Optimal Control of Two Ecological Models

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Outline

1. Introduction to some Optimal Control Ideas

2. River Model
with collaborator, Rebecca Pettit (Dept. of Defense)

3. Black Sea Fishery Model
with collaborator, Mahir Demir (Michigan State U, postdoc in fishery group)

Basic Idea behind Optimal Control

System of ODEs or PDEs (or discrete or integrodifference)
Decide on how to manage this system
—by choosing format and bounds on the controls

Design an appropriate GOAL, objective functional
—balancing opposing factors in functional
—include (or not) terms at the final time

Derive necessary conditions for the optimal control
Compute the optimal control numerically

Optimal Control and Pontryagin's Maximum Principle

- Pontryagin and his collaborators developed optimal control theory for ODEs about 1950.
- Pontryagin's key idea was the introduction of the adjoint variables to attach the differential equations to the objective functional (like a Lagrange multiplier attaching a constraint to an optimization of a function).
- Instead of finding an optimal control to maximize the objective functional subject to dynamic equations, they maximize the Hamiltonian with respect to the control at each time.

After setting up a PDE with a control in a specifed set and an objective functional, proving existence of an optimal control in an appropriate weak solution space is a first step.

To derive the necessary conditions , we need to differentiate the map

control $\rightarrow$ objective functional

Note that the state contributes to the objective functional, so we also must differentiate the map

control $\rightarrow$ state

The "sensitivity" is the derivative of the control-to-state map. The sensitivity solves a PDE, which is linearized version of the state PDE.

The formal **adjoint** of the operator in the sensitivity PDE is found.

Transversality Condition: final time condition $\lambda = 0$ at $t = T$

nonhomogeneous term

$$\frac{\partial(\text{ integrand of J })}{\partial \text{state}}$$

Differentiate the objective functional $J(control)$ with respect to the control.

Use the adjoint problem and the sensitivity problem to simplify and obtain the explicit **characterization** of an optimal control.

First Example: Introduction to River Model

- Model adapted from - Jin, Hilker, Steffler, and Lewis (2014) SIAP
    - Seasonal invasion dynamics in a spatially heterogeneous river with fluctuation flows
    - PDE reaction-diffusion model
    - Incorporates both river and population dynamics
    - Use the water discharge flow to control the species
    - Goal: Use flow control in our model to keep the invasive species downstream and prevent the population from moving upstream

Problem Formulation

STATE PDE: $N(x,t)$ population density in the river at location x and time t

$$N_t = -A_t(x,t)\frac{N}{A(x,t)} + \frac{1}{A(x,t)}\left(D(x,t)A(x,t)N_x\right)_x - \frac{Q(t)}{A(x,t)}N_x + rN\left(1-\frac{N}{K}\right)$$

$$\begin{aligned} &N(0,t) = 0 && \text{on } (0,T), x=0, \ \text{(upstream)} \\ &N_x(L,t) = 0 && \text{on } (0,T), x=L, \ \text{(downstream)} \\ &N(x,0) = N_0(x) && \text{on } (0,L), t=0 \end{aligned}$$

in weak solution space $L^2((0,T); H^1_{\{0\}}(0,L))$ with time

$A(x,t)$ cross-sectional area of river
$Q(t)$ water discharge rate, CONTROL

Our control set is

$$U = \{Q \in L^{\infty}(0, T) \,|\, m \leq Q(t) \leq M\}$$

with $0 \leq m < M$.
Our objective functional to minimize

$$J(Q) = \int_0^T \int_0^L W(x)N(x,t)dxdt + \int_0^T \epsilon Q^2(t)dt$$

where $\epsilon > 0$ is small.

The weight $W(x)$ is large near $x = 0$ to emphasize keeping the population low upstream.

Optimal Control Problem Formulation and Analysis

1. Differentiate the control-to-state map as a directional derivative, sensitivity PDE
2. Find our adjoint PDE from the sensitivity PDE
3. Characterize the Optimal Control by differentiate the map from control-to-J (goal)
4. Numerical simulation of state and adjoint system with optimal control

Mostly just showing a few results here

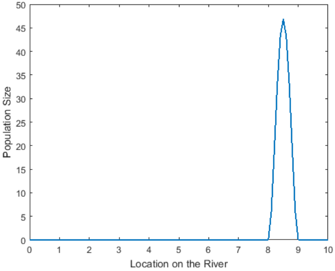
The Adjoint PDE and Optimal Control Characterization

$$
\begin{aligned}
&-\lambda_t - \left(DA\left(\frac{\lambda}{A}\right)_x\right)_x - Q\left(\frac{\lambda}{A}\right)_x + \frac{A_t\lambda}{A} - r\lambda + 2\frac{r\lambda}{K}N^* = W(x) \\
&\lambda(x,T) = 0 \qquad \text{on } (0,L), t = T, \\
&\lambda(0,t) = 0 \qquad \text{on } (0,T), x = 0, \\
&D(L,t)A(L,t)\left(\frac{\lambda(L,t)}{A(L,t)}\right)_x + Q(t)\frac{\lambda(L,t)}{A(L,t)} = 0 \text{ on } (0,T), x = L
\end{aligned}
$$

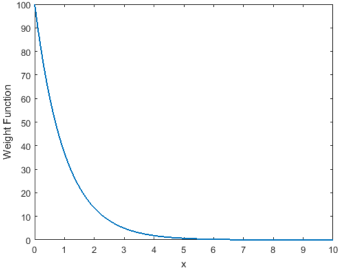
Optimal control characterization

$$
Q^*(t) = \min\left(M, \max\left(\frac{1}{2\epsilon}\int_0^L \frac{\lambda}{A}N_x^*(x,t)dx, m\right)\right)
$$

Initial Condition and Weight Function

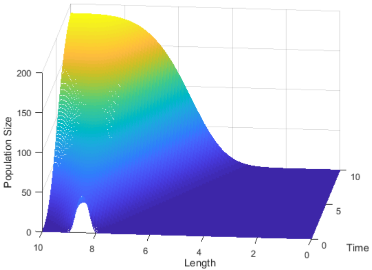


(a) Initial Condition

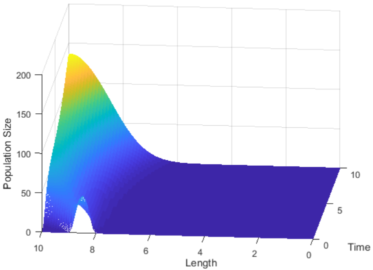


(b) Weight Function

Cross-sectional Area is Constant, Population, Control



(c) No Control



(d) Optimal Control

Figure: Population plots for the no control population and the optimal control population. The parameter values are $T = 10$, $L = 10$, $r = 0.6$, $K = 200$, $D = 0.1$, $A = 20$, $\epsilon = 0.05$, and $0 \leq Q(t) \leq 10$.

Cross-sectional Area is Constant, Downstrean

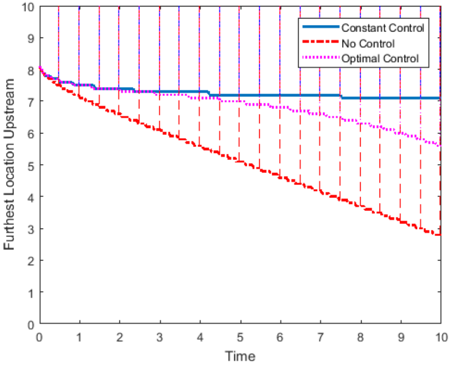
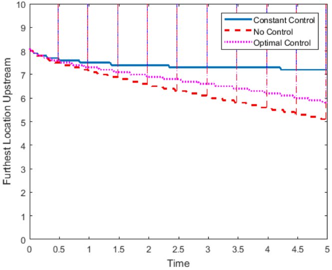


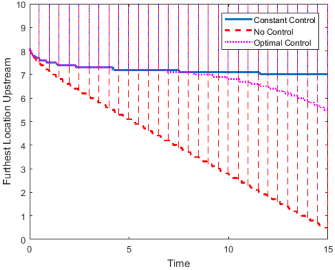
Figure: The upstream location of the constant control population with $Q = 10$ (solid blue line), no control population (red dashed line), and the optimal control population (magenta dotted line).

Detection level greater than 0.5

Cross-sectional Area is Constant - Varying the Final Time



(a) T=5



(b) T=15

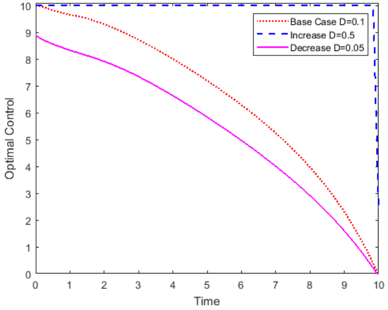
Figure: The upstream location of constant control population (solid blue line), no control population (red dashed line), and optimal control population (magenta dotted line).

Cross-sectional Area is Constant

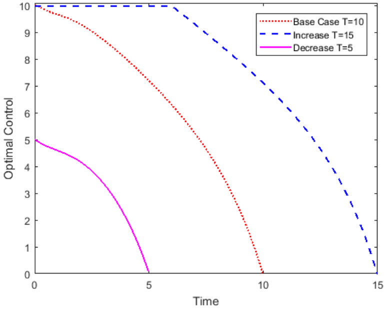
Table: The objective functional outputs for the cases tested where we changed parameter values for K and r (given below) with $T = 10$.

	Base Case	$K = 150$	$K = 250$	$r = 0.3$
No Control	239.52	204.28	269.14	55.18
Constant Control	56.63	56.16	56.97	52.44
Optimal Control	40.12	37.96	41.13	20.29
OC Improv. on CC	29%	32%	28%	61%

Cross-sectional Area is Constant, vary D and T



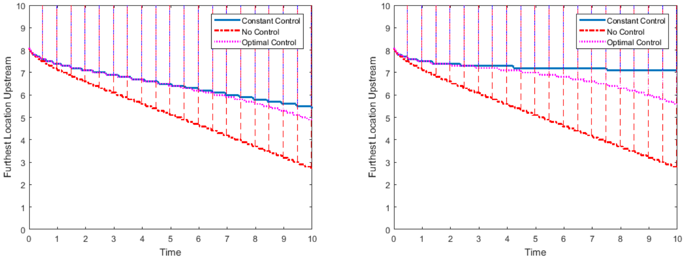
(a) Varying D



(b) Varying T

Figure: The optimal control plots for varying of the parameters D and T. The base case (red dotted line), the increased value (dashed blue line), and the decreased value (solid magenta line).

Cross-sectional Area is Not Constant - $A(x,t) = (0.5x + 25) + (0.2t\,(10 - t))$



(a) $A(x,t)$ **(b)** $A = 20$

Figure: The upstream location of the constant control population (solid), no control population (red dashed), and the optimal control population (magenta) with $T = 10$.

J values ar 66 and 40 respectively.

Approximation of an Optimal Control

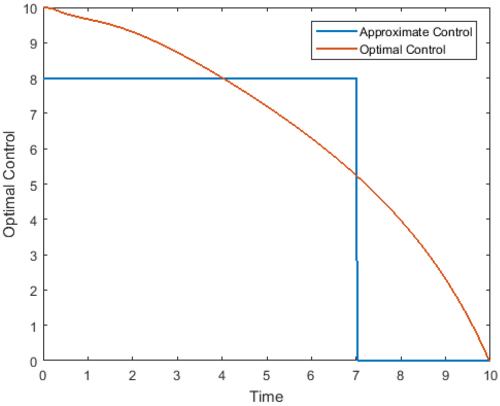


Figure: Comparing the optimal control and the approximate control case when $A = 20$

J(approximate control) is 6% higher than $J(Q^*)$

Conclusions and Future Work

Conclusions

- Successful in illustrating pushing an invasive species downstream compared to the no control case
- Various results with varying parameters, initial conditions, weight function, and the cross-sectional area

Future Work

- Want to use a more realistic $A(x, t)$
- Find data for an invasive species moving upstream
- Restrict flow to certain seasons in a year

Second Example: Some Background about the Black Sea Anchovy

Figure 1: Location of the Black Sea

- The Anchovy family contributes to the global fisheries over 10 % of landing.
- The European anchovy is the third most widely harvested species of the Anchovy family, and about 40% comes from the Black Sea.
- Fishery Season is open on the Turkish Coast of the Black Sea between September 1 and April 14, but for the commercial fishery of anchovy, the fishing season is about 3 months.
- Anchovy **plays a crucial role in the Black Sea pelagic food web** as a prey and predator of many species. It is also an **important consumer of zooplanktons** in the Black Sea.

Main goal of our study:

- The **main goal** is to investigate ecosystem-based optimal fishery management strategies for the anchovy fishing on the southern part of the Black Sea.

- **Tools**:
  - We built a **food chain model** with three trophic levels and with seasonal fishery **to track the effects of the fishery** on the Black Sea food web, and see the effect of predator-prey relations on the anchovy fishery, especially effect of the invasive Jellyfish.
  - Use OC tools to find the optimal harvesting strategy that maximizes the discounted net value of the anchovy population with seasonal harvesting.

Flow Diagram of Consumption in the System

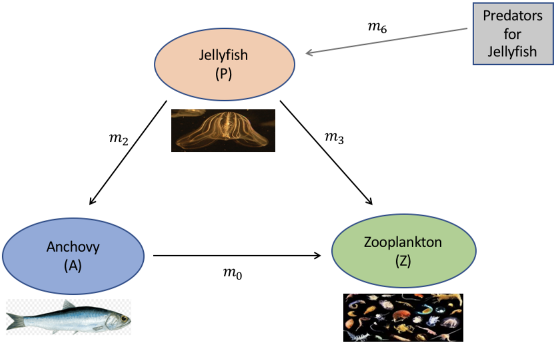


Figure 3: The flow diagram of consumption in our food chain model.

- $A(t)$: Anchovy biomass.
- $P(t)$: Predator biomass of anchovy (jellyfish).
- $Z(t)$: Zooplankton biomass.

Our Food Chain Model with Seasonal Harvesting

$$\frac{dA}{dt} = r_1 A(1 - \frac{A}{K_1}) + m_0 AZ - m_1 PA - hA$$
$$\frac{dP}{dt} = r_2 P(1 - \frac{P}{K_2}) + m_2 PA + m_3 PZ - m_6 P$$
$$\frac{dZ}{dt} = r_3 Z(1 - \frac{Z}{K_3}) - m_4 AZ - m_5 PZ$$

with the initial conditions:

$$A(0) = A_0, \quad P(0) = P_0, \quad Z(0) = Z_0$$

- $h(t)$: Harvest rate (effort), **OUR CONTROL**,
  $h = 0$ in the offseason.
- m_0, m_1, m_2, m_3, m_4, and m_5 are predation rates.
- m_6 denotes the predation rate on the jellyfish, P, from other predators

Objective Functional

$$J(h) = \int_0^T e^{-\alpha t}(hA - \mu_1 h - \mu_2 h^2)dt = \int_\Omega e^{-\alpha t}(hA - \mu_1 h - \mu_2 h^2)dt$$

- $\Omega = \cup_{i=1}^{T}[a_i, b_i]$ time intervals for seasonal harvesting.
- $[a_i, b_i]$ represents the fishery seasons for $i = 1, 2, ..., T$
- $e^{-\alpha t}$ is the discount rate
- hA is the yield of the fishery
- $\mu_1 h + \mu_2 h^2$ denotes the cost of the harvest on Ω, and $h = 0$ on $[0, T] \setminus \Omega$

Find an optimal control, h^* in $\mathcal{A}$ such that

$$J(h^*) = \sup_{h \in \mathcal{A}} J(h)$$

$$\mathcal{A} = \{h : [0, T] \longrightarrow [0, M] \mid \text{h=0 on } [0, T] \setminus \Omega \text{ and h Leb. meas.}\}$$

Use Pontryagin's Maximum Principle and Hamiltonian

$$H = e^{-\alpha t}(hA - \mu_1 h - \mu_2 h^2) + \lambda_A \left[r_1 A - \frac{r_1}{K_1} A^2 + m_0 AZ - m_1 PA - hA \right]$$
$$+ \lambda_P \left[r_2 P - \frac{r_2}{K_2} P^2 + m_2 PA + m_3 PZ - m_6 P \right]$$
$$+ \lambda_Z \left[r_3 Z - \frac{r_3}{K_3} Z^2 - m_4 AZ - m_5 PZ \right]$$

$$\frac{d\lambda_A}{dt} = -\frac{\partial H}{\partial A}$$
$$\frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial P}$$
$$\frac{d\lambda_Z}{dt} = -\frac{\partial H}{\partial Z}$$

together with the transversality conditions, $\lambda_1(T) = \lambda_2(T) = \lambda_3(T) = 0$.

$$h^*(t) = \min \left\{ M, \max \left\{ 0, \frac{A^*(1 - e^{\alpha t}\lambda_A^*) - \mu_1}{2\mu_2} \right\} \right\} \quad \text{on } [0, T]$$

Numerical Methods and Parameter Estimation

- Using the annual **landing and fleets data** of the anchovy population on the southern part of Black Sea, 2003-2016, from
  obtained by the Scientific, Technical and Economic Committee for Fisheries (STECF),
  we estimated the parameters with constant h, and then did optimal control problem.
- **Estimated Parameters** are r_1, r_2, r_3 (intrinsic growth rates), m_i for $i = 0, 1, .., 6$ (interaction coefficients) with h constant.

Optimal Control Case

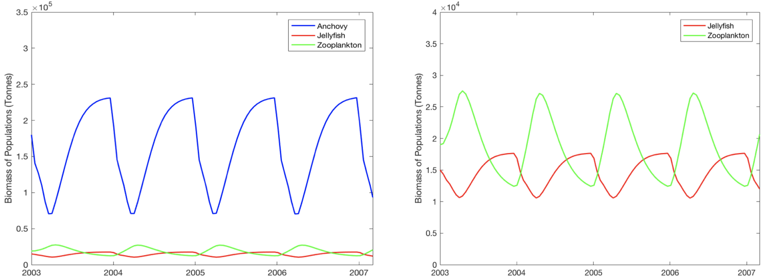


Figure 4: LHS: Landing of the Black Sea anchovy with OC case, $h_{max} = 0.4$. RHS: Biomass of the Black Sea anchovy (blue), Jellyfish (red), and Zooplankton (green) with OC case.

Optimal Control Rates and their Approximation

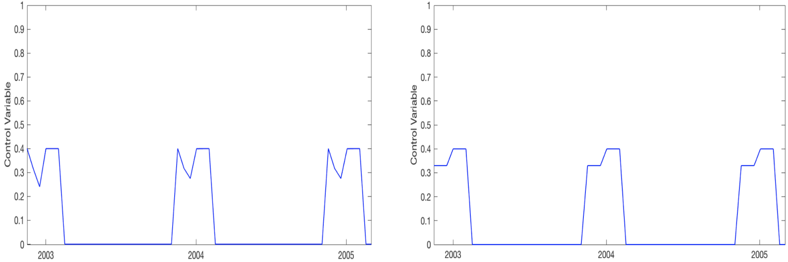


Figure 5: Left: The harvesting effort in OC case, $h_{max} = 0.4$. Right: The approximate harvesting effort with the first half of the fishing season, $h = 0.335$

J(approximate) is 3% less than $J(h^*)$

Estimation of fishing fleets (Effort $\approx h$)

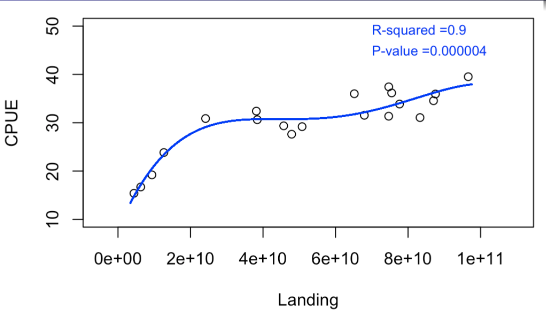


Figure 5: Non-linear regression between CPUE and the landing of the anchovy population depending on the data.

The number of fishing fleets (Effort) is estimated as

$$Effort^* = \frac{Landing^*}{CPUE^*},$$

where, $Landing^*$ is our optimal landing, and $CPUE^*$ is our approximate $CPUE$ obtained from non-linear regression model.

Compare with single DE case with only Anchovy

$$\frac{dA}{dt} = r_1 A(1 - \frac{A}{K_1}) - hA$$

We found the parameters to this model by fitting to the landing data and used the same objective functional.

We found optimal harvest for this model.

Comparison of two models in optimal control case

Comparison of the Models with OC			
Models	**Landing (Tonnes) (Data 2,865,392)**	**D. Net Cost**	**D. Net Profit (Tonnes)**
Food Chain $h_{max} = 0.4$	3,239,600	767,470	2,009,100
Anchovy $h_{max} = 0.4$	3,665,400	772,850	2,368,400
Food Chain$^{(***)}$ $h_{max} = 0.4$	3,234,700	772,850	1,994,200

(***) : We implemented the harvesting strategy of the anchovy equation in our food chain model.

Table 2: Comparison of two models under assumption of about 50% net profit ($\mu_1 = 30700$, and $\mu_2 = 0.1$)

- We got much more profit by using the anchovy equation than by using our food chain model, but it is not realistic. Actual profit would be lower if we took account of biological interactions of the ecosystem .

Conclusions for ODE part

- It is better to **reduce harvesting effort to optimal level** to obtain more profit and to help the anchovy population to reproduce more new individuals for next fishery seasons.
- Taking into account of the food web for the Black Sea anchovy **gives more reliable management information** than only using the anchovy equation.
- Optimal controls with too much variation may be difficult to implement and an approximation of an optimal control may be chosen and implemented effectively.

Ideas for Discussion

- Some actions do not happen continuously and one may need to do impulse actions.
- Other approaches: Viability modeling work of Luc Doyen and Pedro Gajardo, (state constraints)
- Economic-ecology work of Jim Sanchirico, Mike Springborn
- Adaptive management and learning, Paul Fackler, Jim Nichols, Michael Runge.
- NIMBioS Ecosystem Federalism working group studying two patches with external 'federal' control and local control
- THANKS