The non-local Mean-Field equation on an interval.

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Joint work with: A. Hyder, Y. Sire and L. Martinazzi.

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"Nonlinear Geometric PDE's"
BIRS–BANFF

Aim

Study the existence of solution for the Mean field equation

$$(-\Delta)^{\frac{1}{2}} u = \rho \frac{e^u}{\int_I e^u dx}, \text{ on } I = (-1, 1)$$

with zero Dirichlet boundary condition

$$u \equiv 0 \text{ in } \mathbb{R} \setminus I.$$

Motivation

Dimension $n = 2$ (Probability theory and Statistical mechanics)

- Canonical ensemble $\rightarrow$ Probability of the system X being in state x

$$P(X = x) = \frac{1}{Z(\beta)} e^{-\beta E(x)}$$

$Z =$ norm. ctt, $\beta =$ free parameter, $E =$energy at the point

- Gibbs measure $\rightarrow$ generaliz. the canonical ensemble to ∞ systems
- N-vortex system in a bounded domain $\Lambda \rightarrow$ the 1 particle distrib. fcn converges to a superposition of solutions to

$$-\Delta\varphi = \frac{e^{-\beta\varphi}}{\int_\Lambda e^{-\beta\varphi}}, \;\; \varphi|_{\partial\Lambda} = 0$$

as $N \rightarrow \infty$.

Dimension $n = 2$

Local analogue

$$-\Delta u = \rho \frac{e^u}{\int_\Omega e^u dx} \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \quad \Omega \Subset \mathbb{R}^2 \qquad \text{(2D)}$$

$\rho > 0$, $\Omega \equiv$ smoothly bounded.

- Caglioti, Lions, Marchioro & Pulvirenti '92: Variational methods
- Kiessling '93: Probabilistic methods

$\Rightarrow \exists$ solution to (2D) $\forall \rho \in (0, 8\pi)$.

Pohozaev identity $\Rightarrow 8\pi$ is sharp (Ω star-shaped $\Rightarrow \nexists\ \forall \rho \geq 8\pi$)

(Not star-shaped: Ding-Jost-Li-Wang, Struwe-Tarantello, Malchiodi)

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$\Omega \equiv$ smoothly bounded.

$\Rightarrow \exists$ solution of (2D) $\forall \rho \in (0, 8\pi)$.

Imp tool: Studying blowing-up behaviour of (u_k) sol. to (2D) $\rho = \rho_k$.

(Brezis-Merle or Li-Shafrir–Liouville eq.)

Different dimension??

Generalizations of the problem

- $n = 4 \Rightarrow$ Wei '96, Robert-Wei '08
- $n = 2m$, $m \geq 1 \Rightarrow$ Martinazzi-Petrache'10

$\hookrightarrow$ n even

* Weston (Blowing up, 1-point), Esposito-Grossi-Pistoia (Multi-peak), Ding-Jost-Li-Wang (not simply connected)

- n odd $\Rightarrow$ Non-local problem :O

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Main work–Non local 1d

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Study the existence of solution for the Mean field equation with zero Dirichlet boundary condition

$$\begin{cases} (-\Delta)^{\frac{1}{2}} u = \rho \dfrac{e^u}{\int_I e^u dx}, & \text{on } I = (-1,1) \\ u \equiv 0, & \text{in } \mathbb{R} \setminus I. \end{cases} \qquad (1d\text{-MF})$$

Preliminaries

Fractional Laplacian

$\mathcal{S} \equiv$ the Schwartz space of rapidly decreasing smooth functions

- $u \in L_{\frac{1}{2}}(\mathbb{R}) \Rightarrow \langle (-\Delta)^{\frac{1}{2}} u, \varphi \rangle := \int_{\mathbb{R}} u (-\Delta)^{\frac{1}{2}} \varphi dx, \quad \varphi \in \mathcal{S}$

  where $L_{\frac{1}{2}}(\mathbb{R}) = \left\{ u \in L^1_{\text{loc}}(\mathbb{R}) : \int_{\mathbb{R}} \frac{|u(x)|}{1+|x|^2} dx < \infty \right\}$.

- $$\varphi \in \mathcal{S} \Rightarrow \begin{cases} (-\Delta)^{\frac{1}{2}} \varphi := \mathcal{F}^{-1}(|\cdot|\hat{\varphi}). \\ |(-\Delta)^{\frac{1}{2}} \varphi(x)| \le C(1+|x|^2)^{-1} \end{cases}$$

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  where $\hat{\varphi}(\xi) \equiv \mathcal{F}\varphi(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} \varphi(x)\, dx$.

- $u \in C^{0,\alpha}(I) \Rightarrow (-\Delta)^{\frac{1}{2}} u(x) := \frac{1}{\pi} P.V. \int_{\mathbb{R}} \frac{u(x)-u(y)}{(x-y)^2} dy, \quad x \in I.$

$$\hookrightarrow C^{0,\frac{1}{2}}_{\infty}(\mathbb{R}) := C^{0,\frac{1}{2}}(\mathbb{R}) \cap C^{\infty}(I)$$

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Green's representation

$$G_x(y) := \begin{cases} \frac{1}{\pi}\left(\underbrace{log(\sqrt{(1-|x|^2)(1-|y|^2)}+1-xy)}_{:=H(x,y)} - \log|x-y|\right) & x, y \in I \\ 0 & x, \in I, y \in \mathbb{R}\setminus I \end{cases} \tag{G}$$

$$\hookrightarrow (-\Delta)^{\frac{1}{2}} G_x = \delta_x \quad \text{for } x \in I.$$

$$u \in C_\infty^{0,\frac{1}{2}}(\mathbb{R}) \text{ sol } (1d\text{-MF}) \Leftrightarrow u(x) = \rho \int_I G_x(y) \frac{e^{u(y)}}{\int_I e^{u(\xi)} d\xi} dy.$$

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More important facts

$$\begin{cases} (-\Delta)^{\frac{1}{2}} u = \rho \dfrac{e^u}{\int_I e^u dx}, & \text{on } I = (-1,1) \\ u \equiv 0, & \text{in } \mathbb{R} \setminus I. \end{cases} \tag{1d-MF}$$

- $u \in C_\infty^{0,\frac{1}{2}}(\mathbb{R})$ sol (1d-MF) $\Leftrightarrow u(x) = \rho \int_I G_x(y) \frac{e^{u(y)}}{\int_I e^{u(\xi)} d\xi} dy.$
- $u \in C_\infty^{0,\frac{1}{2}}(\mathbb{R})$ sol (1d-MF) $\Rightarrow u > 0$ in I
  (Proof: Green's representation)
- $u \in C_\infty^{0,\frac{1}{2}}(\mathbb{R})$ sol (1d-MF) $\Rightarrow u$ is even & decreasing
  $(u(x) = u(-x)$ & $u(x) \geq u(y)$, $0 \leq x \leq y.)$
  (Proof: Non-local moving plane)

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Pohozaev-type inequality

$$\hat{u} \in C_{\infty}^{0,\frac{1}{2}}(\mathbb{R}), \qquad \hat{u}(x) = \int_I G_x(y) e^{\hat{u}(y)} dy + c, \;\; c \in \mathbb{R}$$

$$\Rightarrow \rho := \int_I e^{\hat{u}(y)}\, dy < 2\pi, \qquad \text{(Poho)}$$

Proof:

- Definition of $G_x(y)$
- $\hat{u} \in C^1(I) \Rightarrow \hat{u}'(x)$
- $I_1 := \int_I x \cdot \hat{u}'(x) e^{\hat{u}(x)} dx = 2e^{\hat{u}(1)} - \int_I e^{\hat{u}} dx = 2e^{\hat{u}(1)} - \rho.$
- $\hat{u} = \hat{u}_{\varphi,\varepsilon}$ with φ_ε cut-off
- Splitting: $I_1 = I_2 + I_3 + I_4 \xrightarrow{\varepsilon \to 0} -\frac{\rho^2}{2\pi}.$

$$\hookrightarrow 2e^{\hat{u}(1)} - \rho \; (> -\rho) \xrightarrow{\varepsilon \to 0} -\frac{\rho^2}{2\pi} \quad \Leftrightarrow \quad \rho < 2\pi$$

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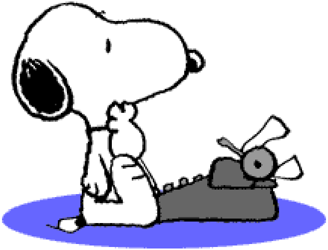

$n = 2 \Rightarrow$ Blow-up analysis of sequence of solutions

Main results

Blow-up analysis

Theorem 1 (DIT-Hyder-Sire-Martinazzi '19)

$\{u_k\} \subset C_\infty^{0,\frac{1}{2}}(\mathbb{R})$ solutions to (1d-MF), $\rho = \rho_k > 0$. $\Rightarrow$ (up to subseq.)

$$\begin{cases} (-\Delta)^{\frac{1}{2}} u_k = \rho_k \dfrac{e^{u_k}}{\int_I e^{u_k}\,dx}, & \text{on } I = (-1,1) \\ u_k \equiv 0, & \text{in } \mathbb{R} \setminus I. \end{cases} \tag{1d-MF}$$

1. either $\|u_k\|_{C^{0,\frac{1}{2}}(\mathbb{R}) \cap C^{\ell}{}_{\mathrm{loc}}(I)} \leq C$, $\forall \ell \in \mathbb{N}$
2. or $\lim_{k\to\infty} u_k(0) = \infty$ & as $k \to \infty$

$$\begin{cases} \rho_k \uparrow 2\pi \quad . \\ u_k(y) \to 2\pi G_0(y) \quad \text{in } C^{0,\sigma}_{\mathrm{loc}}(\mathbb{R}\setminus\{0\}),\ \forall\, 0 < \sigma < \dfrac{1}{2} \end{cases}$$

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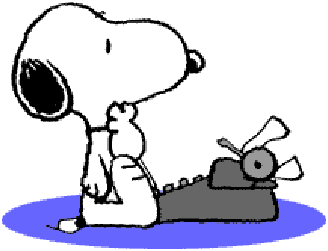

$\exists$ sol of (1d-MF)??

Existence-Non existence result

Theorem 2 (DIT-Hyder-Sire-Martinazzi '19)

$\exists$ non-trivial non-negative $u = u_\rho$ sol to (1d-MF) $\Leftrightarrow$ $\rho \in (0, 2\pi)$.
Moreover,

$$u_\rho(0) \to \infty \quad \text{as } \rho \uparrow 2\pi.$$

Remark: $u_\rho(0) \to \infty$ as $\rho \uparrow 2\pi$. $\Rightarrow$ 2 in Theorem 1 occurs

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Proof:

- $\rho \geq 2\pi. \Rightarrow \nexists$ (Pohozaev (Poho))
- $\rho < 2\pi. \Rightarrow \exists$ (Schauder Fix-point theorem)

  X Banach space, $T : X \to X$ compact mapping

$$\text{If } \exists \begin{cases} C;\ \|u\|_X \leq C, & \forall\, u \in X \\ t \in [0,1] :\ u = tTu \end{cases} \Rightarrow T \text{ has a fix point.} \tag{1}$$

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    $\hookrightarrow T_\rho : X \to X,\ T_\rho(u)(x) := \rho \int_I G_x(y) \frac{e^{u(y)}}{\int_I e^{u(\xi)} d\xi} dy$
    $\Rightarrow \forall\ \rho > 0\ T_\rho$ is compact.

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  - $X := C^0([-1,1])$, $\to$ $T_\rho(u)(x) := \rho \int_I G_x(y) \frac{e^{u(y)}}{\int_I e^{u(\xi)} d\xi} dy$ compact.
  - Fix ρ, take $t_k \in (0,1]$, $u_k \in X$: $u_k = t_k T_\rho(u_k)$
    $\Rightarrow$ u_k sol (1d-MF) with $\rho \sim \rho t_k < 2\pi$
    $\hookrightarrow$Th.1 $\Rightarrow$ $\exists$ $C > 0$, $\|u_k\|_X \leq C$

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  - $t_k \in (0,1]$, $u_k \in X$: $u_k = t_k T_\rho(u_k)$ $\|u_k\|_X \leq C$
  - Schauder Fix-point theorem
    - ⋆ $\forall\ \rho \in (0, 2\pi)\ \exists\ u_\rho$ fixed point T_ρ ,
    - ⋆ (G) $\Rightarrow$ u_ρ is sol to $(1d\text{-MF})$

Existence-Non existence result

Theorem 2 (DIT-Hyder-Sire-Martinazzi '19)

$\exists$ non-trivial non-negative $u = u_\rho$ sol to (1d-MF) $\Leftrightarrow \rho \in (0, 2\pi)$.
Moreover,

$$u_\rho(0) \to \infty \quad \text{as } \rho \uparrow 2\pi.$$

Proof:

- $\rho \geq 2\pi. \Rightarrow \nexists$ (Pohozaev (Poho))
- $\rho < 2\pi. \Rightarrow \exists$
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  $\Rightarrow$ (Poho) + $u_\rho(0) = \max_I u_\rho \Rightarrow u_\rho(0) \to \infty \quad$ as $\rho \uparrow 2\pi$. □

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Blow-up analysis

Theorem 1 (DIT-Hyder-Sire-Martinazzi '19)

$\{u_k\} \subset C_\infty^{0,\frac{1}{2}}(\mathbb{R})$ solutions to (1d-MF), $\rho = \rho_k > 0$. $\Rightarrow$ (up to subseq.)

$$\begin{cases} (-\Delta)^{\frac{1}{2}} u_k = \rho_k \dfrac{e^{u_k}}{\int_I e^{u_k}\,dx}, & \text{on } I = (-1,1) \\ u_k \equiv 0, & \text{in } \mathbb{R} \setminus I. \end{cases} \qquad (1d\text{-MF})$$

1. either $\|u_k\|_{C^{0,\frac{1}{2}}(\mathbb{R}) \cap C^{\ell}{}_{loc}(I)} \le C$, $\forall \ell \in \mathbb{N}$
2. or $\lim_{k\to\infty} u_k(0) = \infty$ & as $k \to \infty$

$$\begin{cases} \rho_k \uparrow 2\pi & . \\ u_k(y) \to 2\pi G_0(y) & \text{in } C^{0,\sigma}_{loc}(\mathbb{R}\setminus\{0\}),\ \forall\, 0 < \sigma < \dfrac{1}{2} \end{cases}$$

Main ideas proof Blow-up analysis

Idea $\to$ $\hat{u}_k := u_k - \alpha_k \quad \alpha_k := \log\left(\frac{\int_I e^{u_k}dx}{\rho_k}\right), \quad \rho_k := \int_I e^{\hat{u}_k}dx$

$$(G) \Rightarrow \begin{cases} u_k(x) = \rho_k \int_I G_x(y) \frac{e^{u_k(y)}}{\int_I e^{u_k(\xi)}d\xi} dy = \int_I G_x(y) e^{\hat{u}_k(y)} dy, \\ \hat{u}_k(x) = \int_I G_x(y) e^{\hat{u}_k(y)} dy - \alpha_k \end{cases} \qquad (\hat{G})$$

- $\hat{u}_k(0) \leq C \Rightarrow \|u_k\|_{C^{0,\alpha}([-1,1])} \leq C$ & $\|u_k\|_{C^{\ell}_{loc}(-1,1)} \leq C$ $\alpha \in [0, \frac{1}{2}]$, $\ell \geq 0$
  $\Rightarrow$ 1 in Th 1 ✓
- $\hat{u}_k(0) \to \infty \Rightarrow$ 2 in Th 1 ??

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Main ideas proof Blow-up analysis

$$\hat{u}_k := u_k - \alpha_k, \quad \alpha_k := \log\left(\frac{\int_I e^{u_k} dx}{\rho_k}\right), \quad \rho_k := \int_I e^{\hat{u}_k} dx$$

- Assuming $\hat{u}_k(0) \to \infty$ & $r_k := 2e^{-\hat{u}_k(0)} \to 0$.
  - i) $r_k u_k(0) \to 0$.
  - ii) $\eta_k(x) := \hat{u}_k(r_k x) + \log(r_k) \to \eta_0(x) := \log \frac{2}{1+x^2}$ in $C^\infty_{\text{loc}}(\mathbb{R})$.
    $\left(\int_{\mathbb{R}} \eta_0(-\Delta)^{\frac{1}{2}}\varphi dx = \int_{\mathbb{R}} e^{\eta_0}\varphi dx \quad \text{for every } \varphi \in \mathcal{S}(\mathbb{R})\right)$
  - iii) $\lim_{R\to\infty} \lim_{k\to\infty} \int_{-Rr_k}^{Rr_k} e^{\hat{u}_k} dx = 2\pi$.
  - iv) $\alpha_k \to \infty$ ($\Rightarrow\Leftarrow$ with $\rho < 2\pi$) .
  - v) $\hat{u}_k \to -\infty$ in $C^0_{loc}(\bar{I} \setminus \{0\})$ $(\hat{u}_k(\varepsilon) + \alpha_k \leq C(\varepsilon)) \Rightarrow u_k(\varepsilon) \to -\infty)$.

* $u_k \to 2\pi G_0$ in $C^{0,\sigma}_{\text{loc}}(\bar{I} \setminus \{0\})$. $\forall \sigma \in (0, \frac{1}{2})$

Thanks for your attention!! :)

Save tomorrow evening! Hiking at 16:15