Nonsymmetric Macdonald polynomials and Demazure characters

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Outline

Schur functions

Ring of symmetric functions $\Lambda_{\mathbb{C}}$ in variables $X = x_1, x_2, x_3, \ldots$ has bases: $m_\lambda, e_\lambda, h_\lambda, p_\lambda, \ldots$

Definition (Schur functions)

The orthonormal basis for $\Lambda_{\mathbb{C}}$ is

$$s_\lambda(X) = \sum_{T \in \mathrm{SSYT}(\lambda)} x_1^{\mathrm{wt}(T)_1} x_2^{\mathrm{wt}(T)_2} \cdots$$

Definition (Semistandard Young tableau)

An $\mathrm{SSYT}(\lambda)$ is a filling $T : \lambda \to \mathbb{N}$ such that

1. $T(c) \leq T(d)$ if c left of d same row
2. $T(c) > T(d)$ for c above d

Example (The set $\mathrm{SSYT}_3(2,1)$ used to compute $s_{(2,1)}(x_1, x_2, x_3)$)

$$\mathrm{SSYT}_3(2,1) = \left\{ \begin{array}{|c|c}\hline 3 & \\ \hline 2 & \multicolumn{1}{c|}{3} \\ \hline \end{array}, \begin{array}{|c|c}\hline 3 & \\ \hline 2 & \multicolumn{1}{c|}{2} \\ \hline \end{array}, \begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array}, \begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline \end{array}, \begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array}, \begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline \end{array}, \begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline \end{array}, \begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline \end{array} \right\}$$

$$s_{(2,1)}(x_1, x_2, x_3) = x_2 x_3^2 + x_2^2 x_3 + x_1 x_3^2 + x_1 x_2 x_3 + x_1 x_2 x_3 + x_1^2 x_3 + x_1 x_2^2 + x_1^2 x_2$$

Expanding a symmetric function in the Schur basis is important in many contexts, for example

- For $\mathbb{S}_\lambda(\mathbb{C}^n)$ an irred. rep. of GL_n, the character is $\mathrm{char}(\mathbb{S}_\lambda(\mathbb{C}^n)) = s_\lambda(x_1, \ldots, x_n)$.
- For Sp_λ an irred. rep. of $\mathcal{S}_n$, the Frobenius character is $\mathrm{ch}(\mathrm{Sp}_\lambda) = s_\lambda(X)$.
- For X_λ a Schubert variety for $\mathrm{Gr}(n,k)$, the Schubert poly is $\mathfrak{S}_{v(\lambda,k)} = s_\lambda(x_1, \ldots, x_k)$.

Fundamental problem: write $g(X) = \sum_\lambda g_\lambda s_\lambda(X)$ and find a combinatorial formula for g_λ.

Hall–Littlewood symmetric functions

Hall–Littlewood symmetric functions $P_\mu(X;t)$ and $H_\mu(X;t)$ generalize Schur functions to $\Lambda_{\mathbb{C}(t)}$

$$s_\lambda(X) = \sum_\mu K_{\lambda,\mu}(t) P_\mu(X;t) \quad \text{and} \quad H_\mu(X;t) = \sum_\lambda K_{\lambda,\mu}(t) s_\lambda(X)$$

The Kostka–Foulkes polynomials $K_{\lambda,\mu}(t) \in \mathbb{C}(t)$ satisfy $K_{\lambda,\mu}(0) = \delta_{\lambda,\mu}$, so $P_\lambda(X;0) = s_\lambda(X)$.

Theorem (Lascoux–Schützenberger 1978, Butler 1986)

$$K_{\lambda,\mu}(t) = \sum_{T \in \mathrm{SSYT}(\lambda),\ \mathrm{wt}(T)=\mu} t^{\mathrm{charge}(T)} \in \mathbb{N}[t]$$

Example (Computing the Schur expansion of $H_{(2,2,1)}(X;t)$)

$$\begin{array}{l}3\\2\ 2\\1\ 1\end{array} \qquad \begin{array}{l}3\\2\\1\ 1\ 2\end{array} \qquad \begin{array}{l}2\ 3\\1\ 1\ 2\end{array} \quad \begin{array}{l}2\ 2\\1\ 1\ 3\end{array} \qquad \begin{array}{l}2\\1\ 1\ 2\ 3\end{array} \quad \begin{array}{l}3\\1\ 1\ 2\ 2\end{array} \qquad \begin{array}{l}1\ 1\ 2\ 2\ 3\end{array}$$

$$s_{(2,2,1)} + t s_{(3,1,1)} + (t+t^2) s_{(3,2)} + (t^2+t^3) s_{(4,1)} + t^4 s_{(5)}$$

Hall–Littlewood polynomials arise in similar contexts as Schur functions, for example

- For χ_λ a unipotent char. of $\mathrm{GL}_n(\mathbb{F}_t)$ and μ a conj. class, $\chi_\lambda(\mu) = t^{n(\mu)} K_{\lambda,\mu}(1/t)$.
- For R_μ the Garsia–Procesi $\mathcal{S}_n$-module, the Frob. char. is $\mathrm{ch}(R_\mu) = t^{n(\mu)} H_\mu(X;1/t)$.
- For B_μ a Springer fiber, the cohomology ring $H^*(B_\mu)$ has Frob. series $t^{n(\mu)} H_\mu(X;1/t)$.

Macdonald symmetric functions

The Macdonald symmetric functions $P_\lambda(X;q,t)$ specialize to the classical bases by

- Schur functions: $P_\lambda(X;q,q) = s_\lambda(X)$
- Hall–Littlewood functions: $P_\lambda(X;0,t) = P_\lambda(X;t)$
- Jack symmetric functions: $\lim_{t\to 1} P_\lambda(X;t^\alpha,t) = J_{\lambda,\alpha}(X)$

Moreover, integral Macdonald symmetric functions are *sort of* Schur positive:

$$J_\mu(X;q,t) = \left(\prod_{c\in\lambda} 1 - q^{\mathrm{arm}(c)} t^{\mathrm{leg}(c)+1}\right) P_\mu(X;q,t) = \sum_\lambda K_{\lambda,\mu}(q,t) s_\lambda[X(1-t)]$$

where $s_\lambda[X(1-t)]$ is the plethystic Schur basis dual to $s_\lambda(X)$: $\langle s_\lambda[X(1-t)], s_\mu(X)\rangle_t = \delta_{\lambda,\mu}$

The transformed Macdonald symmetric functions of Garsia and Haiman are Schur positive

$$H_\mu(X;q,t) = J_\mu[X\left(\frac{1}{1-t}\right);q,t] = \sum_\lambda K_{\lambda,\mu}(q,t) s_\lambda(X)$$

Theorem (Haiman 2001)

The isospectral Hilbert scheme of points in the plane is Cohen-Macaulay, and so the Garsia–Haiman S_n-module has dimension $n!$, and so $K_{\lambda,\mu}(q,t) \in \mathbb{N}[q,t]$.

Nonsymmetric Macdonald polynomials

Nonsymmetric Macdonald polynomials $E_a(x_1, \ldots, x_n; q, t)$ are polynomials indexed by weak compositions that form a basis for the full polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$.

They generalize the symmetric Macdonald polynomials in the following sense:

$$\begin{aligned}
E_{(\lambda_n, \lambda_{n-1}, \ldots, \lambda_1)}(x_1, \ldots, x_n; q, t) &= P_\lambda(x_1, \ldots, x_n; q, t) \\
E_{0^m \times a}(x_1, \ldots, x_m, 0, \ldots, 0; q, t) &= P_{\mathrm{sort}(a)}(x_1, \ldots, x_m; q, t) \\
\lim_{m \to \infty} E_{0^m \times a}(x_1, \ldots, x_m, 0, \ldots, 0; q, t) &= P_{\mathrm{sort}(a)}(x_1, x_2, \ldots; q, t)
\end{aligned}$$

Additional structure in the polynomial ring helps illuminate the symmetric case.

Theorem (Haglund–Haiman–Loehr 2008)

$$E_a(X; q, t) = \sum_{\substack{T : a \to [n] \\ \text{non-attacking}}} q^{\mathrm{maj}(T)} t^{\mathrm{coinv}(T)} X^{\mathrm{wt}(T)} \prod_{c \neq \mathrm{left}(c)} \frac{1 - t}{1 - q^{\mathrm{leg}(c)+1} t^{\mathrm{arm}(c)+1}}$$

Question: Are there any natural positivity results for $E_a(x; q, t)$ parallel to symmetric case?

While there is an integral form for the nonsymmetric Macdonald polynomials, there is no well-defined notion of plethysm in the polynomial ring, so we cannot make this positive.

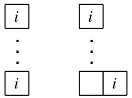
Combinatorial formula

Theorem (Haglund–Haiman–Loehr 2008)

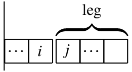
$$E_a(X;q,t) = \sum_{\substack{T:a\to[n] \\ \text{non-attacking}}} q^{\mathrm{maj}(T)} t^{\mathrm{coinv}(T)} X^{\mathrm{wt}(T)} \prod_{c \neq \mathrm{left}(c)} \frac{1-t}{1-q^{\mathrm{leg}(c)+1} t^{\mathrm{arm}(c)+1}}$$

The diagram of a weak composition a has a_i cells in row i. Fill them with positive integers.

attacking fillings

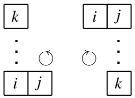


$$\mathrm{maj}(T) = \sum_{T(\mathrm{left}(c))<T(c)} \mathrm{leg}(c)$$



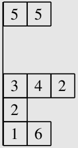
$i < j$

$$\mathrm{coinv}(T) = \#\{\text{co-inv triples}\}$$



$i<j<k$ or
$j<k<i$ or
$k<i<j$

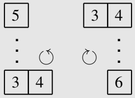
Example (One non-attacking filling for $E_{(2,1,3,0,0,2)}(x_1 \ldots x_6; q, t)$)



3 4 □ $\mathrm{leg}(\boxed{4}) = 2$

1 6 $\mathrm{leg}(\boxed{6}) = 1$

$\mathrm{maj}(T) = 2 + 1 = 3$



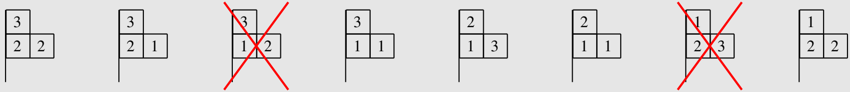
$\mathrm{coinv}(T) = 2$

contribution to $E_{(2,1,3,0,0,2)}(x_1 \ldots x_6; q, t)$

$$q^3\, t^2\, x_1 x_2^2 x_3 x_4 x_5^2 x_6 \cdot \left(\frac{\text{big}}{\text{mess}}\right)$$

Semistandard key tabloids

Example (The eight non-attacking fillings giving terms in $E_{(0,2,1)}(x_1, x_2, x_3; q, t)$)



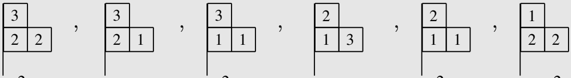
$x_2^2 x_3$ | $x_1 x_2 x_3$ | $qt x_1 x_2 x_3$ | $x_1^2 x_3$ | $q x_1 x_2 x_3$ | $x_1^2 x_2$ | $qt x_1 x_2 x_3$ | $x_1 x_2^2$

Setting $t = 0$ in $E_a(X; q, t)$ has the following nice simplification

$$E_a(X; q, 0) = \sum_{\substack{T : a \to [n] \\ \text{non-attacking}}} q^{\mathrm{maj}(T)} 0^{\mathrm{coinv}(T)} X^{\mathrm{wt}(T)} \prod_{c \neq \mathrm{left}(c)} \frac{1 - 0}{1 - q^{\mathrm{leg}(c)+1} 0^{\mathrm{arm}(c)+1}} = \sum_{\substack{T : a \to [n] \\ \text{non-attacking} \\ \mathrm{coinv}(T) = 0}} q^{\mathrm{maj}(T)} X^{\mathrm{wt}(T)}$$

Example (The six semistandard key tabloids giving terms in $E_{(0,2,1)}(x_1, x_2, x_3; q, 0)$)

$\mathrm{SSKD}(0,2,1) =$



$$E_{(0,2,1)}(X; q, 0) = x_2^2 x_3 + x_1 x_2 x_3 + x_1^2 x_3 + q x_1 x_2 x_3 + x_1^2 x_2 + x_1 x_2^2$$

In search of positivity

Proposition

For ω the symmetric function involution $\omega s_\lambda = s_{\lambda^T}$, where λ^T denotes transpose, we have

$$\lim_{m\to\infty} E_{0^m\times a}(x_1,\ldots,x_m,0,\ldots,0;q,0) = \omega H_{\mathrm{sort}(a)^T}(X;0,q)$$

For example, $\lim_{m\to\infty} E_{0^m\times(0,3,0,2)}(X;q,0) = \omega H_{(2,2,1)}(X;0,q) = \omega H_{(2,2,1)}(X;q)$.

Question: Does this positivity in the limit $\lim_{m\to\infty} E_{0^m\times a}(X;q,0)$ pull back to the polynomial ring?

Recall $P_\mu(X;0,0) = s_\mu(X)$. Consider the nonsymmetric analog $E_a(X;0,0) = \kappa_a(X)$.

The Demazure characters κ_a are a basis for polynomials generalizing Schur polynomials

$$\begin{aligned}
\kappa_{(\lambda_n,\lambda_{n-1},\ldots,\lambda_1)}(x_1,\ldots,x_n) &= s_\lambda(x_1,\ldots,x_n)\\
\kappa_{0^m\times a}(x_1,\ldots,x_m,0,\ldots,0) &= s_{\mathrm{sort}(a)}(x_1,\ldots,x_m)\\
\lim_{m\to\infty}\kappa_{0^m\times a}(x_1,\ldots,x_m,0,\ldots,0) &= s_{\mathrm{sort}(a)}(X)
\end{aligned}$$

These are characters of Demazure modules that arise in the study of Schubert varieties.

Theorem (Assaf 2018)

Writing $E_b(X;q,0) = \sum_a K_{a,b}(q)\kappa_a(X)$, using weak dual equivalence, we have $K_{a,b}(q) \in \mathbb{N}[q]$.

Crystal graphs

Schur polynomials are also characters for finite connected normal $\mathfrak{gl}_n$ crystals.

Crystal basis $\mathcal{B}$, weight map $\mathrm{wt} : \mathcal{B} \to \mathbb{Z}^n$

crystal lowering operators $f_i : \mathcal{B} \xrightarrow{i} \mathcal{B} \cup \{0\}$ such that $\mathrm{wt}(b) - \mathrm{wt}(f_i(b)) = \mathbf{e}_i - \mathbf{e}_{i+1}$.

The character of a crystal is

$$\mathrm{char}\,(\mathcal{B}) = \sum_{b \in \mathcal{B}} x_1^{\mathrm{wt}(b)_1} \cdots x_n^{\mathrm{wt}(b)_n}$$

The **standard $\mathfrak{gl}_n$ crystal** has $\mathrm{wt}\left(\boxed{i}\right) = \mathbf{e}_i$

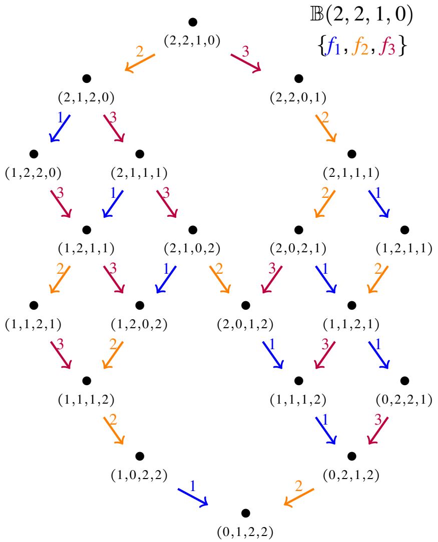
$$\boxed{1} \xrightarrow{f_1} \boxed{2} \xrightarrow{f_2} \boxed{3} \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} \boxed{n}$$

$$x_1 \; + \; x_2 \; + \; x_3 \; + \; \cdots \; + \; x_n$$

The connected (finite, normal) $\mathfrak{gl}_n$ crystals are indexed by dominant weights (partitions).

For $\mathbb{B}(\lambda)$ is the crystal for the irrep $\mathbb{S}_\lambda(\mathbb{C}^n)$

$$\mathrm{char}(\mathbb{B}(\lambda)) = \mathrm{char}(\mathbb{S}_\lambda(\mathbb{C}^n)) = s_\lambda(x_1, \ldots, x_n)$$

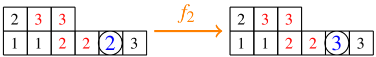


A crystal on tableaux

Define crystal operators e_i on $\mathrm{SSYT}(\lambda)$ that change an $i+1$ to an i in T by

Definition (Pairing rule)

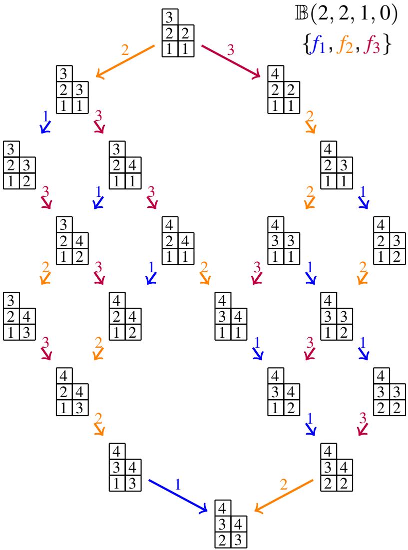
Two cells $\boxed{i}$ and $\boxed{i+1}$ are paired if in the same column or $\boxed{i+1}$ left of $\boxed{i}$ and no unpaired cells $\boxed{i}$ or $\boxed{i+1}$ between.



Definition (Crystal raising operators)

For $T \in \mathrm{SSYT}(\lambda)$ and $1 \leq i < n$, the crystal raising operator e_i acts on T by

- $e_i(T) = 0$ if T has no unpaired $\boxed{i+1}$
- change leftmost unpaired $\boxed{i+1}$ to $\boxed{i}$



Demazure modules

Complex semi-simple Lie algebra $\mathfrak{g}$	$\mathfrak{g} = \mathfrak{gl}_n = \{$invertible matrices$\}$
has a Cartan subalgebra $\mathfrak{h}$	$\mathfrak{h} = \{$invertible diagonal matrices$\}$
and Borel subalgebra $\mathfrak{b} \supset \mathfrak{h}$,	$\mathfrak{b} = \{$invertible upper triangular matrices$\}$
dominant weights Λ^+ indexing f. d. irreps	$\Lambda^+ = \{(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0)\}$
and Weyl group W.	$W = \mathcal{S}_n = \{$permutations of $\{1,2,\ldots,n\}\}$

Finite dimensional irred. representations of $\mathfrak{g}$ decompose into weight spaces $V^\lambda = \bigoplus_a V_a^\lambda$.

The Weyl group acts on extremal weight spaces $\{V_{w\cdot\lambda}^\lambda \mid w \in W\}$, which are all 1-dimensional.

Definition

The Demazure module V_w^λ is the $\mathfrak{b}$-submodule of the irreducible $\mathfrak{g}$-representation V^λ generated by the extremal weight space $V_{w\cdot\lambda}^\lambda$.

Example (Demazure modules)

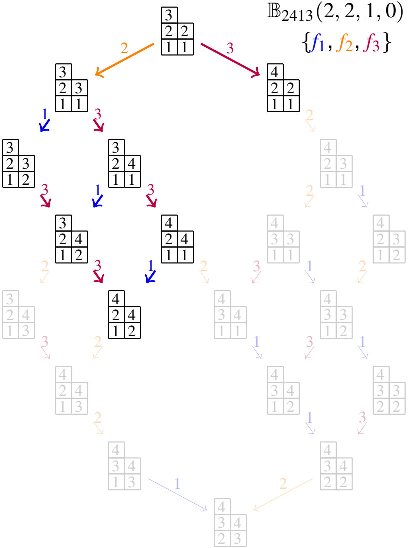
- $V_\lambda^\lambda = V_{\mathrm{id}}^\lambda$ is 1-dim
- $V_{\mathrm{rev}(\lambda)}^\lambda = V_{w_0}^\lambda = V^\lambda$

For $\mathfrak{g} = \mathfrak{gl}_n$, Demazure modules are indexed by weak compositions a by the correspondences

$$(w, \lambda) \mapsto w \cdot \lambda \qquad a \mapsto (w_a, \mathrm{sort}(a))$$

for $w_a =$ shortest permutation such that $w_a \cdot a \in \Lambda$.

Demazure crystals



Define operators $\mathfrak{D}_i$ on subsets $X \subseteq \mathcal{B}$ by

$$\mathfrak{D}_i X = \{b \in \mathcal{B} \mid e_i^k(b) \in X\}$$

For $w = s_1 \cdots s_k$ reduced expression

$$\mathbb{B}_w(\lambda) = \mathfrak{D}_{i_1} \cdots \mathfrak{D}_{i_k}\{u_\lambda\}$$

where u_λ is the highest weight of $\mathbb{B}(\lambda)$.

Theorem (Kashiwara 1993)

The Demazure character κ_a is given by

$$\kappa_a = \mathrm{char}\left(V_w^\lambda\right) = \mathrm{char}\left(\mathcal{B}_w(\lambda)\right)$$

Example (Compute $\mathbb{B}_{2413}(2,2,1,0)$)

For $w = 2413$, we may take $w = s_1 s_3 s_2$

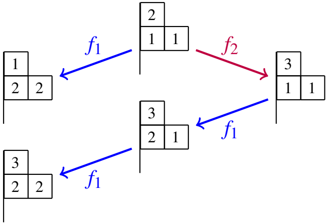
$$\kappa_{(1,2,0,2)} = x_1^2x_2^2x_3 + x_1^2x_2x_3^2 + x_1^2x_2^2x_4 + x_1^2x_2x_3x_4 + x_1^2x_2x_4^2 + x_1x_2^2x_3^2 + x_1x_2^2x_3x_4 + x_1x_2^2x_4^2$$

Semistandard key tableaux

Definition (Assaf 2018, Mason 2009)

An $\mathrm{SSKT}(a)$ is a filling $T : a \to \mathbb{N}$ such that

1. $T(c) \geq T(d)$ if c left of d same row
2. if $T(c) \leq T(d)$ for c above d,
   $\exists\, e$ right of d s.t. $T(c) < T(e)$
3. $T(c) \leq \mathrm{row}(c)$

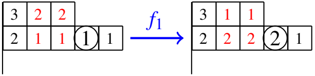


Theorem (Assaf 2018, Mason 2009)

$$\kappa_a(X) = \sum_{T \in \mathrm{SSKT}(a)} x_1^{\mathrm{wt}(T)_1} \cdots x_n^{\mathrm{wt}(T)_n}$$

Definition (Pairings of cells)

Two cells i and $i+1$ are paired if in the same column or i left of $i+1$ and no unpaired cells i or $i+1$ between.



Definition (Crystal raising operators)

For $T \in \mathrm{SSKT}(a)$ and $1 \leq i < n$, $e_i(T)$ is

- $e_i(T) = 0$ if T has no unpaired $i+1$
- change rightmost unpaired $i+1$ to i and change $\begin{array}{|c|}\hline i \\ \hline i{+}1 \\ \hline\end{array} \mapsto \begin{array}{|c|}\hline i{+}1 \\ \hline i \\ \hline\end{array}$ to the left

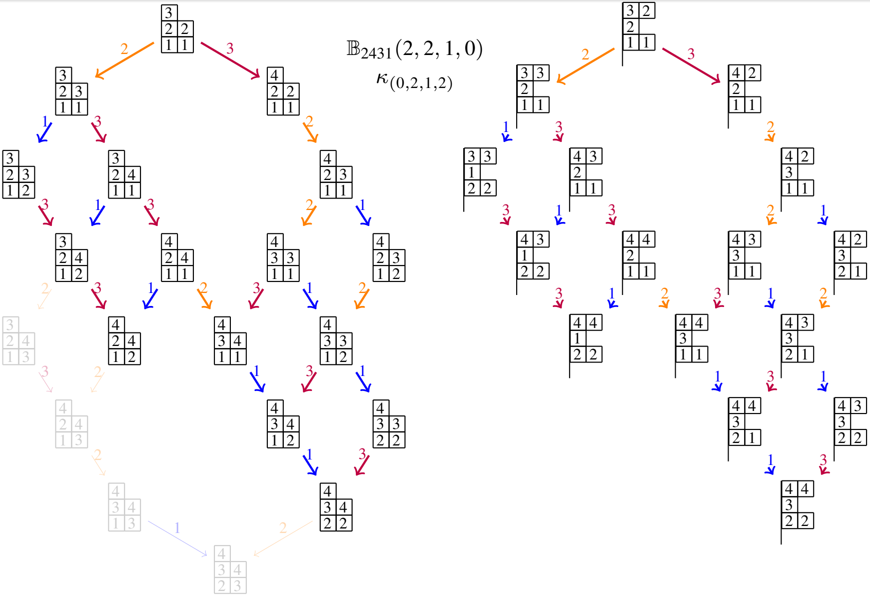
Theorem (Assaf–Schilling 2018)

This is a Demazure crystal for $\mathrm{SSKT}(a)$.

Examples of Demazure crystals on key tableaux

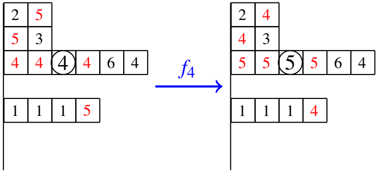
$\mathbb{B}_{2431}(2,2,1,0)$

$\kappa_{(0,2,1,2)}$



Raising operators on key tabloids

Same pairing rule as for key tableaux.



Definition (Crystal raising operators)

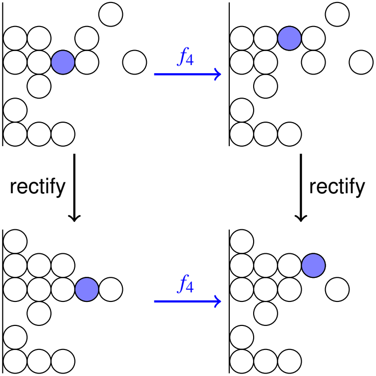
For $T \in \mathrm{SSKD}(a)$ and $1 \leq i < n$, $e_i(T)$ is

- $e_i(T) = 0$ if T has no unpaired $i+1$
- change rightmost unpaired $i+1$ to i and change $\begin{smallmatrix} i \\ i{+}1 \end{smallmatrix} \mapsto \begin{smallmatrix} i{+}1 \\ i \end{smallmatrix}$ to the left and change $\begin{smallmatrix} i{+}1 \\ i \end{smallmatrix} \mapsto \begin{smallmatrix} i \\ i{+}1 \end{smallmatrix}$ to the right

Theorem (Assaf–González 2018)

This is a Demazure crystal for $\mathrm{SSKD}(a)$ *that preserves the major index.*

We construct an explicit bijection between $\mathrm{SSKD}(a)$ and SSKT using Kohnert's algorithm for computing Demazure characters.



The bijection intertwines our operators with the Assaf–Schilling operators.

Demazure subsets

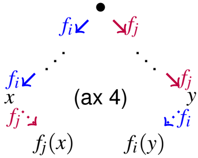
Recall $\mathbb{B}_w(\lambda) = \mathfrak{D}_{i_1} \cdots \mathfrak{D}_{i_k}\{u_\lambda\}$ where u_λ is highest wt and $\mathfrak{D}_i(X) = \{b \in \mathbb{B}(\lambda) \mid e_i^k(b) \in X\}$.

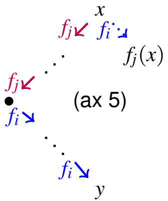
Question: Given a subset $X \subseteq \mathbb{B}(\lambda)$, how can we determine if $X = \mathbb{B}_w(\lambda)$ for some w?

Definition (Assaf–González 2019)

A subset $X \subseteq \mathbb{B}(\lambda)$ is demazure if for $x, y \in X$

1. $u_\lambda \in X$;
2. if $e_i(x) \neq 0$, then $e_i(x) \in X$;
3. if $f_i(x) \neq 0$ and $f_i(x) \notin X$, then $e_i(x) \notin X$;
4. if $e_i^*(x) = e_j^*(y) \in X$, then $f_j(x), f_i(y) \in X$;
5. if $e_j^* e_i^*(x)$ and $f_i(x) \neq 0$, then $f_i(x) \in X$;
6. if $e_i(x) = e_j^* e_i^*(x)$ and $f_{i_1} \cdots f_{i_k}(x) \in X$ with $i_k \neq j$, then $f_{i_1} \cdots f_{i_k}(y) \in X$.

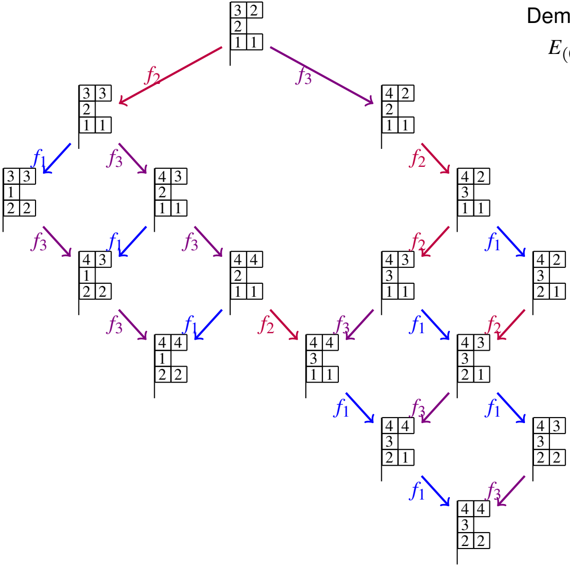




Theorem (Assaf–González 2019)

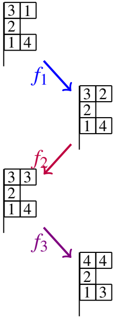
Every Demazure crystal $\mathbb{B}_w(\lambda)$ is a demazure subset of $\mathbb{B}(\lambda)$, and every demazure subset $X \subseteq \mathbb{B}(\lambda)$ is a demazure crystal $X = \mathbb{B}_w(\lambda)$.

Examples of key tabloid crystals



Demazure crystal for

$$E_{(0,2,1,2)}(X;q,0) = \kappa_{(0,2,1,2)} + q\kappa_{(1,1,1,2)}$$



Highest weights

Theorem (Assaf 2018; Assaf–González 2018$^+$)

Nonsymmetric Macdonald polynomials are q-graded sums of Demazure characters.

A connected crystal has a unique highest weight element u characterized by $e_i(u) = 0$ for all i.

$$\mathrm{char}\,(\mathcal{B}) = \sum_{u \in \mathcal{B} \text{ s.t. } e_i(u)=0 \ \forall\ i} s_{\mathrm{wt}(u)}(x_1, \ldots, x_n)$$

Recall $E_{(\mu_n,\mu_{n-1},\ldots,\mu_1)}(X; q, 0) = \omega H_{\mu^T}(X; 0, q)$ and $\kappa_{(\lambda_n,\lambda_{n-1},\ldots,\lambda_1)}(X) = s_\lambda(X)$.

Theorem (Assaf–González 2018$^+$)

$$H_{\mu^T}(X; t) = \sum_{U \in \mathrm{SSKD}(\mu_n,\mu_{n-1},\ldots,\mu_1) \text{ s.t. } e_i(U)=0\ \forall\ i} t^{\mathrm{maj}(U)} s_{\mathrm{wt}(U)^T}(X)$$

Example (The six highest weight elements of $\mathrm{SSKD}(0,0,2,3)$)

2	2	1
1	1	

2	1	1
1	3	

2	2	3
1	1	

2	1	2
1	3	

2	1	4
1	3	

2	4	1
1	3	

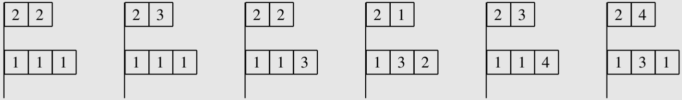
2	4	5
1	3	

$$E_{(0^3,2,3)}(X; q, 0) = \kappa_{(0^3,2,3)} + q\kappa_{(0^2,1,1,3)} + (q+q^2)\kappa_{(0^2,1,2,2)} + (q^2+q^3)\kappa_{(0,1,1,1,2)} + q^4\kappa_{(1,1,1,1,1)}$$
$$H_{(2,2,1)}(X; 0, t) = s_{(2,2,1)} + ts_{(3,1,1)} + (t+t^2)s_{(3,2)} + (t^2+t^3)s_{(4,1)} + t^5 s_{(5)}$$

Demazure lowest weights

Demazure crystals have unique highest weights but $\mathcal{B}_w(\lambda)$ has highest weight λ for every w.

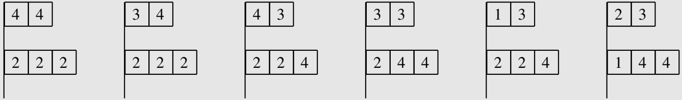
Example (The six highest weight elements of $\mathrm{SSKD}(0,3,0,2)$)



$$\lim_{m\to\infty} E_{0^m\times(0,3,0,2)}(X;q,0) = s_{(3,2)} + qs_{(3,1,1)} + (q+q^2)s_{(2,2,1)} + (q^2+q^3)s_{(2,1,1,1)} + ??$$

$$H_{(2,2,1)}(X;0,t) = s_{(2,2,1)} + ts_{(3,1,1)} + (t+t^2)s_{(3,2)} + (t^2+t^3)s_{(4,1)} + t^4 s_{(5)}$$

Example (The six Demazure lowest weight elements of $\mathrm{SSKD}(0,3,0,2)$)



$$E_{(0,3,0,2)}(X;q,0) = \kappa_{(0,3,0,2)} + q\kappa_{(0,3,1,1)} + q\kappa_{(0,2,1,2)} + q^2\kappa_{(0,1,2,2)} + q^2\kappa_{(1,2,1,1)} + q^3\kappa_{(1,1,1,2)}$$

Refinement of Kostka–Foulkes polynomials

Theorem (Assaf–González 2018+)

The specialized nonsymmetric Macdonald polynomial $E_a(X;q,0)$ is given by

$$E_a(X;q,0) = \sum_{\substack{Z \in \mathrm{SSKD}(a) \\ Z \text{ Demazure lowest weight}}} q^{\mathrm{maj}(Z)} \kappa_{\mathrm{wt}(Z)}(X)$$

$$E_{(0,3,0,2)}(X;q,0) = \underbrace{\kappa_{(0,3,0,2)}}_{s_{(2,2,1)}} + \underbrace{q\kappa_{(0,3,1,1)}}_{ts_{(3,1,1)}} + \underbrace{q\kappa_{(0,2,1,2)} + q^2\kappa_{(0,1,2,2)}}_{(t+t^2)s_{(3,2)}} + \underbrace{q^2\kappa_{(1,2,1,1)} + q^3\kappa_{(1,1,1,2)}}_{(t^2+t^3)s_{(4,1)}}$$

Define nonsymmetric Kostka–Foulkes coefficients $K_{a,b}(q)$ by $E_b(X;q,0) = \sum_a K_{a,b}(q)\kappa_a(X)$

Corollary

For b with column heights μ such that $\mathrm{SSKT}(b)$ has no virtual highest weight elements

$$K_{\lambda,\mu}(t) = \sum_{\mathrm{sort}(a)=\lambda^T} K_{a,b}(t)$$

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Thank You