Multivariate Modules for (m,n)-Rectangular Combinatorics

François Bergeron, LaCIM

PART 0: A SURVEY OF THE STATE OF AFFAIRS

SYMMETRIC FUNCTIONS

$\mu = \mu_1 \geq \mu_2 \geq \cdots \geq \mu_\ell > 0$

$n = \mu_1 + \mu_2 + \cdots + \mu_\ell$

$\mu \vdash n$

- MONOMIAL $\qquad m_\mu(z) = \sum_{\substack{j_i \\ \text{DISTINCTS}}} z_{j_1}^{\mu_1} z_{j_2}^{\mu_2} \cdots z_{j_k}^{\mu_k}$

- COMPLETE HOMOGENOUS $\qquad h_n(z) = \sum_{\mu \vdash n}' m_\mu(z)$

- SCHUR $\qquad s_\mu(z) = \det\left(h_{\mu_i + j - i}(z) \right)_{1 \leq i, j \leq k}$

- ELEMENTARY $\qquad e_n(z) := m_{\underbrace{11\cdots1}_{n}}(z)$

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Graded Frobenius characteristic of an $\mathbb{S}_n$-module

$$\mathcal{W} = \bigoplus_{d \in \mathbb{N}^k} \mathcal{W}_d$$

$$\mathcal{W}_d = \bigoplus_{\mu \vdash n} \mathcal{V}_\mu^{\oplus c_{d,\mu}}$$

$c_{d,\mu}$: number of copies of μ-irreducibles in the d-homogenous component

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GRADED FROBENIUS CHARATERISTIC OF AN $\mathbb{S}_n$-MODULE

$$\mathcal{W}(q_1, q_2, \ldots, q_k; Z) = \sum_{d \in \mathbb{N}^k} q_1^{d_1} q_2^{d_2} \cdots q_k^{d_k} \sum_{\mu \vdash n} c_{d,\mu} \, \mathcal{S}_\mu(Z)$$

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ξ : $(GL_k \times \mathbb{S}_n)$-MODULE

(POLYNOMIAL GL_k-REP)

IRRED. FOR GL_k-ACTION

$$\xi = \bigoplus_{\mu \vdash n} \bigoplus_{\lambda} \left(\mathcal{I}_\lambda \otimes \mathcal{V}_\mu\right)^{\oplus c_{\lambda\mu}}$$

$$\xi(q_1, \ldots, q_k; Z) = \sum_{\mu \vdash n} \sum_{\lambda} c_{\lambda\mu}\, s_\lambda(q_1, \ldots, q_k)\, s_\mu(Z)$$

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Macdonald Polynomials and Operators, Representation Theory

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MACDONALD

$$\widetilde{H}_\mu(q,t;z)$$

$$z = z_1 + z_2 + z_3 + \cdots$$

$$\widetilde{H}_3 = \Delta_3(z) + (q+q^2)\Delta_{21}(z) + q^3\Delta_{111}(z)$$

$$\widetilde{H}_{21} = \Delta_3(z) + (q+t)\Delta_{21}(z) + qt\Delta_{111}(z)$$

$$\widetilde{H}_{111} = \Delta_3(z) + (t+t^2)\Delta_{21}(z) + t^3\Delta_{111}(z)$$

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$\mathcal{H}_n$: THE SPACE OF $\mathbb{S}_n$-HARMONICS

$\mathcal{H}_n$: THE SMALLEST SPACE THAT CONTAINS

$$V_{(n)} = \prod_{i<j} (x_i - x_j)$$

WHICH IS CLOSED UNDER PARTIAL DERIVATIVES

SHEPHARD TODD

1954

$$\mathcal{H}_n(q;z) = \widetilde{H}_n(q;z)$$

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The $n!$ Theorem

$\mu \vdash n$

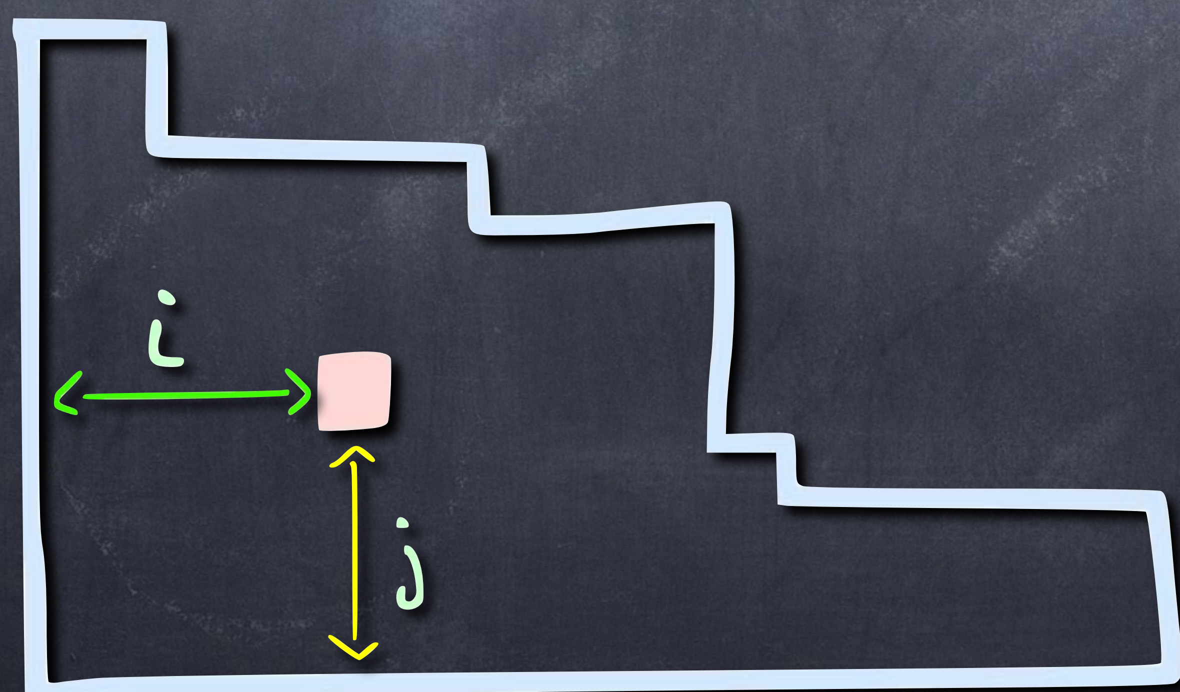
$\mathcal{M}_\mu$: the smallest space that contains

$$V_\mu := \det\left(x_k^{i} y_k^{j}\right)_{\substack{(i,j)\in\mu \\ 1\leq k\leq n}}$$

which is closed under partial derivatives

Garsia

Haiman



$(i,j)\in\mu$

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THE $n!$ THEOREM

$$\dim(\mathcal{M}_\mu) = n!$$

IMPLIES THAT

THE FROBENIUS TRANSFORM OF THE GRADED CHARACTER OF $\mathcal{M}_\mu$ IS EQUAL TO $\widetilde{H}_\mu(q,t;z)$

HAIMAN

THE $n!$ THEOREM

$$\dim(\mathcal{M}_\mu) = n!$$

IMPLIES THAT

$$\mathcal{M}_\mu(q,t;\mathbf{z}) = \widetilde{H}_\mu(q,t;\mathbf{z})$$

∇ Nabla

F.B.

Garsia

$$\nabla(\widetilde{H}_\mu) := \prod_{(i,j)\in\mu} q^i t^j \, \widetilde{H}_\mu$$

$\mathcal{E}_n$: THE SPACE OF DIAGONAL HARMONICS

$\mathcal{E}_n$: THE SMALLEST SPACE THAT CONTAINS

$$V_{(n)} = \prod_{i<j} (x_i - x_j)$$

WHICH IS CLOSED UNDER PARTIAL DERIVATIVES
AND CLOSED FOR POLARIZATION

POLARIZATION OPERATORS

$$\sum_{i=1}^{n} y_i \partial x_i^k$$

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$\mathcal{E}_n$: THE SPACE OF DIAGONAL HARMONICS

THE FROBENIUS TRANSFORM OF THE GRADED CHARACTER OF $\mathcal{E}_n$ IS EQUAL TO $\nabla(e_n)$

$\mathcal{E}_n$: THE SPACE OF DIAGONAL HARMONICS

$$\mathcal{E}_n(q,t;\mathbf{z}) = \nabla(e_n)$$

$$\mathcal{E}_n(q,t;\mathbf{z}) = \nabla(e_n)$$

$$\mathcal{E}_1(q,t;\mathbf{z}) = s_1$$

$$\mathcal{E}_2(q,t;\mathbf{z}) = s_2 + (q+t)s_{11}$$

$$\mathcal{E}_3(q,t;\mathbf{z}) = s_3 + (q^2+qt+t^2+q+t)s_{21} + \underbrace{(q^3+q^2t+qt^2+t^3+qt)}_{q,t\text{-CATALAN}}s_{111}$$

$$\dim(\mathcal{E}_n) = (n+1)^{n-1}$$

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$$\mathcal{E}_n(q,t;\mathbf{z}) = \nabla(e_n)$$

$$\mathcal{E}_1(q,t;\mathbf{z}) = s_1$$

$$\mathcal{E}_2(q,t;\mathbf{z}) = s_2 + (q+t)\,s_{11}$$

$$\mathcal{E}_3(q,t;\mathbf{z}) = s_3 + (q^2+qt+t^2+q+t)\,s_{21} + (\underbrace{q^3+q^2t+qt^2+t^3+qt}_{q,t\text{-CATALAN}})\,s_{111}$$

$$\mathcal{E}_n(q,0;\mathbf{z}) = \widetilde{H}_n(q;\mathbf{z})$$

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Δ_f Operators with Macdonald Polynomials as Joint Eigenfunctions

$$\Delta_f(\widetilde{H}_\mu) := f(\ldots, q^i t^j, \ldots)_{(i,j)\in\mu}\, \widetilde{H}_\mu$$

$$\Delta'_f(\widetilde{H}_\mu) := f(\ldots, q^i t^j, \ldots)_{\substack{(i,j)\in\mu \\ (i,j)\neq(0,0)}}\, \widetilde{H}_\mu$$

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$$\Delta'_{e_0}(e_4) = s_{1111}$$

$$\Delta'_{e_1}(e_4) = (q+t)\, s_{22}$$

$$+ (1+q+t+q^2+qt+t^2)\, s_{211}$$

$$+ (q+t+q^2+qt+t^2+q^3+q^2t+qt^2+t^3)\, s_{1111}$$

THE COEFFICIENTS ARE SYMMETRIC IN q AND t.

IN FACT THEY ARE SCHUR POSITIVE

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$$\Delta'_{e_0}(e_4) = s_{1111}$$

$$\Delta'_{e_1}(e_4) = \underbrace{(q+t)}_{s_1}\, s_{22}$$

$$+\,(1+\underbrace{q+t}_{s_1}+\underbrace{q^2+qt+t^2}_{s_2})\, s_{211}$$

$$+\,(\underbrace{q+t}_{s_1}+\underbrace{q^2+qt+t^2}_{s_2}+\underbrace{q^3+q^2t+qt^2+t^3}_{s_3})\, s_{1111}$$

THE COEFFICIENTS ARE SYMMETRIC IN q AND t.

IN FACT THEY ARE SCHUR POSITIVE

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$$\Delta'_{e_0}(e_4) = s_{1111}$$

$$\Delta'_{e_1}(e_4) = \underbrace{(q+t)}_{s_1} s_{22}$$

$$+ (1 + \underbrace{q+t}_{s_1} + \underbrace{q^2+qt+t^2}_{s_2}) s_{211}$$

$$+ (\underbrace{q+t}_{s_1} + \underbrace{q^2+qt+t^2}_{s_2} + \underbrace{q^3+q^2t+qt^2+t^3}_{s_3}) s_{1111}$$

$$\Delta'_{e_1}(e_4) = s_1 \otimes s_{22} + (1 + s_1 + s_2) \otimes s_{211}$$

$$+ (s_1 + s_2 + s_3) \otimes s_{1111}$$

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$$\Delta'_{e_1}(e_4) = s_1 \otimes s_{22} + (1 + s_1 + s_2) \otimes s_{211} + (s_1 + s_2 + s_3) \otimes s_{1111}$$

$$\begin{aligned}\Delta'_{e_2}(e_4) = {} & (1 + s_1 + s_2) \otimes s_{31} \\ & + (s_{11} + s_1 + s_2 + s_3) \otimes s_{22} \\ & + (s_{11} + s_{21} + s_1 + 2 s_2 + 2 s_3 + s_4) \otimes s_{211} \\ & + (s_{11} + s_{21} + s_{31} + s_3 + s_4 + s_5) \otimes s_{1111}\end{aligned}$$

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$$\Delta'_{e_3}(e_4) = \nabla(e_4) =$$

$$1 \otimes s_4$$

$$+ (s_1 + s_2 + s_3) \otimes s_{31}$$

$$+ (s_{21} + s_2 + s_4) \otimes s_{22}$$

$$+ (s_{11} + s_{21} + s_{31} + s_3 + s_4 + s_5) \otimes s_{211}$$

$$+ (s_{31} + s_4 + s_6) \otimes s_{1111}$$

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Bi-Variate Specialization

$$\nabla(e_3) = 1 \otimes s_3 + (s_1 + s_2) \otimes s_{21} + (s_{11} + s_3) \otimes s_{111}$$

$$(s_{11} + s_3)(q,t) = \underbrace{qt + q^3 + q^2 t + q t^2 + t^3}_{(q,t)\text{-Catalan}}$$

(q,t)-Catalan

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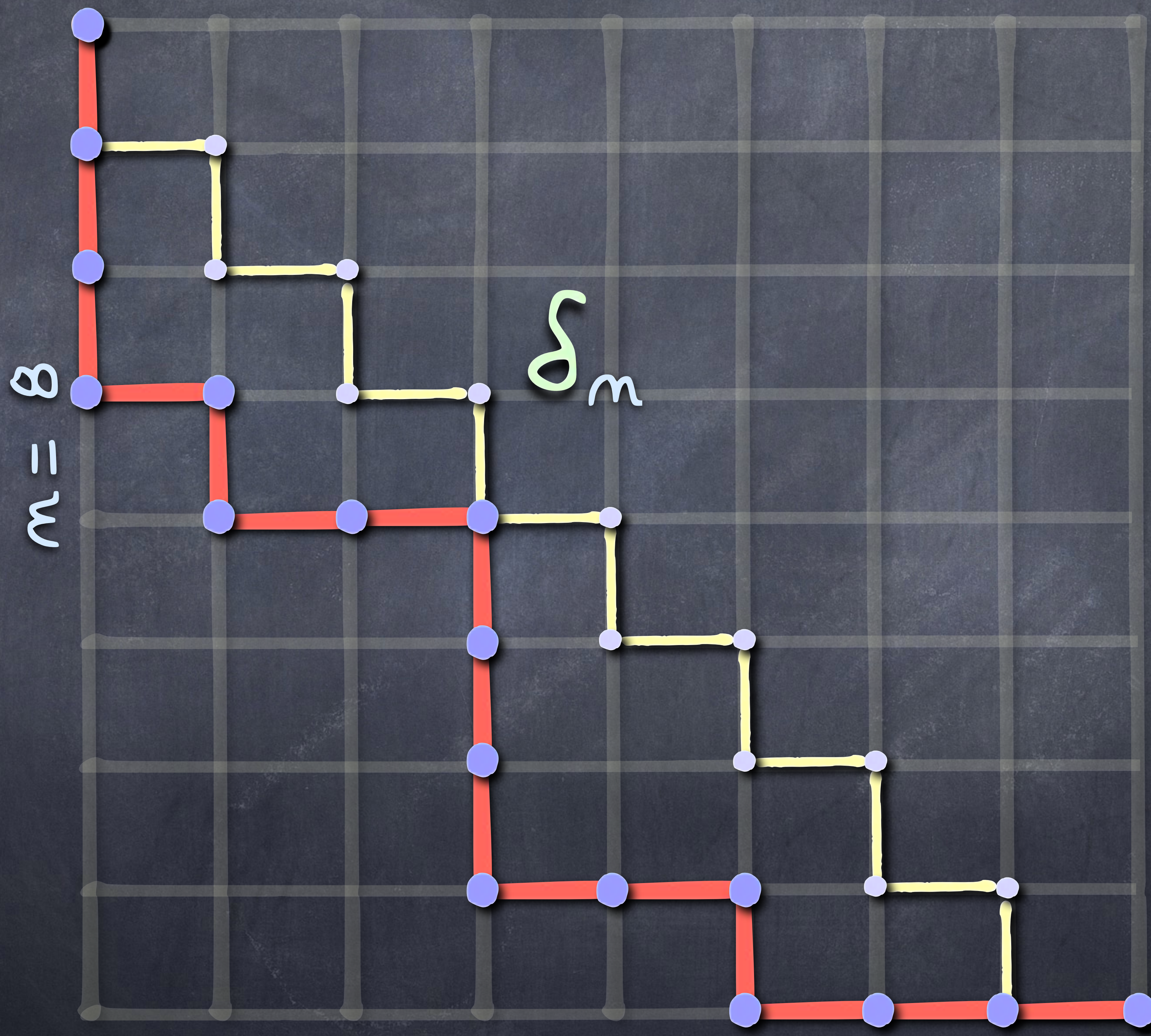
COMBINATORIAL SIDE

m-Dyck Path = Partition μ contained in δ_m



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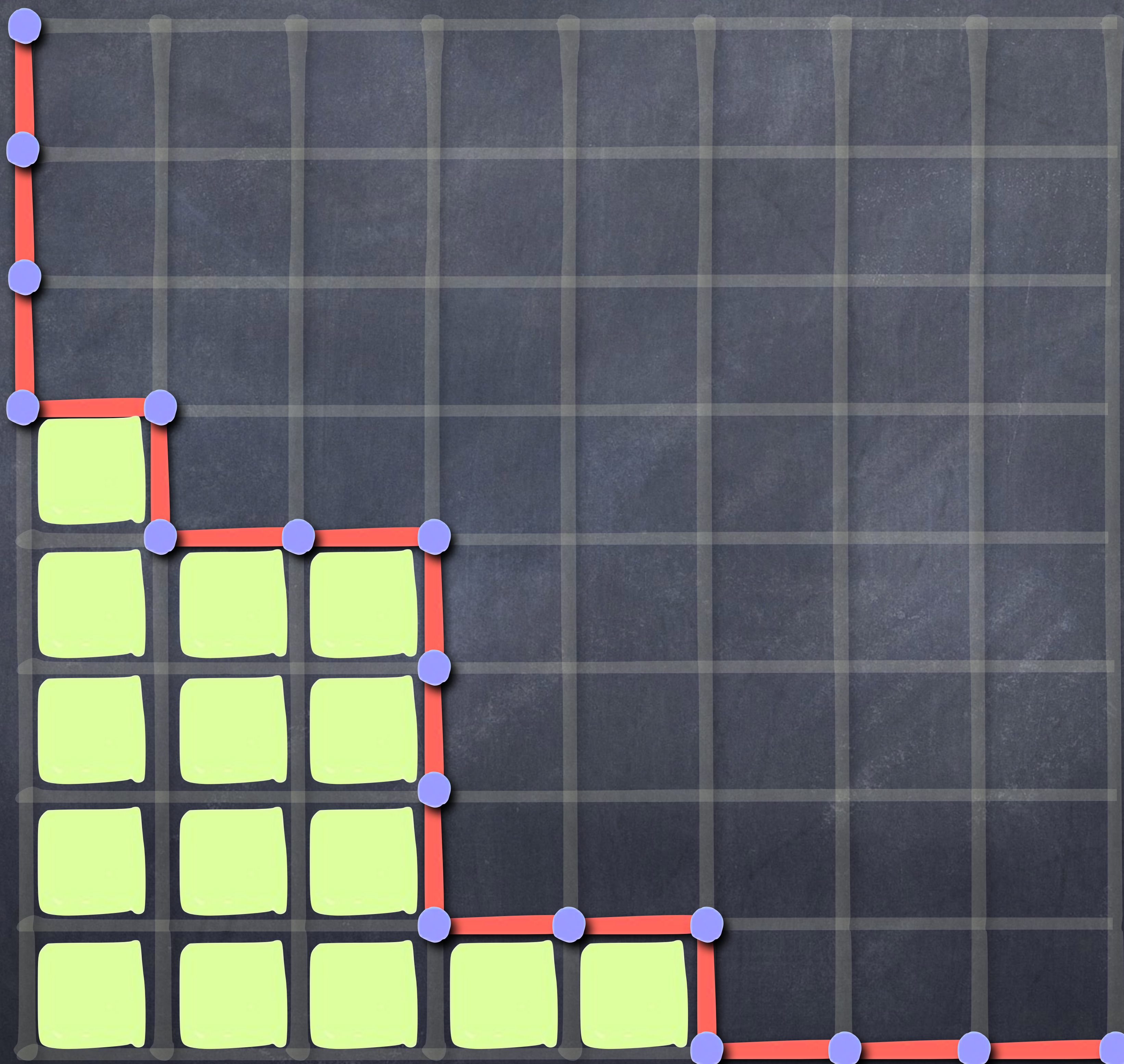
m-DYCK PATH = PARTITION μ CONTAINED IN δ_m



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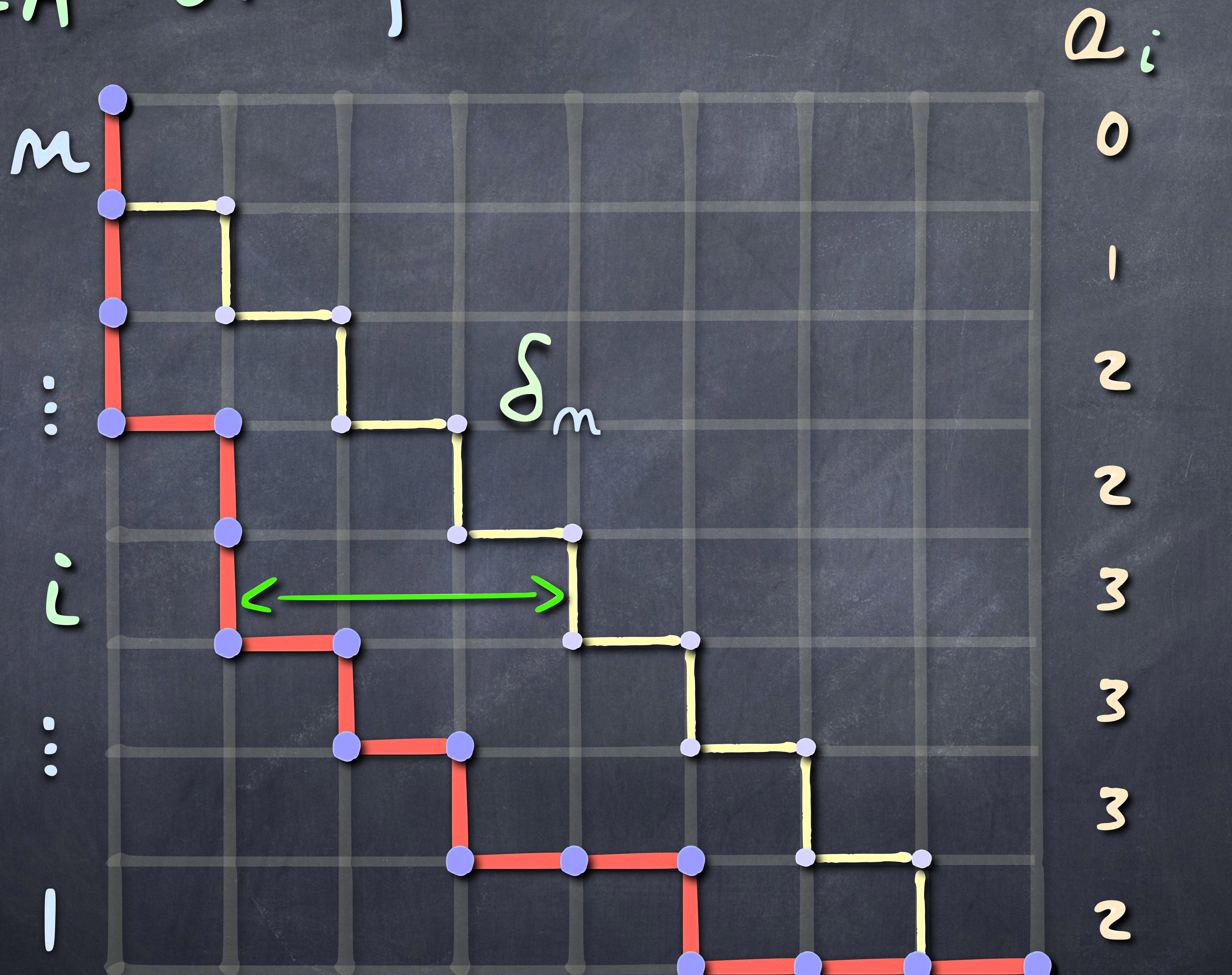
AREA OF μ



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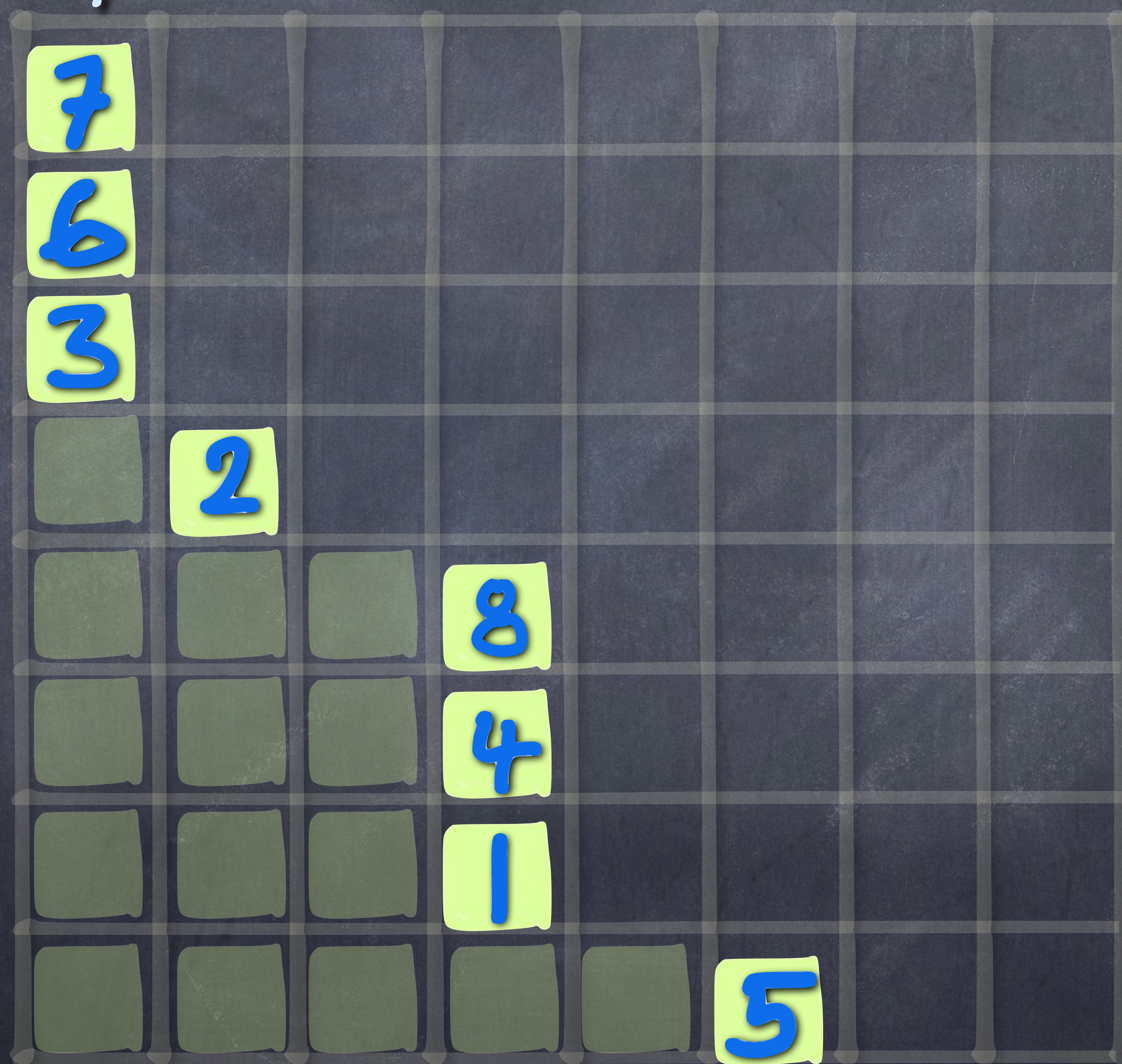
PARKING FUNCTION OF SHAPE μ = STANDARD TABLEAU OF SHAPE $(\mu+1^n)/\mu$



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The Δ-Conjecture

Haglund

$$\Delta'_{e_k}(e_n) = \sum_{\mu \subseteq \delta_n} \Big(\sum_{J} q^{(J|a)}\Big)\, LLT_\mu(t;z)$$

$$\mathrm{Desc}(\mu) \subseteq J \subseteq \{1,2,\dots,n\}, \quad \#J = k$$

$$(J|a) = \sum_{i \in J} a_i$$

$$LLT_\mu(t;z) = \sum_{\tau \in SSYT((\mu+1^n)/\mu)} t^{\mathrm{Dinv}(\tau)}\, z_\tau$$

The Δ-Conjecture

Haglund

$$\Delta'_{e_k}(e_n) = \sum_{\mu \subseteq \delta_n} \Big(\sum_{J} q^{(J\,|\,a)} \Big)\, LLT_\mu(t;z)$$

$$\mathrm{Desc}(\mu) \subseteq J \subseteq \{1,2,\ldots,n\}$$
$$\#J = k$$

$(k=0) \Rightarrow \mu = 0$

$(k=n-1) \Rightarrow (J\,|\,a) = \mathrm{AERA}(\mu)$

The Δ-Conjecture

Haglund

$$\Delta'_{e_k}(e_n) = \sum_{\mu \subseteq \delta_n} \Big(\sum_{J} q^{(J|a)} \Big) \mathrm{LLT}_\mu(t;\mathbf{z})$$

$$\mathrm{Desc}(\mu) \subseteq J \subseteq \{1,2,\ldots,n\}$$
$$\#J = k$$

$$\mathrm{LLT}_\mu(1;\mathbf{z}) = \Delta_{(\mu+1^n)/\mu}(\mathbf{z})$$

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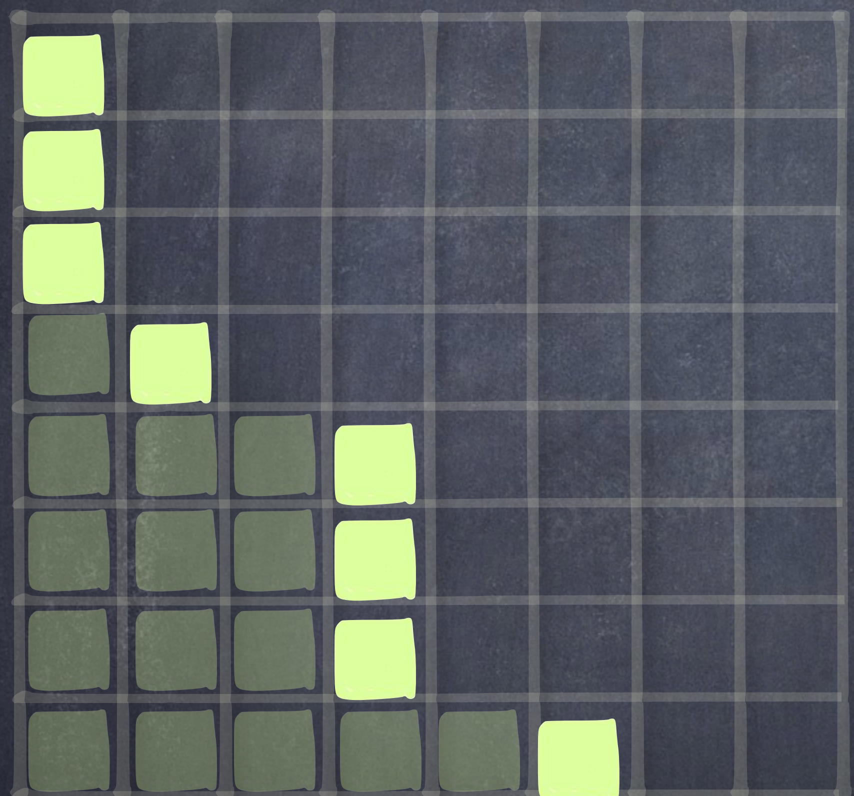
$$LLT_\mu(1;z) = \Delta_{(\mu+1^m)/\mu}(z)$$



$$\Delta_{(\mu+1^m)/\mu}(z) = e_{\rho(\mu)}(z)$$

$\rho(\mu)$: Partition whose parts are the columns of $(\mu+1^m)/\mu$

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$LLT_\mu(t;z)$ is Schur-Positive


Haiman


Grojnovski

Observation

$LLT_\mu(1+t;z)$ is e-Positive

The (m,n)-Shuffle Conjecture

n

Haglund

Haiman

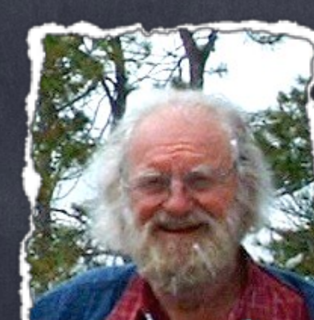
Loehr

Remmel

Ulyanov

(m,n)

F.B.



Garsia

Leven

Xin

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E_{mn} OPERATOR IN THE ELLIPTIC HALL ALGEBRA


BURBAN


VASSEROT


SCHIFFMANN

GENERATED BY

$(-)\cdot e_n$ AND Δ_{e_1}

$$e_{mn}(q,t;z) := E_{mn}\cdot 1$$

E_{mn} "CREATION OPERATOR"

$$e_{2,2} = 1 \otimes \Delta_2 + \Delta_1 \otimes \Delta_{11}$$

$$e_{2,3} = 1 \otimes \Delta_{21} + \Delta_1 \otimes \Delta_{111}$$

$$e_{2,4} = 1 \otimes \Delta_{22} + \Delta_1 \otimes \Delta_{211} + \Delta_2 \otimes \Delta_{1111}$$

$$e_{3,4} = 1 \otimes \Delta_{31} + \Delta_1 \otimes \Delta_{22} + (\Delta_1 + \Delta_2) \otimes \Delta_{211}$$
$$+ (\Delta_{11} + \Delta_3) \otimes \Delta_{1111}$$

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$$e_{1,n} = 1 \otimes s_n$$

$$e_{m+n,n} = \nabla e_{mn}$$

$$e_{rn+1,n} = e_{rn,n} = \nabla^r(e_n)$$

$$e_{2,n} = \sum_{d=0}^{\lfloor n/2 \rfloor} s_{\lfloor n/2 \rfloor - d} \otimes s_{(2^d, 1^{n-2d})}$$

$$e_{2r,2} = s_{r-1} \otimes s_2 + s_r \otimes s_{11}$$

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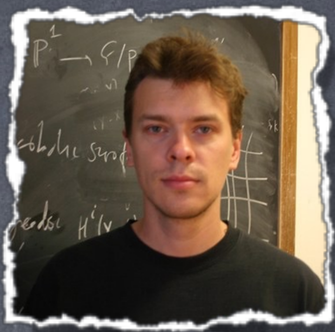
The (m,n)-shuffle theorem



Mellit

Carlsson

$$e_{mn}(q,t;z) = \sum_{\mu \subseteq \delta_{mn}} q^{\mathrm{aera}(\mu)} \sum_{\tau \in SSYT((\mu+1^n)/\mu)} t^{\mathrm{dinv}(\tau)}\, z_\tau$$

Here, both aera and dinv depend on m and n

The (m,n)-shuffle theorem

Mellit

Carlsson

$$e_{mn}(q,t;z) = \sum_{\mu\subseteq\delta_{mn}} q^{\mathrm{AERA}(\mu)} \sum_{\tau\in \mathrm{SSYT}((\mu+1^m)/\mu)} t^{\mathrm{Dinv}(\tau)}\, z_\tau$$

$$e_{mn}(q,1;z) = \sum_{\mu\subseteq\delta_{mn}} q^{\mathrm{AERA}(\mu)}\, \Delta_{(\mu+1^m)/\mu}(z)$$

$$e_{mn}(q,1;z) = \sum_{\mu \subseteq \delta_{mn}} q^{\mathrm{AERA}(\mu)} \Delta_{(\mu+1^m)/\mu}(z)$$

OBSERVATION

$e_{mn}(q,1+t;z)$ is e-POSITIVE

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PART 1: PROPOSED MODULES FOR $e_{mn}(q,t;z)$

(AND MORE)

THE MODULE $\mathcal{E}_{m,n}$

$\mathcal{M}_{m,n}$ SMALLEST MODULE CONTAINING

$$V_{m,n} := \det\left(x_i^a \theta_i^b\right)_{\substack{1\leq i\leq n \\ (a,b)\in\gamma_{m,n}}}$$

CLOSED UNDER

- PARTIAL DERIVATIVES
- POLARIZATION

θ_i INERT VARIABLES (DEGREE 0)

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The Module $\mathcal{E}_{m,n}$

$\mathcal{M}_{m,n}$ smallest module containing

$$V_{m,n} := \det\left(x_i^a \theta_i^b\right)_{\substack{1 \le i \le n \\ (a,b) \in \gamma_{m,n}}}$$

closed under

- Partial derivatives
- Polarization

θ_i inert variables (degree 0)

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$$\boxed{\mathcal{E}_{m,n} := \mathcal{M}_{m,n} / \Lambda^* \mathcal{M}_{m,n}}$$

$\Lambda^* \mathcal{M}_{m,n}$ SMALLEST MODULE CONTAINING

$$F\, V_{m,n}$$

F DIAGONAL SYMMETRIC DERIVATION WITHOUT CONSTANT TERM

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$\gamma_{m,n}$:= LIST OF COORDINATES ASSOCIATED TO PATH


:= LIST OF COORDINATES ASSOCIATED TO PATH
n = 8
00
10
30
40
60
70
90
10,0
$\gamma_{m,n}$
m = 12
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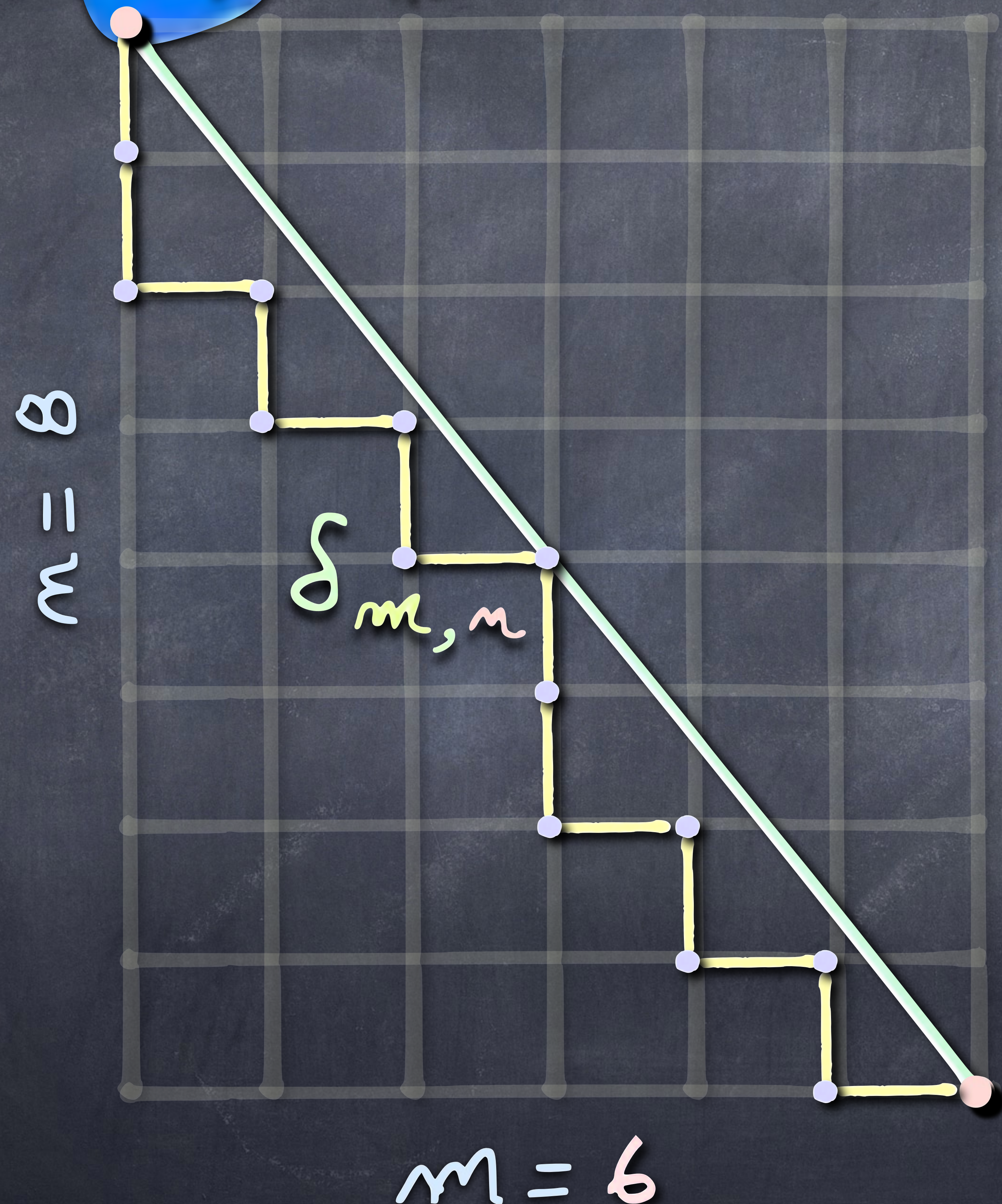
$\gamma_{m,n}$:= LIST OF COORDINATES ASSOCIATED TO PATH



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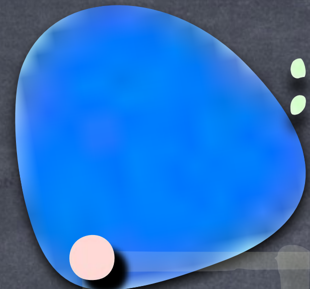 := LIST OF COORDINATES ASSOCIATED TO PATH

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$V_{n-1,n}$

$$V_{n-1,n} := \det \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-2} & \theta_1 \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-2} & \theta_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n & \theta_n \end{pmatrix}$$

A TOY EXAMPLE $\mathcal{E}_{2,3}$

$$V_{2,3}(x) = \det\begin{pmatrix} 1 & x_1 & \theta_1 \\ 1 & x_2 & \theta_2 \\ 1 & x_3 & \theta_3 \end{pmatrix}$$

A TOY EXAMPLE $\mathcal{E}_{2,3}$

$$V_{2,3}(y) = \det \begin{pmatrix} 1 & y_1 & \theta_1 \\ 1 & y_2 & \theta_2 \\ 1 & y_3 & \theta_3 \end{pmatrix}$$

POLARIZATION

A TOY EXAMPLE $\xi_{2,3}$

$$V_{2,3}(y) = \det \begin{pmatrix} 1 & y_1 & \theta_1 \\ 1 & y_2 & \theta_2 \\ 1 & y_3 & \theta_3 \end{pmatrix}$$

POLARIZATION

$$V_{2,3} = (x_3 - x_2)\,\theta_1 - (x_3 - x_1)\,\theta_2 + (x_2 - x_1)\,\theta_3$$

$$\partial x_1 V_{2,3} = \theta_2 - \theta_3 \qquad \partial x_2\, V_{2,3} = \theta_3 - \theta_1$$

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A TOY EXAMPLE $\xi_{2,3}$

$$V^{(y)}_{2,3} = \det \begin{pmatrix} 1 & y_1 & \theta_1 \\ 1 & y_2 & \theta_2 \\ 1 & y_3 & \theta_3 \end{pmatrix}$$

POLARIZATION

$$\xi_{2,3} = \overset{1}{\mathbb{Q}}\{\theta_2-\theta_3,\ \theta_3-\theta_1\} \oplus \mathbb{Q}\{\underset{q}{V^{(x)}_{2,3}}\ \underset{t}{V^{(y)}_{2,3}}\}$$

$$\xi_{2,3} = 1 \otimes \Delta_{21} + (q+t) \otimes \Delta_{111}$$

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A TOY EXAMPLE $\xi_{2,3}$

$$V_{2,3}(y) = \det \begin{pmatrix} 1 & y_1 & \theta_1 \\ 1 & y_2 & \theta_2 \\ 1 & y_3 & \theta_3 \end{pmatrix}$$

POLARIZATION

$$\xi_{2,3} = \overset{1}{\mathbb{Q}\{\theta_2-\theta_3,\ \theta_3-\theta_1\}} \oplus \mathbb{Q}\{\underset{q}{V_{2,3}(x)}\ \underset{t}{V_{2,3}(y)}\}$$

$$\xi_{2,3} = 1 \otimes s_{21} + s_1 \otimes s_{111}$$

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First Conjecture

For all m and n

$$e_{mn}(q,t;z) = \xi_{m,n}(q,t;z)$$

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GENERIC CHARACTER $\xi_{m,n}$

$$\xi_{m,n} = \sum_{\mu \vdash n} \sum_{\lambda} c_{\lambda\mu} (\Delta_\lambda \otimes \Delta_\mu),$$

IRRED. FOR GL_∞ - ACTION (Δ_λ)

IRRED. FOR $\mathbb{S}_n$ - ACTION (Δ_μ)

$c_{\lambda\mu} \in \mathbb{N}$

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$$\mathcal{E}_{2r,2}$$

$$x_2^r - x_1^r, \ldots, x_2^j y_2^{r-j} - x_1^j y_1^{r-j}, \ldots, y_2^r - y_1^r, \ldots, z_2^r - z_1^r$$

$\downarrow$ $\partial x_1 - \partial x_2$

$$x_2^{r-1} + x_1^{r-1}, \ldots, y_2^{r-1} + y_1^{r-1}, \ldots, z_2^{r-1} + z_1^{r-1}$$

$$\mathcal{E}_{2r,2} = \Delta_{r-1} \otimes \Delta_2 + \Delta_r \otimes \Delta_{11}$$

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Structure Theorem

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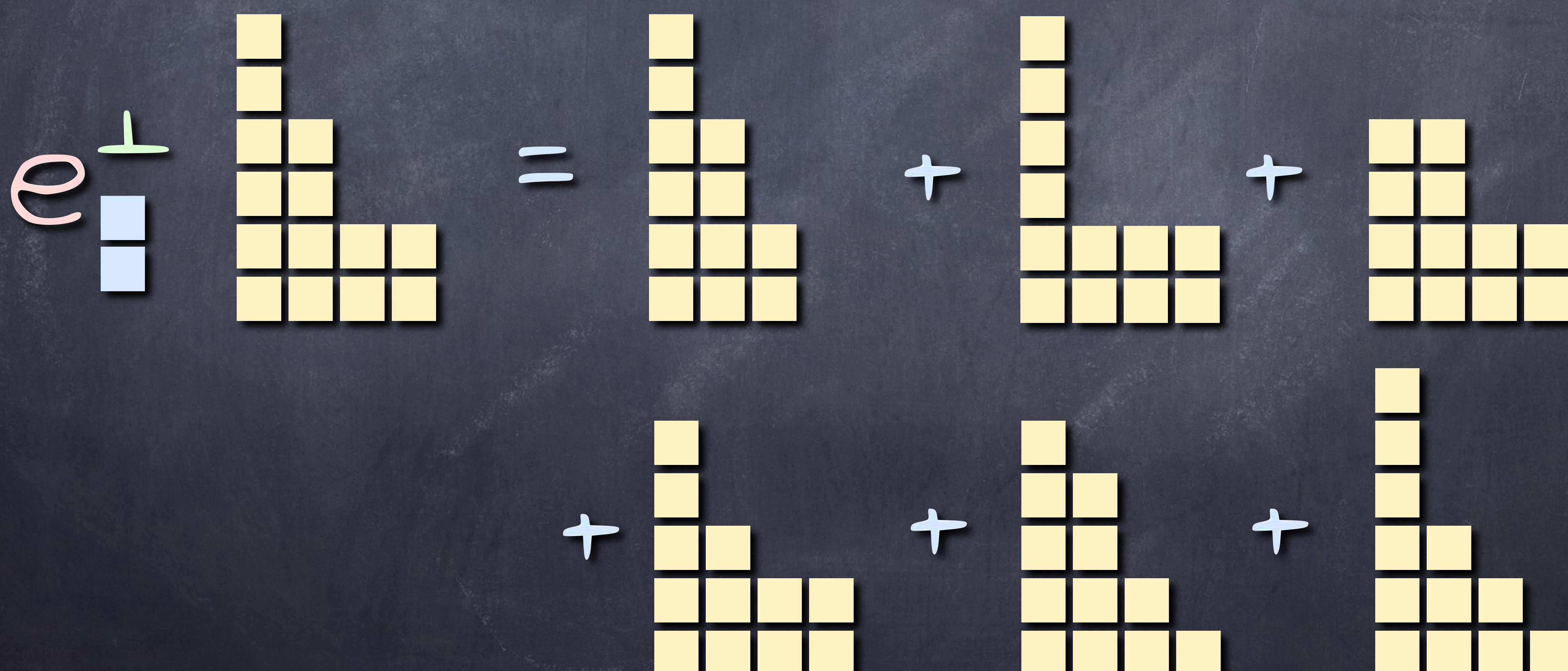
(Dual) Pieri Formula

$$e_k^{\perp}\, s_\lambda = \sum_{\mu \subset_k \lambda} s_\mu$$



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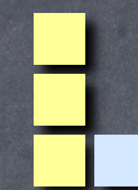
$$(\Delta_{111} + \Delta_{31} + \Delta_{41} + \Delta_{6})$$

$$\downarrow \quad e_1^{\perp}$$



$$(\Delta_{11} + \Delta_{21} + \Delta_{31} + \Delta_{3} + \Delta_{4} + \Delta_{5})$$

$$\begin{aligned}\xi_4 = {} & 1 \otimes \Delta_4 + (\Delta_1 + \Delta_2 + \Delta_3) \otimes \Delta_{31} \\ & + (\Delta_{21} + \Delta_2 + \Delta_4) \otimes \Delta_{22} \\ & + (\Delta_{11} + \Delta_{21} + \Delta_{31} + \Delta_3 + \Delta_4 + \Delta_5) \otimes \Delta_{211} \\ & + (\Delta_{111} + \Delta_{31} + \Delta_{41} + \Delta_6) \otimes \Delta_{1111}\end{aligned}$$

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$$(\Delta_{111} + \Delta_{31} + \Delta_{41} + \Delta_6)$$

$$\downarrow \quad e_2^{\perp}$$

$$(\Delta_1 + \Delta_2 + \Delta_3)$$

$$\begin{aligned}\xi_4 = 1 \otimes \Delta_4 &+ (\Delta_1 + \Delta_2 + \Delta_3) \otimes \Delta_{31} \\ &+ (\Delta_{21} + \Delta_2 + \Delta_4) \otimes \Delta_{22} \\ &+ (\Delta_{11} + \Delta_{21} + \Delta_{31} + \Delta_3 + \Delta_4 + \Delta_5) \otimes \Delta_{211} \\ &+ (\Delta_{111} + \Delta_{31} + \Delta_{41} + \Delta_6) \otimes \Delta_{1111}\end{aligned}$$

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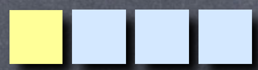
$$(s_{111} + s_{31} + s_{41} + s_6)$$

$$\downarrow \quad e_3^{\perp}$$



$$1$$

$$\begin{aligned}
\xi_4 = {} & 1 \otimes s_4 + (s_1 + s_2 + s_3) \otimes s_{31} \\
& + (s_{21} + s_2 + s_4) \otimes s_{22} \\
& + (s_{11} + s_{21} + s_{31} + s_3 + s_4 + s_5) \otimes s_{211} \\
& + (s_{111} + s_{31} + s_{41} + s_6) \otimes s_{1111}
\end{aligned}$$

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THEOREM

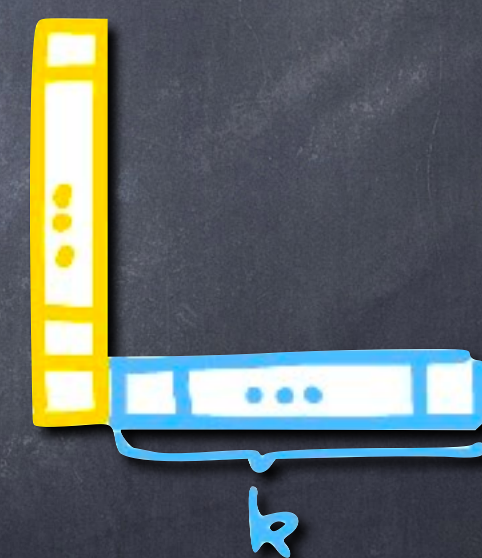
FOR ALL HOOK SHAPES, WE HAVE

$$e_k^{\perp} \langle \mathcal{E}_n, s_{1^n} \rangle = \langle \mathcal{E}_n, s_{(k+1,\, 1^{n-k-1})} \rangle$$

WHERE

COEFFICIENTS OF s_μ

$$\mathcal{E}_n = \cdots + \langle \mathcal{E}_n, s_\mu \rangle \otimes s_\mu + \cdots$$



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Conjecture

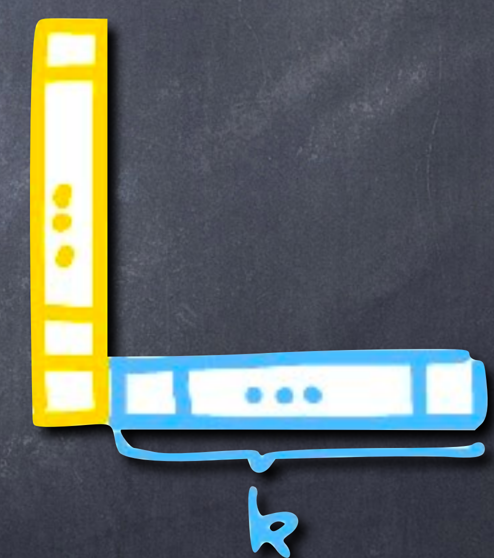
For all hook shapes, we have

$$e_k^{\perp} \langle \xi_{m,n}, \Delta_{1^n} \rangle = \langle \xi_{m,n}, \Delta_{(k+1,1^{n-k-1})} \rangle$$

where

coefficients of Δ_μ
↓

$$\xi_{m,n} = \cdots + \langle \xi_{m,n}, \Delta_\mu \rangle \otimes \Delta_\mu + \cdots$$



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Conjecture

For all hook shapes, we have

$$e_k^\perp \langle \mathcal{E}_{m,n}, s_{1^n} \rangle = \langle \mathcal{E}_{m,n}, s_{(k+1,1^{n-k-1})} \rangle$$

- $$e_k^\perp \langle \mathcal{E}_{m,n}, s_{1^n} \rangle = e_k^\perp \langle \mathcal{E}_{n,m}, s_{1^m} \rangle$$

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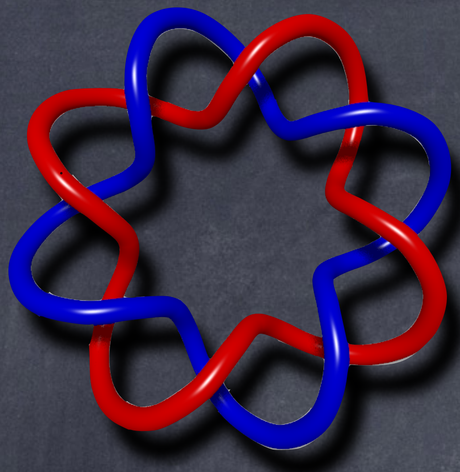

The Superpolynomial of the (m,n)-Torus Link

Khovanov-Rozansky Homology of (m,n)-Torus Links

$$(1+a)\sum_{k=0}^{n-1} \langle \xi_{m,n},\ s_{(k+1,1^{n-k-1})} \rangle\, a^k$$

↑

Evaluated in q, t

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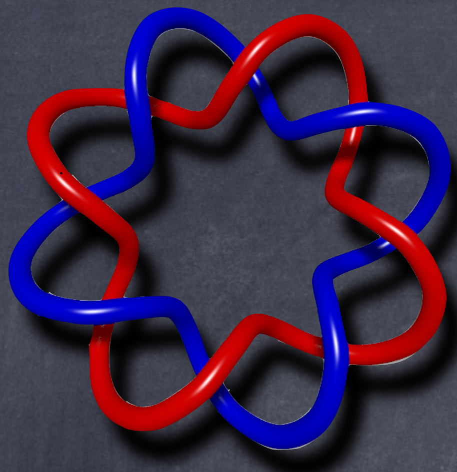

THE SUPERPOLYNOMIAL OF THE (m,n)-TORUS LINK

KHOVANOV-ROZANSKY HOMOLOGY OF (m,n)-TORUS LINKS

$$(1+a)\sum_{k=0}^{n-1} \langle \xi_{m,n}, s_{(k+1,1^{n-k-1})}\rangle\, a^k$$

(m,n)-TORUS LINK $=$ (n,m)-TORUS LINK

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$$\langle \mathcal{E}_{m,n},\ s_{(k+1,\,1^{n-k-1})} \rangle = \langle \mathcal{E}_{n,m},\ s_{(k+1,\,1^{m-k-1})} \rangle$$

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$$\mathcal{E}_{6,4} = s_2 \otimes s_4 + (s_{21} + s_3 + s_{31} + s_4 + s_5) \otimes s_{31}$$
$$+ (s_{111} + s_{22} + s_{31} + s_4 + s_{41} + s_6) \otimes s_{22}$$
$$+ (s_{211} + s_{31} + s_{32} + 2s_{41} + s_5 + s_{51} + s_6 + s_7) \otimes s_{211}$$
$$+ (s_{311} + s_{42} + s_{51} + s_{61} + s_8) \otimes s_{1111}$$

$$\mathcal{E}_{4,6} = s_1 \otimes s_{42} + s_2 \otimes s_{411} + s_2 \otimes s_{33}$$
$$+ (s_{11} + s_{21} + s_2 + 2s_3 + s_4) \otimes s_{321}$$
$$+ (s_{21} + s_{31} + s_3 + s_4 + s_5) \otimes s_{3111}$$
$$+ (s_{21} + s_{31} + s_3 + s_5) \otimes s_{222}$$
$$+ (s_{111} + s_{22} + s_{21} + 2s_{31} + s_{41} + 2s_4 + s_5 + s_6) \otimes s_{2211}$$
$$+ (s_{211} + s_{32} + s_{31} + 2s_{41} + s_{51} + s_5 + s_6 + s_7) \otimes s_{21111}$$
$$+ (s_{311} + s_{42} + s_{51} + s_{61} + s_8) \otimes s_{111111}$$

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RECONSTRUCTION OF $A_n : \langle \xi_{nn}, \Delta_{11\cdots1} \rangle$

$n = 4$

$$A_4 = \Delta_6^+$$

RECONSTRUCTION OF $A_n : \langle \xi_{nn}, \Delta_{11\cdots1} \rangle$

$n = 4$

$$A_4 = \Delta_6^{*}$$

$$e_3^{\perp} A_4 = 1$$

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RECONSTRUCTION OF $A_n : \langle \xi_{nn}, \Delta_{11\cdots1} \rangle$

$n = 4$

$$A_4 = \Delta_6 + \qquad + \Delta_{111}$$

$$e_3^\perp A_4 = 1$$

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Reconstruction of $A_n : \langle \xi_{nn}, \Delta_{11\cdots 1} \rangle$

$n = 4$

$$A_4 = \Delta_6 + \qquad\qquad + \Delta_{111}$$

$$e_2^{\perp} A_4 = \Delta_1 + \Delta_2 + \Delta_3$$

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RECONSTRUCTION OF $A_n : \langle \xi_{nn}, \Delta_{11\cdots1} \rangle$

$n = 4$

$$A_4 = \Delta_6 + \Delta_{31} + \Delta_{41} + \Delta_{111}$$

$$e_2^{\perp} A_4 = \Delta_1 + \Delta_2 + \Delta_3$$

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To Be Continued

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