# Random Matrix Products and Anderson Localization

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In 1977 Philip Anderson shared the Nobel Prize in Physics with his doctoral thesis advisor John van Vleck and his collaborator Nevill Mott. The Nobel Prize was awarded “for their fundamental theoretical investigations of the electronic structure of magnetic and disordered systems”, or, in other words, for the discovery of what is nowadays called *Anderson localization*. In condensed matter physics, Anderson localization is the absence of diffusion of waves in a random (disordered) medium. A popular, though not quite equivalent, mathematical justification of (spectral) Anderson localization is pure point spectrum of the corresponding Schrödinger operator with random potential, along with exponentially decaying eigenfunctions.

Many random models aside from Schrödinger operators with potentials given by iid random variables at each site of a finite-dimensional lattice have been considered, for example sparse random potentials, decaying random potentials, the “trimmed” Anderson model, and the Anderson model on regular trees. But most of the existing methods of proof either require some form of absolute continuity of the randomness (e.g., the Kunz-Souillard method, the fractional moment method, or spectral averaging), or use a highly involved and technically challenging machinery (e.g. multiscale analysis). At the same time, recently it became clear that theory of random matrix products can be used to provide more geometrical and transparent proofs of Anderson Localization, at least in 1D case (see the extended abstracts of talks by Jake Fillman, Victor Kleptsyn, Tom VandenBoom, and Xiaowen Zhu below). New results were obtained in higher dimensional case as well, see the extended abstract of Charles Smart below.

The workshop was organized to bring together people who contributed to the recent progress in the field, as well as both experts and graduate students specializing in the areas that are directly related to random matrix products and/or Anderson Localization.

The report consists of two sections. In the first one we collected some of the open problems that were presented at the problem sessions that were organized during the workshop. Also, the participants were requested to provide the extended abstracts of the talks, that would contain the formal statements of the main results that were presented, and would be useful as an ”entry point” to the subject. The second one consists of the extended abstracts that were provided by the participants.

## 1 Open Problems

Here we provide the list of open problems that were presented during the problem sessions organized at the workshop. The participants were encouraged to share open problems related to the topic of the workshop that they are familiar with and find interesting, even if a problem is well known to the community and was initially formulated long time ago.

**Problems presented by Jake Fillman (Texas State University)**

Given an irrational $\alpha \in \mathbb{R}$, the *skew-shift* is a dynamical system that takes place on the 2-torus $\mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$. The dynamics are given by $T = T_\alpha$:

$$T(\theta_1, \theta_2) := (\theta_1 + \alpha, \theta_1 + \theta_2), \quad (\theta_1, \theta_2) \in \mathbb{T}^2.$$

For each $\boldsymbol{\theta} = (\theta_1, \theta_2) \in \mathbb{T}^2$, one defines a potential $V_{\boldsymbol{\theta}} \in \ell^\infty(\mathbb{Z})$ via

$$V_{\boldsymbol{\theta}}(n) = 2\cos(2\pi(P_2(T^n\boldsymbol{\theta}))), \quad n \in \mathbb{Z},$$

where $P_2(\theta_1, \theta_2) = \theta_2$. Then, for each $\boldsymbol{\theta} \in \mathbb{T}^2$, and $\lambda \in \mathbb{R}$, one has a Schrödinger operator $H_{\boldsymbol{\theta},\lambda}$ given by

$$[H_{\boldsymbol{\theta},\lambda}\psi](n) = \psi(n-1) + \lambda V_{\boldsymbol{\theta}}(n)\psi(n) + \psi_{n+1}.$$

By standard arguments, there is a fixed set $\Sigma_\lambda$ such that $\Sigma_\lambda = \sigma(H_{\boldsymbol{\theta},\lambda})$ for all $\boldsymbol{\theta} \in \mathbb{T}^2$. We proposed the following open problems:

1. Prove or disprove that $\Sigma_\lambda$ is an interval or has at most finitely many gaps for all $\lambda > 0$.
2. Prove or disprove that the Lyapunov exponent of $H_{\boldsymbol{\theta},\lambda}$ is positive for all $\lambda \neq 0$.
3. Prove or disprove that the family $\{H_{\boldsymbol{\theta},\lambda}\}_{\boldsymbol{\theta}\in\mathbb{T}^2}$ enjoys Anderson localization for any $\lambda \neq 0$.

We note that Problem 2 is easily solved for $|\lambda| > 1$ by Herman's estimate; the open problem is then for $\lambda \in [-1, 1] \setminus \{0\}$.

**Problems presented by Anton Gorodetski (University of California, Irvine)**

One of the most basic results in the theory of the 1D ergodic Schrödinger operators is the theorem by Pastur that claims that the spectrum is a compact set that is the same for almost every initial condition (i.e. for almost all phases). In particular, in the case of a potential given by iid random variables, it is known that the almost sure spectrum is a finite union of intervals. The statement certainly cannot hold in general in the non-stationary setting, i.e. for a potential given by independent but not identically distributed variables. At the same time, due to Kolmogorov zero-one law, the almost sure essential spectrum is well defined in this case. *Is it possible to give a description of the essential spectrum (as a set) in this case?* It is known that the essential spectrum does not have to be a finite union of intervals, see the extended abstract of the talk by A.Gorodetski below. *Is it true that in the case of the potential given by independent random variables with variation uniformly bounded away from zero the essential spectrum must contain an interval?*

An attempt to construct a counterexample that would answer the latter question we were trying to consider a potential given by $V_1(n) + V_2(n)$, where $\{V_1(n)\}$ is a Fibonacci potential with large coupling, and $\{V_2(n)\}$ is an Anderson-Bernoulli potential. *What is the spectrum (as a set) of the corresponding Schrödinger operator? Is it a Cantor set? Cantorval? Does it contain an interval?*

**Problems presented by Nishant Rangamani (University of California, Irvine)**

We begin with the set-up of the problem. Let $I$ be a compact interval and suppose for each $E \in I$ we have a family of independent and identically distributed random matrices (i.i.d.) $Y_1^E, \ldots, Y_n^E, \ldots$ in $SL_2(\mathbb{R})$. Suppose further that the smallest closed subgroup generated by the matrices is strongly irreducible and contracting and $\mathbb{E}[\log^+ ||Y_i^E||] < \infty$. Under these conditions, Kingman's subadditive ergodic theorem together with Furstenberg's theorem imply that the Lyapunov exponent which can be defined (for fixed $E$) as $\lim_{n\to\infty} \frac{1}{n} \log ||Y_n^E \cdots Y_1^E||$ exists almost surely and is strictly positive.

The next development along these lines revolved around obtaining further analogs of results for i.i.d. real random variables (e.g. central limit theorem, large deviations, etc.). These developments built around the work of Furstenberg and was carried out by Guivarc'h, Goldsheid, Le Page, and Raugi (among many other authors as well).

In particular, in 1982, Le Page proved analogs of both central limit theorem and large deviation theorems for products of i.i.d. matrices (under appropriate moment condition and a condition on the smallest closed subgroup of $SL_2(\mathbb{R})$ generated by the matrices - strong irreducibility and contracting). These results were later extended to the matrix elements of such products by Tsay. However, these results are all obtained under a moment condition. In particular, it is required that $\mathbb{E}[\exp(\log^+ ||M||)] < \infty$. This is known as having a finite exponential moment. We note that this material and additional background is well covered in the monograph by Bougerol and Lacroix[1].

Thus, it is natural to ask what happens in the absence of such a moment condition.

In particular, can estimates be made as to the speed of the convergence of the products (and their matrix elements) when there is no exponential moment? Towards this end, there has been recent work by Cagri Sert[2] which identifies a convex rate function (under no moment condition) through which a weak large deviation principle can be stated. However, the specifics of the rate function can only be identified in the regime of an exponential moment and it remains desirable to obtain results which link the strength of the moment to the rate at which the random products deviate from their mean.

We state Tsay's theorem for large deviations of the matrix elements below in order to illustrate what types of estimates are available with an exponential moment. We note that it is likely that the rate at which the matrix elements converge to the mean reflects the strength of the moment and in the absence of an exponential moment, one should in turn expect slower convergence to 1 in the probabilities below.

**Theorem.**[3] *Suppose $I$ is a compact interval and for each $E \in I$, $Y_1^E, \ldots, Y_n^E, \ldots$ are i.i.d. random matrices such that the smallest closed subgroup of $SL_2(\mathbb{R})$ generated by the matrices is strongly irreducible and contracting. In addition, suppose $\mathbb{E}[\exp(\log^+ ||Y_1^E||)] < \infty$. Then, for any $\varepsilon > 0$, there is an $\eta > 0$ and an $N$ such that for any $E \in I$, any unit vectors $u, v$, and $n > N$,*

$$\mathbb{P}[e^{(\gamma(E)-\varepsilon)n} \leq |\langle Y_n^E \cdots Y_1^E u, v\rangle| \leq e^{(\gamma(E)+\varepsilon)n}] \geq 1 - e^{-\eta n}. \tag{1}$$

**Problems presented by Xiaowen Zhu (University of California, Irvine)**

Open problem: Prove strong dynamical localization w.r.t. uniform distance in multi-particle model.

The multi-particle model can be defined as follows:

Let $H_\omega : l^2(\mathbb{Z}^{nd}) \to l^2(\mathbb{Z}^{nd})$, where $n$ is the number of the particles and $d$ is the dimension of the space. Let $\mathbf{x} = (x_1, x_2, \cdots, x_n) \in \mathbb{Z}^{nd}$ where each $x_i \in \mathbb{Z}^d$ will denote the position of the $i$-th particle. For future convenience, denote $x_i = (x_i^1, \cdots, x_i^d)$, where $x_i^j \in \mathbb{Z}$.
Let $V_\omega(x_i) = \omega_{x_i}$ be i.i.d random variables with single-site distribution $\mu$ supported on $\mathbb{R}$ that is bounded and non-trivial (supported on more than one point).
Define

$$(H_\omega \psi)(\mathbf{x}) = (\triangle_{nd}\psi)(\mathbf{x}) + \sum_i V_\omega(x_i) + \sum_{1 \leq i < j \leq n} U(x_i - x_j),$$

where $\triangle_{nd}$ is discrete Laplacian on $l^2(\mathbb{Z}^{nd})$, $U : l^2(\mathbb{Z}^d) \to l^2(\mathbb{Z}^d)$ is the short-range interaction, so we require $U(x) = U(-x)$ and $U(x) = 0$ if $|x| > r$ for some constant $r > 0$.

In order to introduce that, we need definition of two different distance for $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^{nd}$, that is the uniform distance $||\mathbf{x} - \mathbf{y}||_\infty$ and Hausdorff distance $d_H(\mathbf{x}, \mathbf{y})$:

$$||\mathbf{x} - \mathbf{y}||_\infty = \max_i \max_j |x_i^j - y_i^j|$$

$$d_H(\mathbf{x}, \mathbf{y}) = \max\{\max_i \min_k ||x_i - y_k||, \max_i \min_k ||y_i - x_k||\}$$

where $||x_i - y_k|| = \max_{1 \leq j \leq n} |x_i^j - y_k^j|$.
Note that $d_H(\mathbf{x}, \mathbf{y}) \leq ||\mathbf{x} - \mathbf{y}||_\infty$. And with this notation, $d_H(\mathbf{x}, \mathbf{y}) < L$ if and only if $\forall x_i$, $\exists y_k$ such that

[1] P. Bougerol and J. Lacroix, *Products of random matrices with applications to Schrodinger operators*, Progress in Probability and Statistics Vol. 8, Birkhause, Boston, 1985.
[2] C. Sert, Large deviation principle for random matrix products, *ArXiv e-prints*, (2018).
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$||x_i - y_k|| < L$, and $\forall y_i$, $\exists x_k$ such that $||y_i - x_k|| < L$.

What we know about this model is Anderson localization and strong dynamical localization w.r.t. Hausdorff distance, i.e.

$$\mathbb{E}(\sup_{t\in\mathbb{R}} | < \delta_{\mathbf{x}}, e^{-itH_\omega}\delta_{\mathbf{y}} > |) \leq e^{-Cd_H(\mathbf{x},\mathbf{y})}$$

What is unknown is strong dynamical localization w.r.t. infinity distance, i.e.

$$\mathbb{E}(\sup_{t\in\mathbb{R}} | < \delta_{\mathbf{x}}, e^{-itH_\omega}\delta_{\mathbf{y}} > |) \leq e^{-C||\mathbf{x}-\mathbf{y}||_\infty}$$

This problem is interesting because of the physics interpretation: We know in the random system in $\mathbb{Z}^d$, if there are only one particle in low energy, the probability that it escapes to some point far away decay exponentially by the strong dynamical localization. Now consider n particles, if one starts with the intial state where all $n$ particals are gathered at the same position, say 0, and wondered what's the probability of one of them escaping to some point far away, according to strong dynamical localization w.r.t. uniform distance, you get no information since the Hausdorf distance of this two states of positions are 0. But since even for just one particle, the probability of escape is exponentially small, it makes sense to guess that in this case, the probability of escape is exponentially small, which will be implied by strong dynamical localization w.r.t. uniform distance. One could also think about the case you start with all particles gathered together and all except one has escaped to some where far away, which should be even smaller. This can't be recognized through dynamical localization w.r.t. Hausdorff distance as well.

## 2 Presentation Highlights

Here we provide the extended abstracts of the talks that were given by the participants of the workshop.

**Random matrix products and random dynamical systems**
**by Peter Baxendale (University of Southern California)**

This tutorial and historical survey was in two parts. The first part considered the connection between random matrix products and random dynamical systems. The random matrices are assumed to form a stationary sequence of $d \times d$ matrices; the restriction to $SL(d, \mathbb{R})$ is unnecessary and is not applicable to dissipative stochastic systems. The concept of multiplication of random matrices extends to composition of random mappings, and hence to random dynamical systems. An important concept is that of the associated skew-product flow. It is in this setting that the multiplicative ergodic theorem (giving rise to the Lyapunov spectrum) and the associated local stable manifold theorem (justifying the use of the linearized system to approximate the underlying non-linear system) can be applied. See [1, 4].

The second part showed how some of these ideas can be used in the analysis of a stochastic bifurcation scenario for a damped and random excited non-linear harmonic oscillator. The linearized system is a 2-dimensional linear stochastic differential equation with top Lyapunov exponent $\lambda = \lambda(\beta, \sigma)$ depending on the coefficient of linear dissipation $\beta$ and the noise intensity $\sigma$. The calculation of $\lambda$ using the Fursteberg-Khas'minskii formula will be discussed. When $\lambda < 0$ it is shown that both the linearization and the underlying non-linear system are almost-surely stable. However when $\lambda > 0$ the non-linear effects become significant, and the behavior of the non-linear system for small positive $\lambda$ will be determined using information from the moment Lyapunov function. The scenario when the parameters $(\beta, \sigma)$ are varied in such a way that $\lambda$ passes through 0 is a stochastic version of the deterministic Hopf bifurcation. See [2, 3].

## References

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**Random perturbations of hyperbolic dynamics**

**by Florian Dorsch (Friedrich-Alexander-Universität Erlangen-Nürnberg, Germany)**

We consider a random dynamical system on an $\mathsf{L}$-dimensional sphere $\mathbb{S}^{\mathsf{L}}$, $\mathsf{L} \geq 2$, given by

$$v_n \;=\; \mathcal{T}_n \cdot v_{n-1}\,, \qquad n \in \mathbb{N}\,, \tag{2}$$

where the action $\cdot\;: \mathrm{GL}(\mathsf{L}+1,\mathbb{R}) \times \mathbb{S}^{\mathsf{L}} \to \mathbb{S}^{\mathsf{L}}$ of the general linear group is

$$\mathcal{T} \cdot v \;=\; \mathcal{T}v \|\mathcal{T}v\|^{-1}\,, \tag{3}$$

and the random matrices $\mathcal{T}_n$ are of the form

$$\mathcal{T}_n \;=\; \mathcal{R}\left(\mathbf{1} \,+\, \lambda r_n U_n\right)\,, \qquad n \in \mathbb{N}\,. \tag{4}$$

Here, the matrix $\mathcal{R}$ is supposed to be deterministic and hyperbolic, *i.e.*, it is of the form

$$\mathcal{R} = \mathrm{diag}(\kappa_{\mathsf{L}+1}, \dots, \kappa_1)\,, \qquad\qquad \kappa_1 \geq \cdots \geq \kappa_{\mathsf{L}+1} > 0 \tag{5}$$

and $\{r_n\}_{n\in\mathbb{N}}$ and $\{U_n\}_{n\in\mathbb{N}}$ are assumed to be sequences of independent and identically distributed random variables taking on values in $[0,1]$ and $\mathrm{O}(\mathsf{L}+1)$, respectively. Moreover, we assume that $r_n \not\equiv 0$ and that the $U_n$ are distributed according to the Haar measure on $\mathrm{O}(\mathsf{L}+1)$.

Each vector $v = (v_1, \dots, v_{\mathsf{L}+1})^{\mathsf{T}} \in \mathbb{R}^{\mathsf{L}+1}$ is split into its upper part $\mathfrak{a}(v) \in \mathbb{R}^{\mathsf{L}_\mathfrak{a}}$, middle part $\mathfrak{b}(v) \in \mathbb{R}^{\mathsf{L}_\mathfrak{b}}$ and lower part $\mathfrak{c}(v) \in \mathbb{R}^{\mathsf{L}_\mathfrak{c}}$ via

$$\mathfrak{a}(v) \;=\; (v_1, \dots, v_{\mathsf{L}_\mathfrak{a}})^{\mathsf{T}}\,, \qquad \mathfrak{b}(v) \;=\; (v_{\mathsf{L}_\mathfrak{a}+1}, \dots, v_{\mathsf{L}_\mathfrak{a}+\mathsf{L}_\mathfrak{b}})^{\mathsf{T}}\,, \qquad \mathfrak{c}(v) \;=\; (v_{\mathsf{L}_\mathfrak{a}+\mathsf{L}_\mathfrak{b}+1}, \dots, v_{\mathsf{L}+1})^{\mathsf{T}}\,,$$

where $(\mathsf{L}_\mathfrak{a}, \mathsf{L}_\mathfrak{b}, \mathsf{L}_\mathfrak{c}) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}$ are such that $\mathsf{L}_\mathfrak{a} + \mathsf{L}_\mathfrak{b} + \mathsf{L}_\mathfrak{c} = \mathsf{L}+1$. Associated to that partition, let us introduce the *macroscopic gap* $\gamma = \gamma\left(\mathcal{R}, \mathsf{L}_\mathfrak{b}, \mathsf{L}_\mathfrak{c}\right)$ between the upper and lower parts by

$$\gamma \;=\; \min\left\{1\,,\, \frac{\kappa^2_{\mathsf{L}_\mathfrak{c}}}{\kappa^2_{\mathsf{L}_\mathfrak{b}+\mathsf{L}_\mathfrak{c}+1}} \,-\, 1\right\} \;\in\; [0,1]\;.$$

If the macroscopic gap $\gamma$ is positive, the entries of the upper part $\mathfrak{a}$ can be seen as the repulsive entries of the hyperbolic action. Therefore, the deviation of the random path $\{v_n\}_{n\in\mathbb{N}}$ from the attractive part of the phase space can be measured as the norm of the upper part $\|\mathfrak{a}(v_n)\|$. The main result provides a quantitative bound on the expectation value of $\|\mathfrak{a}(v_N)\|^2$ for sufficiently large $N$.

**Theorem [1].** *Suppose that* $(\mathsf{L}_\mathfrak{a}, \mathsf{L}_\mathfrak{b}) \neq (1,1)$ *and* $\gamma > 0$. *Then, for all* $0 < \lambda \leq \frac{1}{4}$ *there exist* $N_0 = N_0(\mathsf{L}, \mathsf{L}_\mathfrak{c}, \lambda) \in \mathbb{N}$ *such that*

$$\mathbb{E}\,\|\mathfrak{a}(v_N)\|^2 \;\leq\; 2\left(\frac{\mathsf{L}+1}{\mathsf{L}_\mathfrak{a}+\mathsf{L}_\mathfrak{b}}\right)^{\frac{\mathsf{L}_\mathfrak{a}+\mathsf{L}_\mathfrak{b}-2}{\mathsf{L}_\mathfrak{c}+2}} \left(\frac{6}{\gamma}\,\frac{\mathsf{L}_\mathfrak{a}}{\mathsf{L}_\mathfrak{c}}\,\lambda^2\right)^{\frac{\mathsf{L}_\mathfrak{c}}{2+\mathsf{L}_\mathfrak{c}}} \tag{6}$$

*for all* $N \geq N_0$ *and* $v_0 \in \mathbb{S}^{\mathsf{L}}$.

# References

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**Sufficient criteria for the application of Fürstenberg's theorem**
**by Jake Fillman (Texas State University)**

The talk describes simple sufficient criteria that enable one to apply the classical theorem of H. Fürstenberg on products of random matrices to deduce positive Lyapunov exponents. To begin, let us recall a few relevant definitions. A subgroup $G \subseteq \mathrm{SL}(2,\mathbb{R})$ is called *strongly irreducible* if there does not exist a finite set $\emptyset \neq \Lambda \subseteq \mathbb{RP}^1$ such that $g\Lambda = \Lambda$ for every $g \in G$. We call $G$ a *type-F* subgroup if $G$ is closed, non-compact, and strongly irreducible.

Given a Borel probability measure $\nu$ supported in $\mathrm{SL}(2,\mathbb{R})$, consider a sequence $A_1, A_2, \ldots$ of iid random variables with common distribution $\nu$. One is interested in the Lyapunov exponent:

$$L(\nu) = \lim_{n\to\infty} \frac{1}{n} \int_{\mathrm{SL}(2,\mathbb{R})^n} \log \|A_n A_{n-1} \cdots A_1\| \, d\nu^n.$$

In essentially all proofs of localization in the 1D Anderson model, positivity of the Lyapunov exponent supplies the key input to begin a localization proof, although there are notable exceptions, (e.g. [9, 10]). Classically, one used multi-scale analysis to prove localization [3], but there have been several modern approaches using one-dimensional tools (hence yielding simpler proofs) [1, 7, 8].

**Theorem 1** (Fürstenberg [6])**.** *Let $\nu$ be a probability measure supported in $\mathrm{SL}(2,\mathbb{R})$ with $\mathbb{E}(\log\|g\|) < \infty$ and let $G_\nu$ denote the smallest closed subgroup of $\mathrm{SL}(2,\mathbb{R})$ containing $\mathrm{supp}\nu$. If $G_\nu$ is a type-F subgroup of $\mathrm{SL}(2,\mathbb{R})$, then $L(\nu) > 0$.*

The main theorem of [2] gives simple criteria to check whether one may apply Theorem 1 for analytic one-paramter families.

**Theorem 2** (Bucaj, Damanik, F., Gerbuz, VandenBoom, Wang, Zhang [2])**.** *Let $A, B : \mathbb{C} \to \mathrm{SL}(2,\mathbb{C})$ be entire functions such that:*

(i) *If $z \in \mathbb{R}$, then $A(z), B(z) \in \mathrm{SL}(2,\mathbb{R})$,*

(ii) *$\mathrm{Tr}\, A(z)$ and $\mathrm{Tr}\, B(z)$ are non-constant,*

(iii) *if $\mathrm{Tr}\, A(z) \in [-2,2]$ or $\mathrm{Tr}\, B(z) \in [-2,2]$, then $z \in \mathbb{R}$, and*

(iv) *$[A(z), B(z)] := A(z)B(z) - B(z)A(z) \neq 0$ for at least one $z \in \mathbb{C}$.*

*Then, there is a discrete set $D \subseteq \mathbb{R}$ with the property that the closed subgroup generated by $A(x)$ and $B(x)$ is a type-F subgroup of $\mathrm{SL}(2,\mathbb{R})$ for any $x \in \mathbb{R} \setminus D$.*

The key observation is that one can encode the failure of the hypotheses of Fürstenberg's theorem by the vanishing of analytic quantities. Namely, if $\mathrm{Tr}\, A$, $\mathrm{Tr}\, B$, and $\det[A, B]$ are all nonzero, then one can show that the closed group generated by $A$ and $B$ is of type F.

Critically, hypotheses (i)–(iii) are automatically satisfied in essentially any model based on Schrödinger operators, leaving (iv) as the relevant "nontriviality condition". One can use Theorem 2 to easily verify positivity of the Lyapunov exponent in several models, such as the continuum Anderson model [2], Schrödinger operators arising from a decomposition of radial metric tree graphs [4], and 1D Schrödinger operators with random point interactions [5]. [Joint work with V. Bucaj, D. Damanik, V. Gerbuz, T. VandenBoom, F. Wang, Z. Zhang]

# References


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**Exponential growth of products of non-stationary Markov-dependent matricest**

**by Ilya Goldsheid (Queen Mary, University of London)**

## 1. Introduction

Let $(g_n)_{n\geq 1}$ be a sequence of matrices, $g_n \in SL(m,\mathbb{R})$ and set

$$S_n = g_n ... g_1 \tag{7}$$

In the seminal 1963 paper [2], H. Furstenberg proved, among others, the following fact.

**Theorem 1.** *Suppose that:*

*(a) $(g_n)_{n\geq 1}$ is a sequence of independent identically distributed (i.i.d.) random matrices with distribution $\mu$.*

*(b) The group $\bar{G}_\mu$ generated by the support of $\mu$ does not preserve any probability measure on the unit sphere in $\mathbb{R}^m$. (The relevant definitions can be found below.)*

*Then the following limit (called the top Lyapunov exponent of the product $S_n$) exists with probability 1 and is strictly positive:*

$$\lim_{n\to\infty} \frac{1}{n} \ln \|S_n\| = \gamma > 0. \tag{8}$$

In 1980 A. Virtser [5] extended this result result to the case of stationary Markov chains.

The purpose of this work is to establish sufficient conditions for exponential growth of products of Markov-dependent matrices in the case when the underlying Markov chain is non-stationary.

**Remark.** *In fact, Furstenberg proved, under the additional condition of strong irreducibility of $\bar{G}_\mu$, that for any unit vector $x$ a.s. $\lim_{n\to\infty} \frac{1}{n} \ln \|S_n x\| = \gamma$. We don't discuss this aspect of Furstenberg's result here.*

Products of *independent* non-identically distributed matrices were considered in the past. Here are some references.

Paper [1] by Delyon-Simon-Souillard deals with matrices arising in the theory of localization for Anderson model in dimension one with a potential decaying at infinity. These matrices are of the form

$$g_n = \begin{pmatrix} \lambda a_n q_n & -1 \\ 1 & 0 \end{pmatrix}, \tag{9}$$

where $q_n$ are i.i.d. random variables, $\lambda > 0$ is a constant, and $a_n$ satisfy $C_1|n|^{-\alpha} < |a_n| < C_2|n|^{-\alpha}$ for some positive constants $C_1,\ C_2,\ \alpha$ $(n \neq 0)$. The technique of [1] depends heavily on $q_n$ being i.i.d. with a 'good' probability density function.

An earlier paper [3] by Simon, even though it does not explicitly consider products of matrices, easily implies interesting estimates for the speed of growth for products of matrices (9) with $\alpha = \frac{1}{2} - \varepsilon$ $(\varepsilon > 0)$. The independence of $q_n$'s and the existence of their densities are extensively used in the proofs in this paper while the fact that the densities are the same is not that important (see remarks in [3, page 254]).

Finally, the results of papers by Shubin-Vakilian-Wolff [4], Wolff-Shubin [6], and Wolff [7] are perhaps most closely related to the results of this work. Paper [4] provides constructive estimates for the norm of an operator which is the average of a certain representation of $SL(2,\mathbb{R})$, where the average is computed over the distribution of the matrices. In tern, this result implies a constructive estimate for the rate of growth of products of matrices (9) with $\lambda = 1$, $a_n = 1$ and the distribution of $q_n$'s being non-trivial (not concentrated at one point). We note that the exponential growth of such products follows from Furstenberg's theorem. However, the constructive estimates established in [4] imply more than that. Namely, they imply, under natural conditions, the exponential growth of the product of independent non-identically distributed matrices. The situation with the proof of localization is similar: formally speaking, the proof in [4] is given for the case of i.i.d. potentials; in fact, their proof works also for non-identically distributed potentials (see comments in [4, page 943]).

It is yet to be established which results from [1] can be extended to the case when the $q_n$'s do not have a density function by using the estimates from [4].

In summary, the exponential growth of products of $m \times m$ matrices which are independent but not necessarily identically distributed can be deduced from the results obtained in [4], [6], and especially [7] (we shall comment on this statement later).

The non-stationary Markov-dependent sequences of matrices form a new class of matrices for which exponential growth of their products can be established. They include independent matrices as a particular case. In the case of independent matrices, our proofs are simpler than those in [7].

## 2. Statement of the main result

Our setting is as follows.

*The Markov chain.* Let $(X, \mathcal{B})$ be a measurable set (with $\mathcal{B}$ being the sigma-algebra of measurable subsets of the set $X$). Consider a Markov chain $\xi_n$, $n \geq 1$, with phase space $X$ and initial distribution $\mu_1$. For any $B \in \mathcal{B}$, set

$$k_n(x, B) = \mathbb{P}(\xi_{n+1} \in B \,|\, \xi_n = x).$$

We write $k_n(x, dy)$ for the corresponding transition kernel of the chain $\xi_n$.

Let $\mu_n$ be the distribution of $\xi_n$. As usual, for $n \geq 2$ and $B \in \mathcal{B}$ we have

$$\mu_n(B) = \mathbb{P}(\xi_n \in B) = \int_X \mu_{n-1}(dx) k_n(x, B).$$

We thus have a sequence of 'Markov related' measure spaces $(X, \mathcal{B}, \mu_n)$. Denote $H_n$ the Hilbert space of $\mu_n$-square integrable complex valued functions,

$$H_n = \{f : f : X \mapsto \mathbb{C}, \int_X |f(x)|^2 \mu_n(dx) < \infty\}$$

with the standard inner product: if $f,\ h \in H_n$ then

$$< f, h >_{H_n} = \int_X f(x)\bar{h}(x)\mu_n(dx).$$

Set

$$H_n^{(0)} = \{f \in H_n : \int_X f(x)\mu_n(dx) = 0\}.$$

The integral with respect $\mu_n$ will be denote $\mathbb{E}_n : \mathbb{E}_n(f) \equiv \int_X f(x)\mu_n(dx)$

Let $K_n : H_{n+1} \mapsto H_n$ be the operator defined by

$$(K_n f)(x) = \int_X k_n(x, dy)f(y).$$

Note that the operator $K_n$ 'computes' the conditional expectation of $f(\xi_{n+1})$ conditioned on $\xi_n = x$ and it is easy to see that if $f \in H_{n+1}$ then $K_n f \in H_n$.

Denote $K_n^0$ the restriction of $K_n$ to $H_{n+1}^0$. Note that if $\mathbb{E}_{n+1}(f) = 0$ then $\mathbb{E}_n(K_n f) = 0$, that is $K_n^0 : H_{n+1}^0 \mapsto H_n^0$.

*The matrices.* Let $g : X \mapsto SL(m, \mathbb{R})$ be a matrix-valued $\mathcal{B}$-measurable function on $X$. Define a sequence of random matrices $g_j$ by setting $g_j = g(\xi_j),\ j \geq 1$. Let $\nu_j$ be the distribution of $g_j$, that is for a Borel subset $\Gamma$, $\Gamma \subset SL(m, \mathbb{R})$, we set

$$\nu_j(\Gamma) = \mathbb{P}(g(\xi_j) \in \Gamma).$$

By $\mathrm{supp}(\nu_j) \subset SL(m, \mathbb{R})$ we denote the support of $\nu_j$.

For a distribution $\nu$ on $SL(m, \mathbb{R})$ we define a group $G_\nu$ as follows:

$$G_\nu = \text{ closed group generated by the set } \{g\bar{g}^{-1} : g,\ \bar{g} \in \mathrm{supp}(\nu)\}. \tag{10}$$

By $\mathcal{S}$ we denote the unit sphere in $\mathbb{R}^m$.

**Definition.** *For $g \in SL(m, \mathbb{R})$ and $u \in \mathcal{S}$ we define $g.u = gu/||gu||$. The induced action of $g$ the set of probability measures on $\mathcal{S}$ is defined by $g\kappa(B) = \kappa(g.^{-1}B)$, where $\kappa$ is a probability measure on $\mathcal{S}$ and $B$ is a Borel subset of $\mathcal{S}$. We say that a probability measure $\kappa$ on $\mathcal{S}$ is preserved by $g$ if $\kappa(B) = (g\kappa)(B)$ for any Borel $B$. A group $G$ preserves the measure $\kappa$ on $\mathcal{S}$ if every $g \in G$ preserves $\kappa$.*

We suppose that the Markov chain $\xi$ and the function $g$ satisfy the following assumptions:

I. For some $c$, $0 \leq c < 1$, for all $n \geq 1$

$$\|K_n^0\| \leq c.$$

II. There is a set $M$ of probability measures on $SL(m, \mathbb{R})$ which is compact with respect to weak convergence and such that:

(a) all $\nu_n$ belong to $M$,

(b) for any measure $\nu \in M$ the group $G_\nu$ does not preserve any probability measure on $\mathcal{S}$.

We are now in a position to state our main result:

**Theorem 2.** *Suppose that assumptions I, II are satisfied. Then there is a $\lambda > 0$ such that with probability 1*

$$\liminf_{n\to\infty} \frac{1}{n} \ln \|g_n \ldots g_1\| \geq \lambda. \tag{11}$$

**Remark.** With a slight abuse of notation, we use $\|\cdot\|$ to denote the norm of matrices, functions, and operators. This makes many formulae look less cumbersome while their meaning is always obvious from the context.

## 3. The idea of the proof explained in a simplified setting

Suppose that matrices $g_n,\ n \geq 1$ are independent, $\mu_n$ is the distribution of $g_n$, $\mu_n \in M$. In this setting, $X = \mathcal{S}$ and therefore $\mu_n = \nu_n$. Our probability space is $\left(\mathcal{S}^{\mathbb{N}}, \prod_{j=1}^{\infty} \mu_j\right)$, where $\prod_{j=1}^{\infty} \mu_j = \mu_1 \times \mu_2 \times ....$. The abbreviation a.s. means almost surely with respect to this product measure.

As before, we suppose that $M$ is a compact set of probability measures on $SL(m, \mathbb{R})$ and that for any $\mu \in M$ the group $G_\mu$ does not preserve any probability measure on the unit sphere $\mathcal{S}$.

In this simplified setting, we prove Theorem 1 in three steps.

*Step 1.* Note that in order to prove (11) it suffices to show that there is a $c > 0$ such that

$$\mathbb{E}(\|S_n\|^{-\frac{m}{2}}) \leq e^{-cn}. \tag{12}$$

Indeed, by the Markov inequality for any $\varepsilon > 0$

$$\mathbb{P}(\|S_n\| \leq e^{\varepsilon n}) = \mathbb{P}(\|S_n\|^{-\frac{m}{2}} \geq e^{-\frac{m}{2}\varepsilon n}) \leq e^{\varepsilon \frac{m}{2} n}\mathbb{E}(\|S_n\|^{-\frac{m}{2}}) \leq e^{(\varepsilon \frac{m}{2} - c)n}.$$

If $\varepsilon < 2c/m$ the the Borel-Cantelli lemma implies that the set $\{n : \|S_n\| \leq e^{\varepsilon n}\}$ is a.s. finite. This means that (11) holds a.s. for any $\gamma < 2c/m$.

*Step 2.* Let $L_2(\mathcal{S}, dx)$ be the Hilbert space of complex valued functions on $\mathcal{S}$ equipped with the Lebesgue measure $dx$ which is normalized to 1. The inner product of $f,\ h \in L_2(\mathcal{S}, dx)$ is given by

$$< f, h >= \int_{\mathcal{S}} f(x)\bar{h}(x)dx.$$

Let $\mathbb{V}$ be the set of unitary operators in $L_2(\mathcal{S})$. Consider a unitary 'representation' $\rho : SL(m, \mathbb{R}) \mapsto \mathbb{V}$: the unitary operator $\rho(g) \equiv V_g$ acts on $f \in L_2(\mathcal{S})$ as follows:

$$V_g f(x) = f(g.x)||gx||^{-\frac{m}{2}}. \tag{13}$$

It is easy to verify that

$$||f|| = ||V_g f|| \quad \text{and that} \quad V_{g_1 g_2} = V_{g_2} V_{g_1}. \tag{14}$$

For $\mu \in M$, put

$$W_\mu = \int_{SL(m,\mathbb{R})} V_g d\mu(g).$$

**Theorem 3.** *If no probability measure on $\mathcal{S}$ is preserved by $G_\mu$ then $||W_\mu|| < 1$.*

**Idea of the proof.** We shall show that if $||W_\mu|| = 1$, then there is a probability measure on $\mathcal{S}$ which is preserved by $G_\mu$.

The idea of the proof becomes particularly transparent if there is a function $f \in L_2(\mathcal{S}, dx)$ with $||f|| = 1$ and such that $||W_\mu f|| = 1$. So, let this be the case. Define $f_g = V_g f$ and $\varphi = W_\mu f = \int_{SL(m,\mathbb{R})} f_g d\mu(g)$. Then

$$1 = ||W_\mu f|| = \left|\left|\int_{SL(m,\mathbb{R})} f_g d\mu(g)\right|\right| \leq \int_{SL(m,\mathbb{R})} ||f_g|| d\mu(g) = 1$$

and hence

$$\left|\left|\int_{SL(m,\mathbb{R})} f_g d\mu(g)\right|\right| = 1.$$

Since the space $L_2(\mathcal{S}, dx)$ is uniformly convex, this equality takes place if and only if $f_g = \text{const}$ for $\mu$-almost all $g \in SL(m, \mathbb{R})$. For instance, if we put

$$\varphi(x) = W_\mu f(x) = \int_{SL(m,\mathbb{R})} f_g(x) d\mu(g) \tag{15}$$

then for $\mu$-almost all $g \in SL(m,\mathbb{R})$ and almost all $x \in \mathcal{S}$

$$\varphi(x) = f_g(x). \tag{16}$$

Equality (16) implies equality of measures with densities $|\varphi(x)|^2$ and $|f(g.x)|^2||gx||^{-m}$ respectively:

$$\int_{\mathcal{S}} \psi(x)|\varphi(x)|^2 dx = \int_{\mathcal{S}} \psi(x)|f(g.x)|^2||gx||^{-m} dx, \tag{17}$$

where $\psi$ is any continuous function on $\mathcal{S}$. But then $\varphi = f_g$ for $\mu$-almost all $g$ (where once again the equality means the that the corresponding measures are equal). Since for any probability measure $\kappa$ the measure $g\kappa$ is weakly continuous in $g$, we have that $f_{g_1} = f_{g_2}$ for any $g_1$ and $g_2$ from the support of $\mu$ and hence $f_{g_2 g_1^{-1}} = f$, which means that a measure with the density $|f|^2$ is preserved by any $g_2 g_1^{-1}$ with $g_1,\ g_2 \in \mathrm{supp}\mu$.

If $\|W_\mu f\| < 1$ for every $f \in L_2(\mathcal{S}, dx)$ but $\|W_\mu\| = 1$ then there is a sequence of functions $f_n \in L_2(\mathcal{S}, dx)$ with $\|f_n\| = 1$ and such that $\lim_{n\to\infty} \|W_\mu f_n\| = 1$. We shall view the functions $|f_n|^2$ as densities of measures $\kappa_n$ on $\mathcal{S}$, $d\kappa_n(x) = |f_n(x)|^2 dx$. Since $\mathcal{S}$ is a compact set, any such sequence has a weakly converging subsequence. So, we shall suppose from now on that $\lim_{n\to\infty} \kappa_n = \kappa$. The equalities which were used above would now hold only in the limit $n \to \infty$ and one concludes that $g_1\kappa = g_2\kappa$. This completes the proof of Theorem 3. $\Box$

*Step 3.* Since $\|S_n\| \geq \|S_n x\|$, $x \in \mathcal{S}$, (12) would follow from

$$\int_{\mathcal{S}} \mathbb{E}(\|S_n x\|^{-\frac{m}{2}}) dx \leq e^{-cn}. \tag{18}$$

Note next that

$$\|g_n...g_1 x\|^{-\frac{m}{2}} = (V_{g_n...g_1}\mathbf{1})(x) = (V_{g_1}...V_{g_n}\mathbf{1})(x),$$

where $\mathbf{1}$ is the function on $\mathcal{S}$ which takes value 1 at every $x \in \mathcal{S}$. Therefore

$$\int_{\mathcal{S}} \mathbb{E}(\|S_n x\|^{-\frac{m}{2}}) dx = \mathbb{E}\left(\int_{\mathcal{S}} \|g_n...g_1 x\|^{-\frac{m}{2}} dx\right)$$
$$= \mathbb{E}\left(\int_{\mathcal{S}} (V_{g_1}...V_{g_n}\mathbf{1})(x) dx\right) = \mathbb{E}\left(< V_{g_1}...V_{g_n}\mathbf{1}, \mathbf{1} >\right) =< \mathbb{E}\left(V_{g_1}...V_{g_n}\right)\mathbf{1}, \mathbf{1} > .$$

Since the operators $V_{g_1}, ..., V_{g_n}$ are independent we obtain

$$\mathbb{E}\left(V_{g_1}...V_{g_n}\right) = \mathbb{E}\left(V_{g_1}\right)...\mathbb{E}\left(V_{g_n}\right) = W_{\mu_1}...W_{\mu_n}$$

Finally,

$$\int_{\mathcal{S}} \mathbb{E}(\|S_n x\|^{-\frac{m}{2}}) dx =< W_{\mu_1}...W_{\mu_n}\mathbf{1}, \mathbf{1} >\leq \|W_{\mu_1}\|...\|W_{\mu_n}\| \leq e^{-cn}, \tag{19}$$

where $c = \inf_{\mu\in M}(-\ln \|W_\mu\|)$. $\Box$

## 4. Additional comments

1. Theorem 3 is the main ingredient of the above proof. We could, instead of proving it, use a more general result from [7] (see Theorem 1 in this paper). However, our goal is to prove the exponential growth and for that we have to control the single concrete mapping (13) from $SL(m,\mathbb{R})$ into the space $V$ which has properties (14) (and which is not quite a representation - though the difference is trivial). Our reliance on the general representation theory is almost non-existent and this, together with the simplicity of the above proof and the fact that it makes this work more self-contained justifies our approach.

2. In the context of products of matrices, operators $W_\mu$ were first explicitly defined in [5] where it was proved that the spectral radius of $W_\mu$ is less than 1. In the case of identically distributed independent $g_n$ this fact implies Theorem 2. In fact, [5] starts with a more complicated version of this operator which allows one to control products of stationary Markov dependent matrices and, once again, the positivity of the Lyapounov exponent follows from the fact that the corresponding spectral radius is less than 1.

The full proof of our Theorem 2 requires an approach which makes use of a simplified version of constructions introduced and is more geometric than in that in this paper.

3. For matrices $g_j$ of the form (9), important constructive estimates of $||W_\mu W_{\tilde{\mu}}||$ were obtained in [4]. They are seem to be optimal in some natural sense.

With a little more work, one could obtain constructive estimates also for $||W_\mu||$.

4. Furstenberg's theorem for the i.i.d. case follows from Theorem 2. If the identity matrix $I$ is in the support $\mathrm{supp}\mu$ then our $G_\mu$ and the $\bar{G}_\mu$ from Theorem 1 are the same group. If $I \notin \mathrm{supp}(\mu)$, then we can apply our theorem to the measure $\tilde{\mu} = \frac{1}{2}\mu + \frac{1}{2}\delta_I$. It is easy to see that the Lyapunov exponent for $\tilde{\mu}$ is positive if and only if the Lyapunov exponent for $\mu$ is positive. This observation completes the proof of Furstenberg's theorem.

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**Sums of Cantor sets and non-stationary Anderson-Bernoulli Model**

**by Anton Gorodetski (University of California, Irvine)**

Questions on the structure of Sums of Cantor sets appear naturally in many areas of dynamical systems, number theory, and spectral theory. One can use the known machinery to give an example of a non-stationary Anderson-Bernoulli potential such that the almost sure essential spectrum of the corresponding discrete Schrödinger operator $H : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ intersects an open interval at a Cantor set of zero measure. Construction is very explicit. Namely, choose any sequence $\{n_k\}_{k\in\mathbb{N}}$ of integers such that

$$n_k \to \infty \text{ and } n_{k+1} - n_k \to \infty \quad \text{as} \quad k \to \infty.$$

We define the random potential in the following way:

$$V(n) = \begin{cases} 0 \text{ or } 1 \text{ with probability } 1/2, & \text{if } n \notin \{n_k\}; \\ 0 \text{ or } 100 \text{ with probability } 1/2, & \text{if } n \in \{n_k\}. \end{cases}$$

**Theorem 1.** *Almost sure essential spectrum of the operator $H$ with the potential $\{V(n)\}$ defined above is a union of the interval $[-2,3]$ and a Cantor set contained in the interval $[98,102]$.*

To characterize the spectrum of an operator it will be convenient to use the following criterion:

**Proposition 1.** *Let $\{V(n)\}_{n\in\mathbb{Z}}$ be a bounded potential of the discrete Schrödinger operator $H$ acting on $\ell^2(\mathbb{Z})$ via*

$$[Hu](n) = u(n+1) + u(n-1) + V(n)u(n). \tag{20}$$

*Then we have the following:*

*1) Energy $E \in \mathbb{R}$ belongs to the spectrum of the operator $H$ if and only if there exists $K > 0$ such that for any $N \in \mathbb{N}$ there is $m \in \mathbb{Z}$ and a unit vector $\bar{u}$, $|\bar{u}| = 1$, such that $|T_{[m,m+i],E}\,\bar{u}| \le K$ for all $|i| \le N$, where $T_{[m,m+i],E}$ is the product of transfer matrices given by*

$$T_{[m,m+i],E} = \begin{cases} \Pi_{m+i-1,E}\ldots\Pi_{m,E}, & \text{if } i>0; \\ \mathrm{Id}, & \text{if } i=0; \\ \Pi^{-1}_{m+i,E}\ldots\Pi^{-1}_{m-1,E}, & \text{if } i<0, \end{cases}$$

*and* $\Pi_{n,E} = \begin{pmatrix} E - V(n) & -1 \\ 1 & 0 \end{pmatrix}$.

*2) Energy $E \in \mathbb{R}$ belongs to the essential spectrum of the operator $H$ if and only if there exists $K > 0$ such that for any $N \in \mathbb{N}$ there is a sequence $\{m_j\}_{j\in\mathbb{N}}, m_j \in \mathbb{Z}$, with $|m_j - m_{j'}| > 2N$ if $j \ne j'$, and unit vectors $\bar{u}_j$, $|\bar{u}_j| = 1$, such that $|T_{[m_j,m_j+i],E}\,\bar{u}_j| \le K$ for all $|i| \le N$ and all $j \in \mathbb{N}$.*

For each $\omega \in \{0,1\}^{\mathbb{Z}}$ consider an operator $H_\omega : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ given by the potential

$$V_\omega(n) = \begin{cases} 100, & \text{if } n = 0; \\ \omega_n, & \text{if } n \ne 0. \end{cases}$$

There are uncountably many operators of this form. Each of them has exactly one eigenvalue in the interval $[98, 102]$. Let us denote this eigenvalue by $E_\omega$.

Intersection of the almost sure essential spectrum of the operator $H$ given by the potential $\{V(n)\}$ with the interval $[98, 102]$ is exactly $\cup_{\omega\in\{0,1\}^{\mathbb{Z}}} E_\omega$.

Notice that if $A > 2$, then the matrix of the form $\begin{pmatrix} A & 1 \\ -1 & 0 \end{pmatrix}$ has two eigenvalues, namely $\frac{A+\sqrt{A^2-4}}{2} > 1$ and $\frac{A\sqrt{A^2-4}}{2} = \left(\frac{A+\sqrt{A^2-4}}{2}\right)^{-1} < 1$. Let us denote the proectivizations of the corresponding eigenvectors by $x_1(A)$ and $x_2(A)$.

For an operator $H_\omega$ each transfer matrix $\Pi_{n,E}$, $n \ne 0$, must be either $\begin{pmatrix} E & 1 \\ -1 & 0 \end{pmatrix}$, or $\begin{pmatrix} E-1 & 1 \\ -1 & 0 \end{pmatrix}$, and we are interested in the regime where $E \in [98, 102]$. Let us denote by $I_1(E)$ the interval on $S^1$ between the points $x_1(E)$ and $x_1(E-1)$, and by $I_2(E)$ the interval between the points $x_2(E)$ and $x_2(E-1)$. Denote by $f_{n,E}$ the proectivization of the map $\Pi_{n,E}$. Then if $n \ne 0$, we have $f_{n,E}(I_1(E)) \subset I_1(E)$, and $f^{-1}_{n,E}(I_2(E)) \subset I_2(E)$. Moreover, $f_{n,E}|_{I_1(E)}$ and $f^{-1}_{n,E}|_{I_2(E)}$ are contractions for each $n \ne 0$. For a given $\omega \in \{0,1\}^{\mathbb{Z}}$ there exists exactly one point $z_\omega(E) \in I_1(E)$ such that

$$z_\omega(E) = \cap_{n\in\mathbb{N}} f_{-n,E} \circ \ldots \circ f_{-1,E}(I_1(E)).$$

Notice that if the vector $\bar{w} \in \mathbb{R}^2$, $|\bar{w}| = 1$, correspond to the direction defined by $z_\omega(E)$, then

$$\left(\Pi_{-n,E}\ldots\Pi_{-1,E}\right)^{-1}(\bar{w}) \to 0 \ \text{ as } \ n \to \infty,$$

and for any vector $\bar{v} \nparallel \bar{w}$

$$\left|\left(\Pi_{-n,E}\ldots\Pi_{-1,E}\right)^{-1}(\bar{v})\right| \to \infty$$

exponentially fast as $n \to \infty$. The set $K(E) = \cup_{\omega\in\{0,1\}^{\mathbb{Z}}} z\omega(E)$ is a dynamically defined Cantor set inside of $I_1(E)$. Notice that $\left|f'_{n,E}|_{I_1(E)}\right| \sim \frac{1}{E^2}$, and in our regime $E \sim 100$. Hence Hausdorff dimension of $K(E)$ is small, $\dim_H K(E) = \dim_B K(E) \ll 1/2$.

Similarly, the set

$$C(E) = \cup_{\omega\in\{0,1\}^{\mathbb{Z}}} \left(\cap_{n\in\mathbb{N}} f^{-1}_{1,E} \circ \ldots \circ f^{-1}_{n,E}(I_2(E))\right)$$

is a dynamically defined Cantor set, and $\dim_H C(E) = \dim_B C(E) \ll 1/2$.

A given point $E \in [98, 102]$ is an eigenvalue of an operator $H_\omega$ for some $\omega \in \{0,1\}^{\mathbb{Z}}$ if $f_{0,E}(K(E)) \cap C(E) \ne \emptyset$. Now Proposition 1 follows from the following statement:

**Lemma 1.** *Let $K(E)$ and $C(E)$ be two families of dynamically defined Cantor sets on $\mathbb{R}^1$, $E \in [0, 1]$. Suppose that the following properties hold:*

1. *The Cantor set $K(E)$ is generated by two $C^1$-smooth (both in $x \in \mathbb{R}^1$ and $E \in [0,1]$) orientation preserving contractions $f_{1,E}, f_{2,E} : \mathbb{R}^1 \to \mathbb{R}^1$;*

2. *The Cantor set $C(E)$ is generated by two $C^1$-smooth (both in $x \in \mathbb{R}^1$ and $E \in [0,1]$) orientation preserving contractions $g_{1,E}, g_{2,E} : \mathbb{R}^1 \to \mathbb{R}^1$;*

3. $\max(K(0)) < \min(C(0))$ *and* $\min(K(1)) > \max(C(1))$*;*

4. *There exists $\delta > 0$ such that*
$$\frac{\partial f_{i,E}(x)}{\partial E} > \delta, \quad \frac{\partial g_{i,E}(x)}{\partial E} < -\delta$$
*for all $E \in [0,1]$, $i = 1,2$, and $x \in \mathbb{R}^1$;*

5. *We have*
$$\max_{E\in[0,1]} dim_B\, C(E) + \max_{E\in[0,1]} dim_B\, K(E) < 1.$$

*Then*
$$\{E \in [0,1] \mid C(E) \cap K(E) \neq \emptyset\}$$
*is a Cantor set of box counting dimension not greater than*
$$\left(\max_{E\in[0,1]} dim_B\, C(E) + \max_{E\in[0,1]} dim_B\, K(E)\right).$$

Notice that the question on the structure of the set of translations of one Cantor set that have non-empty intersections with another is closely related to the questions about the structure of the difference of two Cantor sets. Sums (and differences) of dynamically defined Cantor sets were heavily studied, e.g. see [1] and references therein. But in our case we needed to work with two Cantor sets that depend on a parameter, so the question about the set of parameters that correspond to a non-empty intersection of the sets cannot be directly reduced to considering the difference of the Cantor sets, and therefore we need Lemma 1 above.

The reported results were obtained as a joint project with Victor Kleptsyn.

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**Random Hamiltonians with Arbitrary Point Interactions: Positivity of the Lyapunov Exponent**

**by Mark Helman and Jacob Kesten (Rice University)**

We consider disordered Hamiltonians with arbitrary point interactions under minimal assumptions on the randomness. Such operators are realized via self-adjoint vertex conditions imposed on a discrete set of points in the real line. However, contrary to all previously considered Kronig–Penney type random models, we make no assumptions on the regularity of the probability distribution of the i.i.d. random variables in question, which is essential in the study of several random quantum graph models.

In our model, the disordered Hamiltonians are given by the Laplace operator subject to arbitrary random self-adjoint singular perturbations which are supported on a random discrete subset of the real line. Here, the underlying one-step transfer matrix takes a much more general form than in the previous studies. In this setting, we managed to prove the following dichotomy: Either every realization of the random operator has purely absolutely continuous spectrum or spectral and exponential dynamical localization hold. The core of such proof of Anderson Localization for those operators is our new result of the positivity of the Lyapunov exponent for all energies outside of a discrete set.

In particular, we verify the assumptions of Theorem 2.1 in [1], with the matrices being the one step-transfer matrices from the above model, which are given by $\mathcal{M}^E(\ell, B) := B \begin{bmatrix} \cos\sqrt{E}\ell & \frac{\sin\sqrt{E}\ell}{\sqrt{E}} \\ -\sqrt{E}\sin\sqrt{E}\ell & \cos\sqrt{E}\ell \end{bmatrix}$, where $B \in \mathrm{SL}_2(\mathbb{R})$ and $\ell \in \mathbb{R}_{>0}$. This verification boiled down to showing that the commutator of 2 such transfer matrices, $\mathcal{M}^E(\ell_1, B_1)$ and $\mathcal{M}^E(\ell_2, B_2)$, is a non-identically zero function of the energy, $E$, over $\mathbb{C}$, except for the trivial cases when $\ell_1 = \ell_2$ and $B_1 = \pm B_2$, or $B_i = \pm I_2$ for $i = 1, 2$. Then, [1, Theorem 2.1] gives that the Lyapunov Exponent of the model is positive away from a discrete set of energies $E \in \mathbb{R}$, thus allowing us to conclude that spectral and exponential dynamical localization holds for all but a discrete set of energies, on which the spectrum of the operator will be purely absolutely continuous.

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**Cantor Spectrum for CMV and Jacobi Matrices with Coefficients arising from Generalized Skew-Shifts**

**by Hyunkyu Jun (Rice University)**

Let $X$ be a compact metric space and let $T : X \to X$ be a strictly ergodic homeomorphism, which fibers over an almost periodic dynamical system (generalized skew-shifts). This means there exists an infinite compact abelian group $\mathbb{G}$ and an onto continuous map $h : X \to \mathbb{G}$ such that $h(T(x)) = h(x) + g$ for some $g \in \mathbb{G}$. We consider CMV matrices and Jacobi matrices whose Verblunsky coefficients and respectively, Jacobi coefficients are obtained by a continuous sampling map along an orbit of $T$.

Our interest is to investigate spectral properties of CMV and Jacobi matrices. Let $f \in C^0(X, \mathbb{D})$ where $\mathbb{D}$ is the unit disk in the complex plane. Define the bi-infinite Verblansky coefficients $\{\alpha_n\}_{n\in\mathbb{Z}}$ as $\alpha_n := f(T^n x)$ where $x \in X$. Let $\mathcal{C}_x$ be the associated bi-infinite CMV matrix. By minimality of $T$, there exists $\Sigma \subset \partial\mathbb{D}$ such that $\sigma(\mathcal{C}_x) = \Sigma$ for all $x \in X$. Moreover, in Damanik *et al* [2], the authors show

$$\partial\mathbb{D} \setminus \Sigma = \{z \in \partial\mathbb{D} : (T, \overline{A}_{f,z}) \text{ is uniformly hyperbolic}\}$$

where

$$\overline{A}_{f,z}(x) := \frac{1}{z^{-1/2}\sqrt{1 - |f(x)|^2}} \begin{bmatrix} z & -\bar{f}(x) \\ -f(x)z & 1 \end{bmatrix}.$$

One of our results states:

**Theorem 1.** *For a generic $f \in C^0(X, \mathbb{D})$, we have that $\partial\mathbb{D} \setminus \Sigma$ is dense; that is, the associated CMV operators have a Cantor spectrum.*

Here, by saying $f$ is generic, this means $f$ is an element of countable intersection of open dense subsets of $C^0(X, \mathbb{D})$.

For the Jacobi case, define the bi-infinite Jacobi coefficients $\{a_n\}_{n\in\mathbb{Z}}$ and $\{b_n\}_{n\in\mathbb{Z}}$ by $a_n = f_a(T^n(x))$ and $b_n = f_b(T^n x)$, respectively. Let $\mathcal{J}_x$ be the associated Jacobi matrix. By minimality of $T$, there exists $\Sigma' \subset \mathbb{R}$ such that $\sigma(\mathcal{J}_x) = \Sigma'$ for all $x \in X$. Moreover, in Marx [3], the author shows

$$\mathbb{R} \setminus \Sigma' = \{E \in \mathbb{R} | (T, A_{E,f_a,f_b}) \text{ is uniformly hyperbolic}\}$$

where

$$A_{E,f_a,f_b}(x) = \frac{1}{f_a(x)} \begin{bmatrix} E - f_b(x) & -1 \\ f_a(x)^2 & 0 \end{bmatrix}$$

The other major result states:

**Theorem 2.** *Let $f_a \in C^0(X, \mathbb{R})$ with $f_a(x) > 0$ for all $x \in X$. For generic $f_b \in C^0(X, \mathbb{R})$, we have that $\mathbb{R} \setminus \Sigma'$ is dense; that is, the associated Jacobi matrices have Cantor spectrum.*

Here, by saying $f_b$ is generic, this means $f_b$ is an element of a countable intersection of open dense subsets of $C^0(X, \mathbb{R})$.

The proofs heavily builds upon the results in Avila *et al* [1]. Moreover, the proof of Theorem 2 is a direct application of results in [1]. In Avila *et al* [1], the authors prove that an $SL(2, \mathbb{R})$-cocycle with a certain property can be perturbed so that it is uniformly hyperbolic. This implies that if a cocycle associated to a CMV matrix, $(T, \overline{A}_{f,z})$, is not uniformly hyperbolic, it can be perturbed so that it is a uniformly hyperbolic $SL(2, \mathbb{R})$-cocycle. Our proof converts this perturbed $SL(2, \mathbb{R})$-cocylce to one associated with a CMV matrix while the uniformly hyperbolicity is preserved.

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**Furstenberg theorem: now with a parameter!**

**by Victor Kleptsyn (CNRS, Institut de Recherche Mathématique de Rennes)**

Let $(\Omega, \mu)$ be a probability space, $J \subset \mathbb{R}$ be a compact interval of parameters, and $F : \Omega \times J \to SL(2, \mathbb{R})$ be a bounded measurable (and continuous in second argument) map that to any $\omega \in \Omega$ puts in correspondence a matrix $F_a(\omega)$ that depends continuously on the parameter $a \in J$. One of the main application of our results is given by products of transfer matrices for 1D Anderson Model, where the role of the parameter is played by the value of energy $E$. For a given sequence $\bar{\omega} \in \Omega^{\mathbb{N}}, \bar{\omega} = \omega_1 \omega_2 \dots$ denote

$$T_{n,a,\bar{\omega}} = F_a(\omega_n) F_a(\omega_{n-1}) \dots F_a(\omega_1).$$

Furstenberg-Kesten Theorem [1] implies that for each value of the parameter $a \in J$ there is a subset $\Omega_a \subseteq \Omega^{\mathbb{N}}$ with $\mu^{\mathbb{N}}(\Omega_a) = 1$ such that for any $\bar{\omega} \in \Omega_a$ the limit

$$\lambda_F(a) := \lim_{n \to \infty} \frac{1}{n} \log \|T_{n,a,\bar{\omega}}\|$$

exists.

*Is it possible to choose $\Omega_a$ uniformly in the parameter? In other words, is it true that $\mu^{\mathbb{N}}$-almost surely the limit above exists for all values of the parameter $a \in J$?* It turns out that the answer to these questions is drastically different depending of presence or absence of uniform hyperbolicity.

**Definition** A collection of $SL(2, \mathbb{R})$ (or $SL(k, \mathbb{R})$) matrices $\{M_\alpha\}_{\alpha \in \mathcal{A}}$ is called *uniformly hyperbolic* if there exists a constant $\eta > 1$ such that for any finite sequence of matrices $M_{\alpha_1}, M_{\alpha_2}, \dots, M_{\alpha_n}$ we have $\|M_{\alpha_1} M_{\alpha_2} \dots M_{\alpha_n}\| > \eta^n$.

There is a number of equivalent ways to describe uniform hyperbolicity of $SL(2, \mathbb{R})$ (or $SL(k, \mathbb{R})$) cocycles, such as an invariant splitting into stable and unstable directions, or the absence of a Sacker-Sell solution. In particular, existence of invariant one-dimensional stable and unstable directions for uniformly hyperbolic $SL(2, \mathbb{R})$ cocycles combined with Birkhoff Ergodic Theorem immediately implies the following statement:

**Proposition** *In the setting above, assume that for each $a \in J$ the collection of matrices $\{F_a(\omega)\}_{\omega\in\Omega}$ is uniformly hyperbolic. Then, for $\mu^{\mathbb{N}}$-a.e. $\bar{\omega} \in \Omega^{\mathbb{N}}$ the limit*

$$\lim_{n\to\infty} \frac{1}{n} \log \|T_{n,a,\bar{\omega}}\| = \lambda_F(a) > 0$$

*exists for all $a \in J$.*

**Remark** *In the case of $SL(k, \mathbb{R})$, $k > 2$, even uniform hyperbolicity does not guarantee the convergence uniformly in parameter, or even pointwise convergence for all parameters. More restrictive assumptions (e.g. positivity of all entries of the matrices) are needed.*

The case of positive Lyapunov exponent in absence of uniform hyperbolicity is usually referred to as *non-uniformly hyperbolic case.*

From now on, we will proceed under the following standing assumptions:

(A1) **(Furstenberg condition)** Denote by $\mu_a$ the measure $\mu_a = (F_a)_*(\mu)$. We assume that for each $a \in J$ the measure $\mu_a$ on $SL(2, \mathbb{R})$ satisfies the (individual) Furstenberg non-degeneracy condition, that is, its support is not contained in any compact subgroup of $SL(2, \mathbb{R})$, and there is no supp $\mu_a$-invariant finite union of proper subspaces of $\mathbb{R}^2$.

(A2) **($C^1$-boundedness)** The maps $F_a(\omega)$ are $C^1$-smooth in the parameter $a \in J$, with uniformly bounded $C^1$-norm, i.e. there exists $M > 0$ such that for all $\omega \in \Omega$ and all $a \in J$

$$\|F_a(\omega)\|, \left\| \frac{d}{da} F_a(\omega) \right\| \leq M.$$

(A3) **(Non-uniform hyperbolicity)** For each $a \in J$ the collection of matrices $\{F_a(\omega)\}_{\omega\in\Omega}$ is *not* uniformly hyperbolic.

(A4) **(Monotonicity)** There exists $\delta > 0$ such that

$$\frac{d}{da} \arg(F_a(\omega)\bar{v}) > \delta > 0$$

for all $a \in J, \omega \in \Omega, \bar{v} \in \mathbb{R}^2 \backslash \{0\}$. In other words, as we increase the parameter, the image of any given vector $\bar{v}$ spins in the positive direction with a speed that is bounded from below.

Our main result is the following theorem, describing the behaviour of the random parameter-dependent products of $SL(2, \mathbb{R})$ matrices:

**Theorem 1** (Parametric version of Furstenberg Theorem)**.** *Under the assumptions $(A1) - (A4)$ above, for $\mu^{\mathbb{N}}$-almost every $\bar{\omega} \in \Omega^{\mathbb{N}}$ the following holds:*

- ***(Regular upper limit)*** *For every $a \in J$ we have*

$$\limsup_{n\to\infty} \frac{1}{n} \log \|T_{n,a,\bar{\omega}}\| = \lambda_F(a) > 0.$$

- ***($G_\delta$-vanishing)*** *The set*

$$S_0(\bar{\omega}) := \left\{ a \in J \mid \liminf_{n\to\infty} \frac{1}{n} \log \|T_{n,a,\bar{\omega}}\| = 0 \right\}$$

*is a (random) dense $G_\delta$-subset of the interval $J$.*

- ***(Hausdorff dimension)*** *The (random) set of parameters with exceptional behaviour,*

$$S_e(\bar{\omega}) := \left\{ a \in J \mid \liminf_{n\to\infty} \frac{1}{n} \log \|T_{n,a,\bar{\omega}}\| < \lambda_F(a) \right\},$$

*has zero Hausdorff dimension:*

$$\dim_H S_e(\bar{\omega}) = 0.$$

Notice that in this case existence of a dense subset of energies in the spectrum for which the limit that defines the Lyapunov exponent does not exist was shown in [2, Theorem 6.2].

Other related results as well as the complete proof of Theorem 1 can be found in [3].

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**Non-stationary versions of Anderson Localization and Furstenberg Theorem on random matrix products**

**by Victor Kleptsyn (CNRS, Institut de Recherche Mathématique de Rennes)**

The asymptotic behavior of sums of i.i.d. random variables is very well studied in the classical probability theory. Analogous questions on random products of matrix-valued i.i.d. random variables were initially formulated in the simplest case of $2 \times 2$ matrices with positive entries by Bellman. Later these questions attracted lots of attention due to the results by Furstenberg and Kesten [1] who showed that exponential rate of growth of the norms of the random products (aka Lyapunov exponent) is well defined almost surely, and Furstenberg [2, 3], where it was shown that under some non-degeneracy conditions Lyapunov exponent must be positive.

The most famous and classical result is the following Furstenberg Theorem:

**Theorem 1.** *Let $\{X_k, k \geq 1\}$ be independent and identically distributed random variables, taking values in $SL(d,\mathbb{R})$, the $d \times d$ matrices with determinant one, let $G_X$ be the smallest closed subgroup of $SL(d,\mathbb{R})$ containing the support of the distribution of $X_1$, and assume that*

$$E[\log \|X_1\|] < \infty.$$

*Also, assume that $G_X$ is not compact, and there exists no $G_X$-invariant finite union of proper subspaces of $\mathbb{R}^d$. Then there exists a positive constant $\lambda_F$ such that with probability one*

$$\lim_{n\to\infty} \frac{1}{n} \log \|X_n \dots X_2 X_1\| = \lambda_F > 0.$$

In the first part of this paper we generalize Furstenberg Theorem to the case when the random variables $\{X_k, k \geq 1\}$ do not have to be identically distributed. Here is our setting:

Let $\{\nu_\alpha\}_{\alpha\in K}$, $\operatorname{supp}\nu_\alpha \subset SL(d,\mathbb{R})$, be a collection of compactly supported probability measures, indexed by a parameter $\alpha$ from a compact metric space $K$. We assume that dependence of $\nu_\alpha$ on $\alpha$ is continuous (in weak-* topology). As a partial case, one can consider a finite collection $\{\nu_i\}_{i=1,\dots,k}$ of probability measures on $SL(d,\mathbb{R})$.

For any $A \in SL(d,\mathbb{R})$ we will denote by $f_A : \mathbb{RP}^{d-1} \to \mathbb{RP}^{d-1}$ the induced projective transformation. We make the following

**Standing Assumption:** *We assume that for any $\alpha \in K$ there are no Borel probability measures $\mu_1$, $\mu_2$ on $\mathbb{RP}^{d-1}$ such that $(f_A)_*\mu_1 = \mu_2$ for $\nu_\alpha$-almost every $A \in SL(d,\mathbb{R})$.*

Let us fix some sequence $\{\alpha_i\}_{i\in\mathbb{N}}$, $\alpha_i \in K$, and let $A_i \in SL(d,\mathbb{R})$ be chosen randomly with respect to distribution $\nu_{\alpha_i}$. Set $T_n = A_n A_{n-1} \dots A_1$, and denote

$$L_n = \mathbb{E} \log \|T_n\|, \tag{21}$$

where the expectation is taken over the distribution $\nu_{\alpha_1} \times \nu_{\alpha_2} \times \dots \times \nu_{\alpha_n}$.

**Theorem 2.** *Under the Standing Assumption above, for any fixed sequence $\{\alpha_i\}_{i\in\mathbb{N}} \in K^{\mathbb{N}}$ we have*

$$\liminf_{n\to\infty} \frac{1}{n} L_n > 0.$$

A statement similar to Theorems 2 was previously announced by I. Goldsheid, see the extended abstract of his talk above.

In the case of $SL(2,\mathbb{R})$ matrices one can actually say much more.

**Theorem 3.** *In the case $d = 2$ (i.e. in the case of random non-stationary products of $SL(2,\mathbb{R})$ matrices) almost surely additionally to the statement of Theorem 2 the following hold:*

*1) $\lim_{n\to\infty} \frac{1}{n} (\log \|T_n\| - L_n) = 0$;*

*2) There exists a unit vector $\bar{v} \in \mathbb{R}^2$ such that $|T_n \bar{v}| \to 0$ as $n \to \infty$. Moreover,*

$$\lim_{n\to\infty} \frac{1}{n} (\log |T_n \bar{v}| + L_n) = 0$$

The statement of Theorem 2 and the first part of Theorem 3 in the case of products of i.i.d. random matrices correspond to the classical Furstenberg Theorem.

We will prove Theorem 3 via Large Deviations Estimates Theorem, that is also of independent interest:

**Theorem 4.** *In the case $d = 2$, for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all sufficiently large $n \in \mathbb{N}$ we have*

$$\mathbb{P}\left\{|\log \|T_n\| - L_n| > \varepsilon n\right\} < e^{-\delta n},$$

*where $\mathbb{P} = \nu_{\alpha_1} \times \nu_{\alpha_2} \times \ldots \times \nu_{\alpha_n}$. Moreover, the same estimate holds for the lengths of random images of any given initial unit vector $v_0$:*

$$\forall v_0 \in \mathbb{R}^2,\ |v_0| = 1 \quad \mathbb{P}\left\{|\log \|T_n v_0\| - L_n| > \varepsilon n\right\} < e^{-\delta n}.$$

The reported results were obtained in collaboration with A. Gorodetski.

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**Phase transition of capacity for uniform $G_\delta$ sets**

**by Fernando Quintino (University of California, Irvine)**

In a joint work with Victor Kleptsyn, we consider a family of dense $G_\delta$ subsets of $[0,1]$, defined as intersections of unions of small uniformly distributed intervals, and study their capacity. That is, given a (sufficiently fast) decreasing sequence $r_n \to 0$, for every $n$ we consider a union of $n$ equally spaced intervals of length $r_n$:

$$V_n := \bigcup_{j=1}^{n} J_{k,n}, \tag{22}$$

where $J_{k,n}$ is an open interval of length $r_n$ centered at $c_{k,n} = \frac{k+(1/2)}{n}$:

$$J_{k,n} := (c_{j,n} - \frac{r_n}{2}, c_{j,n} + \frac{r_n}{2}), \quad c_{j,n} = \frac{2j+1}{2n}, \quad j = 0, 1, \ldots, n-1. \tag{23}$$

Then we define the *uniform $G_\delta$-set $S$*, corresponding to the sequence $r_n$, by

$$S := \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} V_n; \tag{24}$$

it is immediate to see that $S$ is indeed a $G_\delta$-subset of $[0,1]$. Such an example is interesting for us for two reasons. First, in [2] we found that by considering different decrease speed for the lengths $r_n$, we observed a sharp phase transition: while for a fast decrease this set is of zero capacity, for a slower one it turns out to be of *full* capacity (that is, equal to the capacity of $[0,1]$ itself). Given a compactly supported measure $\mu$ on $\mathbb{C}$, one defines its (Coulomb) *energy* as a double integral:

$$I(\mu) := \iint -\log|z-w|\, d\mu(z)d\mu(w). \tag{25}$$

The logarithmic capacity of a bounded subset $X \subset \mathbb{C}$ is then defined by minimizing this energy:

**Definition.** Let $\mathcal{P}(X)$ be the space of probability measures, supported on a (bounded) set $X \subset \mathbb{C}$. The *logarithmic capacity* of this set is

$$\mathrm{Cap}(X) := \exp(-\inf\{I(\mu) \mid \mu \in \mathcal{P}(X)\}).$$

**Theorem 1** (Phase transition, V. Kleptsyn, F. Quintino)**.** *For $r_n = e^{-n^\alpha}$,*

1. *if $\alpha > 2$, then $\mathrm{Cap}(S) = 0$,*
2. *if $\alpha < 2$, then $\mathrm{Cap}(S) = \mathrm{Cap}([0,1])$.*

Second, such a $G_\delta$ set can be considered as a toy model for the set of exceptional energies in the parametric version of the Furstenberg theorem on random matrix products[1]. A more general $G_\delta$-sets can be constructed in the following way. Consider the set

$$\widetilde{S} = \bigcap_{m} \bigcup_{k \geq m} I_k,$$

where $I_k$ are intervals of length $r'_k$. In this general setting a similar pattern seems to be at play.

**Theorem 2** (V. Kleptsyn, F. Quintino)**.** *If the series $\sum_n \frac{1}{|\log r'_n|}$ converges, then the set $\widetilde{S}$ is of zero capacity.*

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**Exponential Dynamical Localization for Random Word Models**

**by Nishant Rangamani (University of California, Irvine)**

We give a new proof of spectral localization for the one-dimensional Schrodinger operators whose potentials arise by randomly concatenating words from an underlying set. We then show that once one has the existence of a complete orthonormal basis of eigenfunctions (with probability one), the same estimates used to prove it naturally lead to a proof of exponential dynamical localization in expectation (EDL) on any compact interval not containing a finite set of critical energies.

The random word models we consider are defined on $l^2(\mathbb{Z})$ and are given by

$$H_\omega \psi(n) = \psi(n+1) + \psi(n-1) + V_\omega(n)\psi(n).$$

The potential $V$ is a family of random variables defined on a probability space $\Omega$. To construct the potential $V$ above, we consider words (vectors in $\mathbb{R}^n$ with $1 \leq n \leq m$), $\ldots \omega_{-1}, \omega_0, \omega_1, \ldots$, so that $V_\omega(0)$ corresponds to the $k$th entry in $\omega_0$. A precise construction of the probability space $\Omega$ and the random variables $V_\omega(n)$ is carried out in [1]. In particular, the authors show that there is a finite set $D$ so that the Lyapunov exponent is positive outside of $D$.

Motivated by recent proofs of spectral and dynamical localization given for the Anderson model in [2] and a proof of exponential dynamical localization in expectation given in [3], we demonstrate the application of these techniques in the random word case to obtain the two theorems listed at the end of this section.

We note that the proofs given in [2] and [3] use positivity and large deviations of the Lyapunov exponent to replace parts of the multi-scale analysis. The major improvement in this regard (aside from a shortening of the length and complexity of localization proofs in one-dimension) is that the complement of the event where the Green's function decays exponentially can be shown to have exponentially rather than sub-exponentially small probability. These estimates were implicit in the proofs of spectral and dynamical localization given in [2] and were made explicit in [3]. The authors in [3] then used these estimates to prove EDL for the Anderson model and we extend these techniques to the random word case.

There are several issues one encounters when adapting the techniques developed for the Anderson model in [2] and [3] to the random word case. Firstly, in the Anderson setting, a uniform large deviation estimate is immediately available using a theorem in [4]. Since random word models exhibit local correlations, there are additional steps that need to be taken in order to obtain suitable analogs of large deviation estimates used in [2] and [3]. Secondly, random word models may have a finite set of energies where the Lyapunov exponent vanishes and this phenomena demands some care in obtaining estimates on the Green's functions analogous to those in [2,3].

The aforementioned results are:

**Theorem [5]** The spectrum of $H_\omega$ is almost surely pure point with exponentially decaying eigenfunctions.

**Theorem [5]** *There is a finite $D \subset \mathbb{R}$ such that if $I$ is a compact interval and $D \cap I = \emptyset$, then there are $C > 0$ and $\alpha > 0$ such that* $\mathbb{E}\left[\sup_{t\in\mathbb{R}} \left|\langle \delta_p, P_I(H_\omega)e^{-itH_\omega}\delta_q\rangle\right|\right] \leq Ce^{-\alpha|p-q|}$ *for any $p, q \in \mathbb{Z}$.*

Here $P_I$ denotes the spectral projection onto the interval $I$.

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**Unique continuation and localization on the planar lattice**

**by Charles Smart (University of Chicago)**

Recall that the Anderson–Bernoulli model is a random linear operator on $\ell^2(\mathbb{Z}^d)$ given by

$$H = -\Delta + \beta V,$$

where $\Delta$ is the graph Laplacian, $\beta > 0$ is the noise strength, and $V : \mathbb{Z}^d \to \{0, 1\}$ is a Bernoulli potential. We discuss the following two results.

**Theorem 1** (Ding–Smart)**.** *If $d = 2$, then $H$ almost surely has pure-point spectrum in* $[0, \varepsilon]$.

**Theorem 2** (Li–Zhang)**.** *If $d = 3$, then $H$ almost surely has pure-point spectrum in* $[0, \varepsilon]$.

These results advance the state of the art by establishing localization for singular noise in dimensions larger than one. Following the program of Bourgain–Kenig, the key ingredients of these theorems are the following unique continuation results.

**Theorem 3** (Ding–Smart)**.** *The following holds for all $\alpha > 1 > \varepsilon > 0$ and sufficiently large $L > 0$. If $d = 2$, $|\bar{\lambda}| < \alpha$, and $Q = [-L, L]^2 \cap \mathbb{Z}^2$, then*

$$\mathbb{P}[\mathcal{E}] \geq 1 - e^{-L^{1/4-\varepsilon}}$$

*where $\mathcal{E}$ is the event that*

$$H\psi = \lambda\psi \quad \text{in } Q \quad \text{and} \quad |\lambda - \bar{\lambda}| \leq e^{-L^{1/2+\varepsilon}}$$

*implies*

$$\#\{x \in Q : |\psi(x)| \geq e^{-L^{1+\varepsilon}} |\psi(0)|\} \geq L^{3/2-\varepsilon}.$$

**Theorem 4** (Li–Zhang)**.** *There is a $p > 0$ such that, for all $\alpha > 1 > \varepsilon > 0$, the following holds for sufficiently large $L > 0$. If $d = 3$, $|\Delta\psi| \leq \alpha|\psi|$ holds in $Q = [-L, L]^3 \cap \mathbb{Z}^3$, then*

$$\#\{x \in Q : |\psi(x)| \geq e^{-L^{1+\varepsilon}} |\psi(0)|\} \geq L^{3/2+p}.$$

Both of these unique continuation theorems use ideas from recent work of Buhovsky–Logunov–Malinnikova–Sodin.

## Anderson localization for radial trees
### by Selim Sukhtaiev (Rice University)

We establish spectral and dynamical localization for several Anderson–Bernoulli models on metric and discrete radial trees. The localization results are obtained on compact intervals contained in the complement of discrete sets of exceptional energies. All results are proved under the minimal hypothesis on the type of disorder: the random variables generating the trees assume at least two distinct values. This level of generality, in particular, allows us to treat radial trees with disordered geometry as well as Schrödinger operators with Bernoulli-type singular potentials. Our methods are based on an interplay between graph-theoretical properties of radial trees and spectral analysis of the associated random differential and difference operators on the half-line.

More specifically, let us denote the common distribution of single sites random variables by $\mu$ and the continuum Kirchhoff–Laplacian by $H$. Assume that supp$\mu$ is a bounded set containing at least two elements. Then there exists a discrete set of exceptional energies $D$ such that:

(i) The operator $H_\omega$ exhibits Anderson localization at all energies outside of $D$. That is, almost surely, $H_\omega$ has pure point spectrum and any eigenfunction of $H_\omega$ corresponding to an energy $E \in R \setminus D$ enjoys an exponential decay estimate of the form

$$|f(x)| \leq \frac{Ce^{-\lambda|x|}}{\sqrt{w_o(|x|)}}$$

with $C > 0$ and $\lambda > 0$, where $w_o(|x|)$ denotes the number of vertices in the generation of $x$, i.e., $w_o(|x|) = \#\{y \in \mathcal{V} : gen(y) = gen(x)\}$.

(ii) For every compact interval $I \in R \setminus D$ and every $p > 0$, there exists a set $\Omega^* \subset \Omega$ with $\mu(\Omega^*) = 1$ such that for every $\omega \in \Omega^*$ and every compact set $K \subset \Gamma_{b_\omega,\ell_\omega}$ one has

$$\sup_{t>0} \left\| |X|^p \chi_I(H_\omega) e^{-itH_\omega} \chi_K \right\|_{L^2(\Gamma_{b_\omega,\ell_\omega})} < \infty,$$

where $\chi_I(H_\omega)$ is the spectral projection corresponding to $I$, and $|X|^p$ denotes the operator of multiplication by the radial function $f(x) := |x|^p$, $x \in \Gamma_{b_\omega,\ell_\omega}$, where $|x|$ denotes the distance from $x$ to the root $o$.

This result together with its discrete version is established in [1].

# References

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**Diophantine properties of matrices**

**by Yuki Takahashi (Tohoku University)**

Let $d \geq 2$, and let $\mathcal{A} = \{A_i\}_{i\in\Lambda}$ be a finite collection of $GL_d(\mathbb{R})$ matrices. Write $A_{\mathbf{i}} = A_{i_1} \cdots A_{i_n}$ for $\mathbf{i} = i_1 \cdots i_n$. We say that the set $\mathcal{A}$ is *Diophantine* if there exists a constant $c > 0$ such that for every $n \in \mathbb{N}$, we have

$$\mathbf{i}, \mathbf{j} \in \Lambda^n, A_{\mathbf{i}} \neq A_{\mathbf{j}} \implies \|A_{\mathbf{i}} - A_{\mathbf{j}}\| > c^n.$$

The set $\mathcal{A}$ is strongly Diophantine if there exists $c > 0$ such that for all $n \in \mathbb{N}$,

$$\mathbf{i}, \mathbf{j} \in \Lambda^n, \mathbf{i} \neq \mathbf{j} \implies \|A_{\mathbf{i}} - A_{\mathbf{j}}\| > c^n.$$

Clearly, $\mathcal{A}$ is strongly Diophantine if and only if it is Diophantine and generates a free semigroup. For any collection of linearly independent vectors $v_1, \cdots, v_d$ in $\mathbb{R}^d$ consider the cone

$$\Sigma = \Sigma_{v_1,\cdots,v_d} = \{x_1 v_1 + \cdots x_d v_d : x_1, \cdots, x_d \geq 0\}.$$

If a matrix $A \in GL_d(\mathbb{R})$ satisfies

$$A(\Sigma \setminus \{0\}) \subset \Sigma^\circ,$$

we say that $\Sigma$ is *strictly invariant* for $A$. Given a cone $\Sigma = \Sigma_{v_1,\cdots,v_d}$, denote by $\mathcal{X}_{\Sigma,m}$ the set of all $GL_d(\mathbb{R})$ $m$-tuples of matrices for which $\Sigma$ is strictly invariant. We consider $\mathcal{X}_{\Sigma,m}$ as an open subset of $\mathbb{R}^{d^2 m}$.

Let $\Sigma = \Sigma_{v_1,\cdots,v_d}$ be a cone in $\mathbb{R}^d$ and $m \geq 2$. Together with B. Solomyak, the author proved the following in [1]: For a.e. $\mathcal{A} \in \mathcal{X}_{\Sigma,m}$, the $m$-tuple $\mathcal{A}$ is strongly Diophantine. In particular, a.e. $m$-tuple of positive $GL_d(\mathbb{R})$ matrices is strongly Diophantine.

# References

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**Localization for the one-dimensional Anderson model via positive Lyapunov exponents and a Large Deviation Theorem**

**by Tom VandenBoom (Yale University)**

It is well-known that the one-dimensional Anderson model is almost-surely Anderson localized – that is to say, the operator

$$H_\omega = \Delta + \omega$$

almost surely (in $\omega = (\omega_n)_{n\in\mathbf{Z}}$) has pure point spectrum with exponentially decaying eigenfunctions provided the terms $\omega_n$ are sampled i.i.d. and randomly. However, proofs of this fact in one dimension have until recently (cf. [2, 7, 8]) utilized the sophisticated multi-dimensional machinery of Multi-Scale Analysis (MSA) to handle highly singular probability distributions [3, 9]. In this talk, we demonstrate a simplified proof of Lyapunov behavior for all generalized eigenfunctions of an almost-sure $H_\omega$ [2].

To state our result precisely, we require some notation: let $\tilde{\mu}$ be a probability measure with compact real support $A \subset \mathbf{R}$, and denote by $(\Omega, \mu) = (A^{\mathbf{Z}}, \tilde{\mu}^{\mathbf{Z}})$ the associated probability space on the full shift over $A$. Letting $\omega \in \Omega$ and $E \in \mathbf{C}$, define the Schrödinger transfer matrix $M_n(E,\omega)$ as the unique $SL(2,\mathbf{R})$ matrix such that $u \in \mathbf{C}^{\mathbf{Z}}$ solves $H_\omega u = Eu$ if and only if $[u_n \ u_{n-1}]^\top = M_n(E,\omega)[u_0 \ u_{-1}]^\top$. Then the Lyapunov exponent $L(E) = \lim_{n\to\infty} \frac{1}{n}\int_\Omega \log \|M_n(E,\omega)\| d\mu(\omega)$ exists and is positive for all $E \in \mathbf{R}$ by Furstenberg's theorem. Our result is that, for $\mu$-almost every $\omega \in \Omega$, for any generalized eigenvalue $E(\omega)$ of $H_\omega$, the norms of the transfer matrices $M_n(E(\omega),\omega)$ grow at precisely the Lyapunov rate:

$$\lim_{n\to\infty} \frac{1}{n} \log \|M_n(E(\omega),\omega)\| = L(E(\omega)).$$

Our proof can also be extended to prove dynamical localization via the standard SULE techniques [4].

Historically, proofs of Anderson localization in complete generality involved three key ingredients: first, an initial length-scale estimate coming from positivity of the Lyapunov exponent; second, a Wegner estimate on the density of states; and finally, the MSA machinery. Our proof has similar first ingredients; namely, we achieve an initial length-scale estimate using positive Lyapunov exponents from Furstenberg's Theorem [5, 6], and then prove a Large Deviation Theorem (LDT) (which serves the same role as the Wegner estimate). Where our proof differs significantly from previous proofs is in the final step, whereby we eliminate long-range "double resonances": distant pairs of intervals supporting simultaneous localization. We eliminate such pairs using our LDT and independence; this step is the focus of this talk. With the double resonances eliminated, one can apply an Avalanche Principle argument to the transfer matrices to conclude exponential decay of the generalized eigenfunctions. This strategy, initially observed by Bourgain and Schlag in application to strongly mixing potentials [1], is very general and applicable to a variety of one-dimensional models.

# References

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**A short proof of Anderson localization for the 1-d Anderson model**
**by Xiaowen Zhu (University of California, Irvine)**

The proof of Anderson localization for 1D Anderson model with arbitrary (e.g. Bernoulli) disorder, originally given by Carmona-Klein-Martinelli in 1987, is based on the Furstenberg theorem and multi-scale analysis. This topic has received a renewed attention lately, with two recent new proofs, exploiting the one-dimensional nature of the model. At the same time, in the 90s it was realized that for one-dimensional models with positive Lyapunov exponents some parts of multi-scale analysis can be replaced by considerations involving subharmonicity and large deviation estimates for the corresponding cocycle, leading to nonperturbative proofs for 1D quasiperiodic models. Here we present a proof along these lines, for the Anderson model. It is a joint work with S. Jitomirskaya. Our entire proof of spectral localization fits in three pages and we expect to present almost complete detail during the talk. I will also present my proof of Anderson localization for the OPUC (Orthogonal polynomial on the unit circle) with any nontrivial i.i.d random Verblunsky coefficients, in the spirit of the work above. This proof was commissioned by Barry Simon for the new edition of his OPUC book.