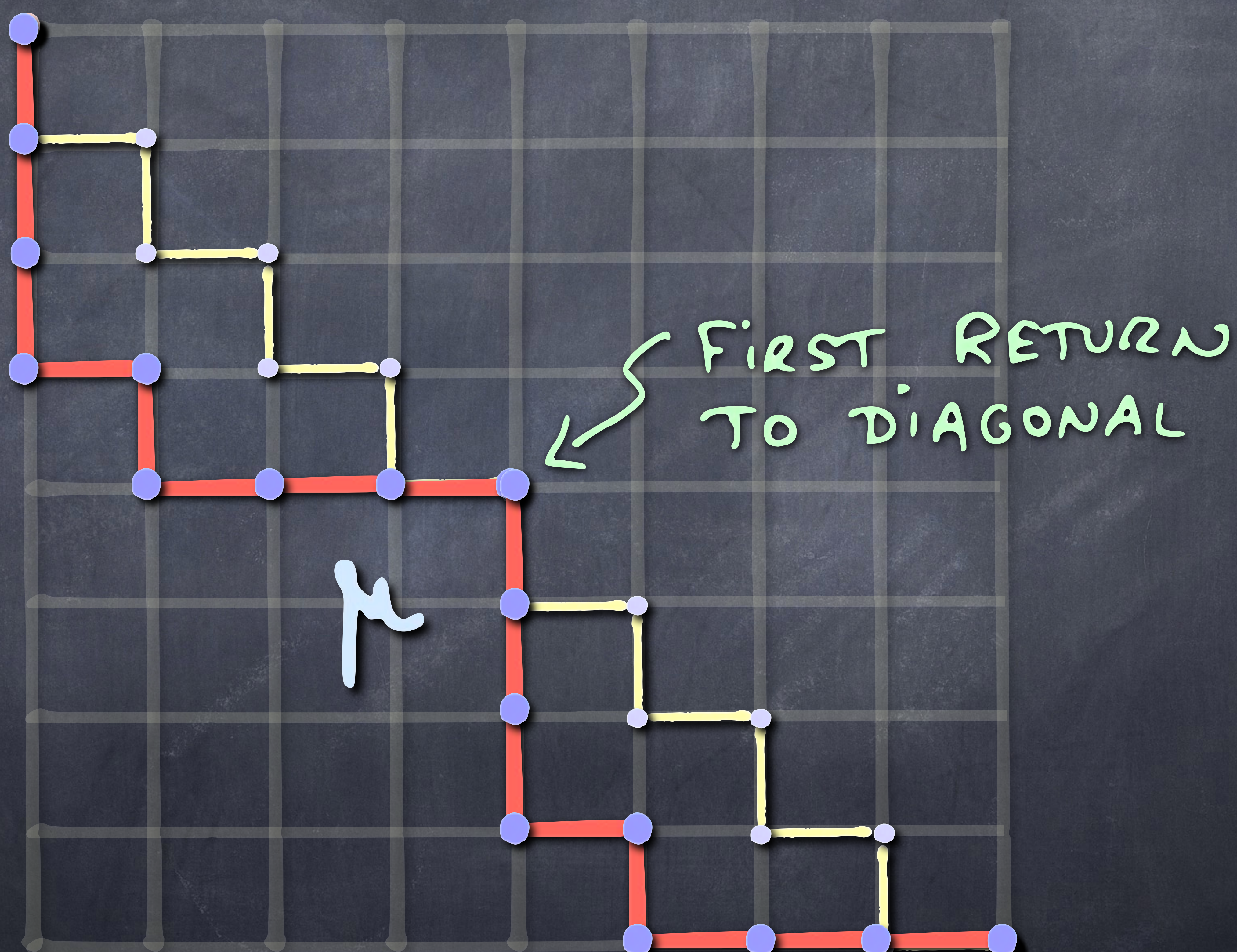
There seems to exists a family of Schur-positive functions $\sigma_n(\mu)$ such that

$$\mathcal{F}_n = \sum_{\mu \vdash n} \sigma_n(\mu) \otimes \Delta_{(\mu + 1^n)/\mu}$$

F.B. Nantel Ceballos Pilaud

Properties

- $\ell(\sigma_n(\mu)) = n - \ell(\Delta_{(\mu+1^n)/\mu})$
- $\sigma_n(0) = \langle \xi_{q_{nn}}, \Delta_{11\cdots 1} \rangle$
- $\sigma_n(\mu) = \sigma_n(\mu')$
- $e_k^\perp \sigma_n(0) = \sum_{\mathrm{DESC}(\mu)=[k]} \sigma_n(\mu)$
- $\sigma_n(0)[Q+1] = \sum_{\mu \leq \delta_n} \sigma_n(\mu)$



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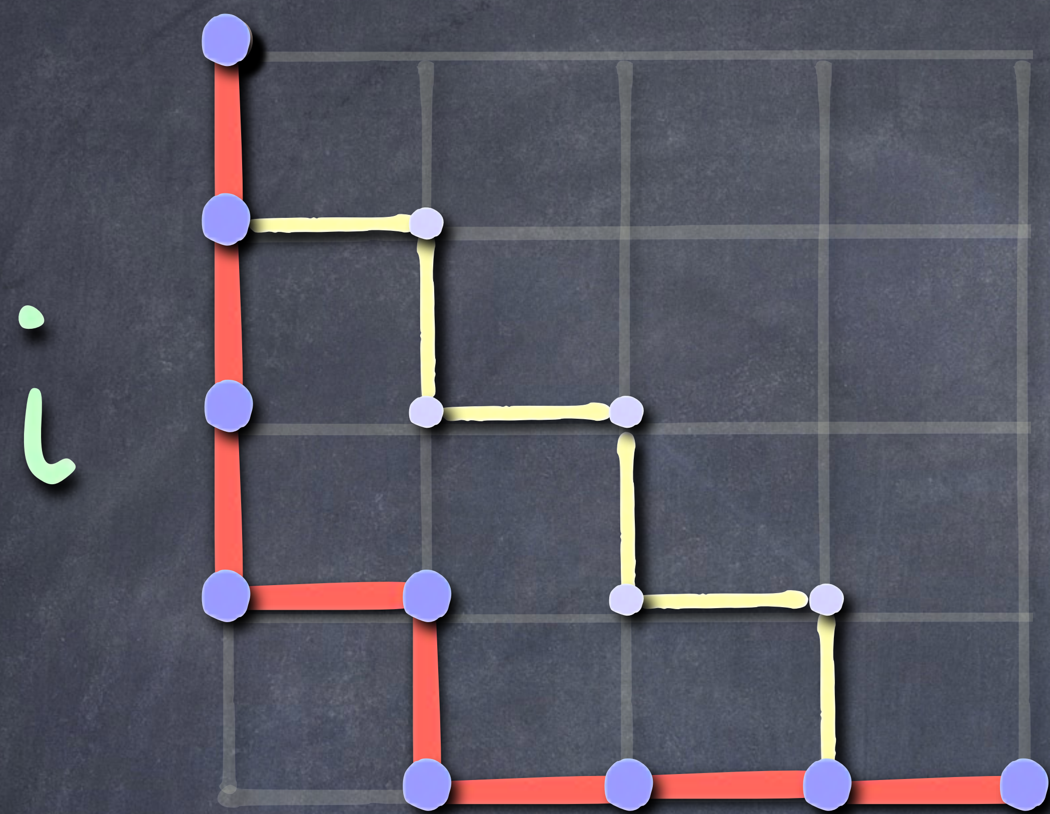
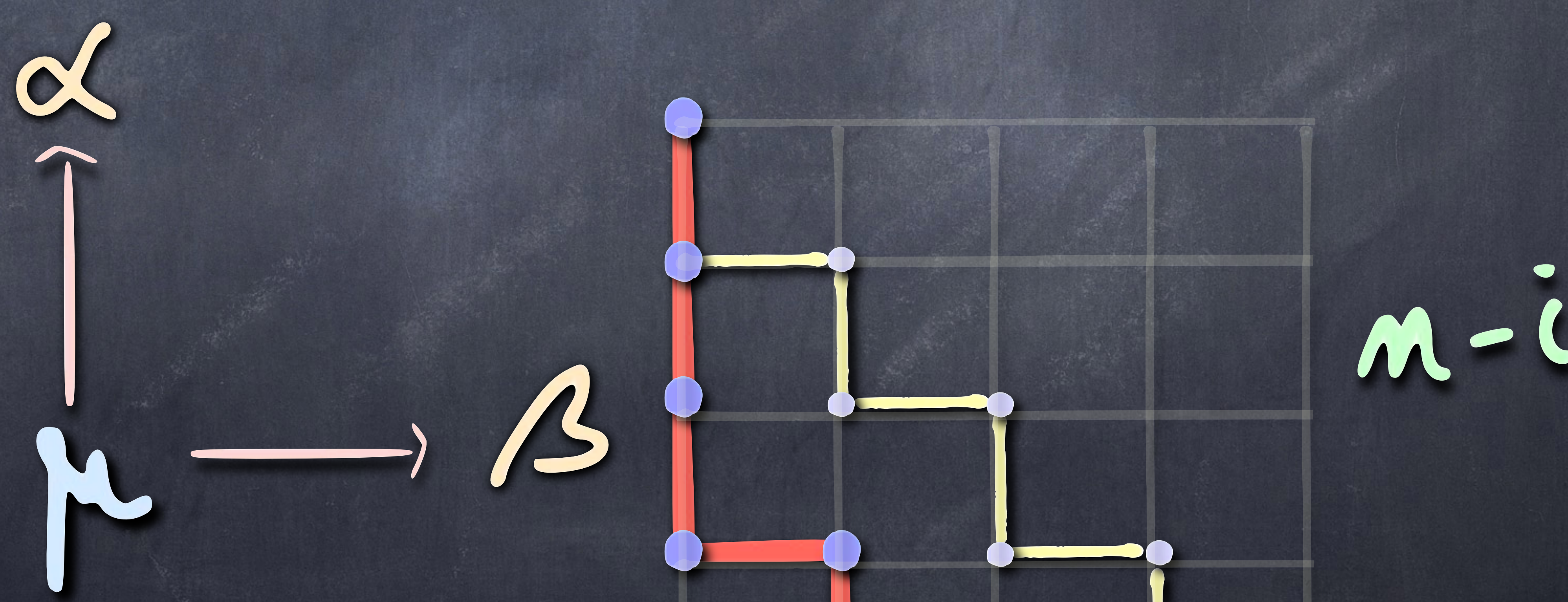
FIRST RETURN SPLIT

$\mapsto (\; , \;)$





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MULTIPLICATION PROPERTY

$$\sigma_n(\mu) = \sigma_i(\alpha)\,\sigma_{n-i}(\beta)$$

FIRST RETURN SPLIT

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LOCAL Δ-CONJECTURE

$$e_k^\perp \sigma_m(\mu) =_1 \sum_J \Delta_{(J|a)}$$

$\#J = m-1-k$

$\mathrm{DESC}(\mu) \subseteq J$

in PARTICULAR

$$e_0^\perp \sigma_m(\mu) =_1 \Delta_a$$

$a = \mathrm{AREA}(\mu)$

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$$\sigma_n(0) = s_{111} + s_{31} + s_{41} + s_6, \qquad \sigma_n(1) = s_{31} + s_5,$$
$$\sigma_n(2) = s_{21} + s_4, \qquad \sigma_n(11) = s_{21} + s_4,$$
$$\sigma_n(3) = s_{11} + s_3, \qquad \sigma_n(111) = s_{11} + s_3,$$
$$\sigma_n(21) = s_3, \qquad \sigma_n(22) = s_{11} + s_2,$$
$$\sigma_n(31) = s_2, \qquad \sigma_n(211) = s_2,$$
$$\sigma_n(32) = s_1, \qquad \sigma_n(221) = s_1,$$
$$\sigma_n(311) = s_1, \qquad \sigma_n(321) = 1.$$

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$$\mathcal{F}_n = \sum_{\mu \vdash n} \sigma_n(\mu) \otimes \Delta_{(\mu+1^n)/\mu}$$

$$\xi_n = \sum_{\mu \vdash n} \sigma_n(\mu)[Q-1] \otimes \Delta_{(\mu+1^n)/\mu}$$

$$Q = q_1 + q_2 + \cdots$$

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Fin