Dimers and circle patterns

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Joint works with:

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Marianna Russkikh (MIT)

Dimers, Ising Model and their Interactions
Banff, November 20 2019

Dimers and circle patterns

Part 1

Dimers and circle patterns

Part 1

Part 2

Dimers and circle patterns

Part 1   Part 3   Part 2

Dimers and circle patterns

Part 1 Part 3 Part 2

Applications — Part 4

Dimers and circle patterns

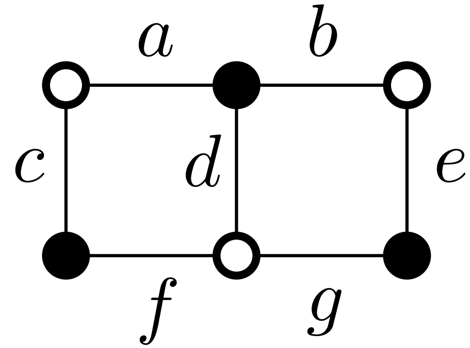
Part 1 Part 3 Part 2

Applications — Part 4

Benoît Laslier and Dmitry Chelkak later today:
Perfect t-embeddings and convergence to GFF

1 The dimer model

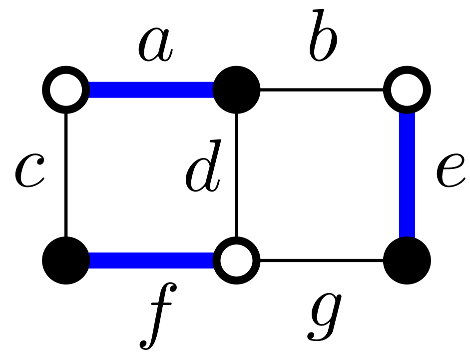
- *Planar bipartite* graphs with positive edge weights.



- *Dimer covering*: subset of edges such that each vertex is incident to exactly one edge.

- *Boltzmann measure*: draw a dimer covering at random with probability proportional to its weight.

- *Planar bipartite* graphs with positive edge weights.



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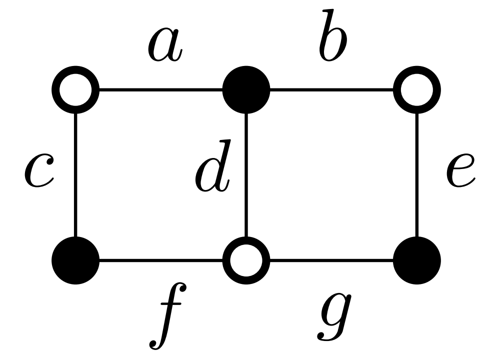
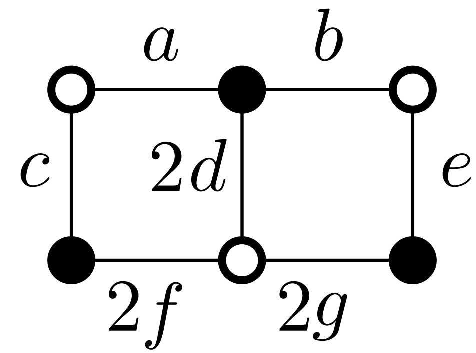


- Multiplying by $\lambda > 0$ the weight of every edge incident to a given vertex (*gauge transformation*) does not change the probability measure.

- Alternating products of edge weights around faces are coordinates on the space of edge weights modulo gauge.

- *Planar bipartite* graphs with positive edge weights.



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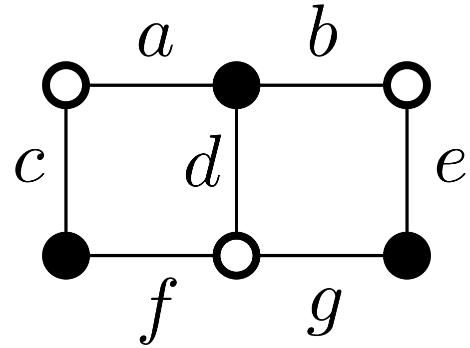

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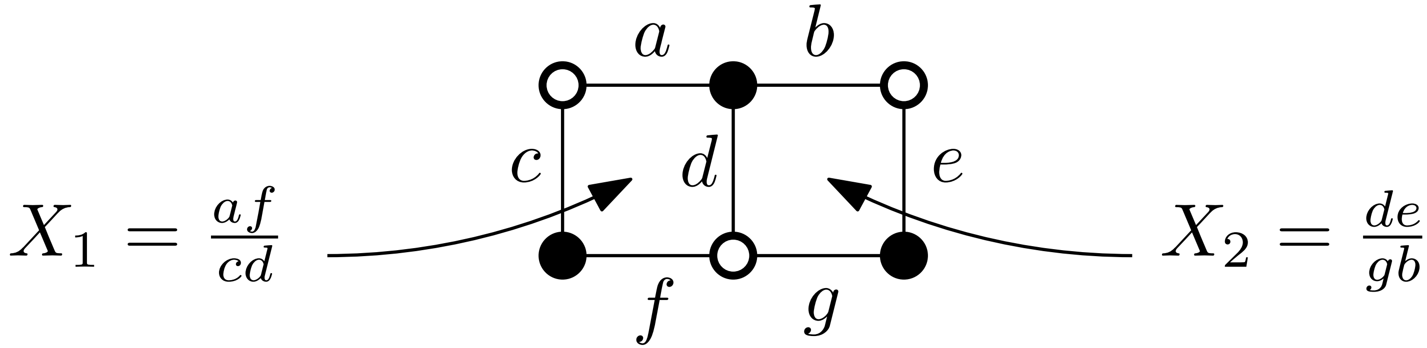

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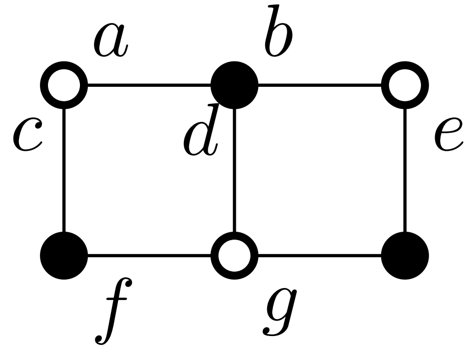
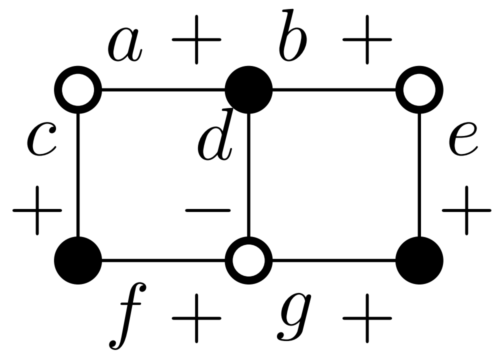
The Kasteleyn matrix K



- *Kasteleyn signs*: assign a sign to each edge such that the number of minus signs around a face of degree 2 mod 4 (resp. 0 mod 4) is even (resp. odd).

- K: weighted signed adjacency matrix with rows (resp. columns) indexed by white (resp. black) vertices.

The Kasteleyn matrix K



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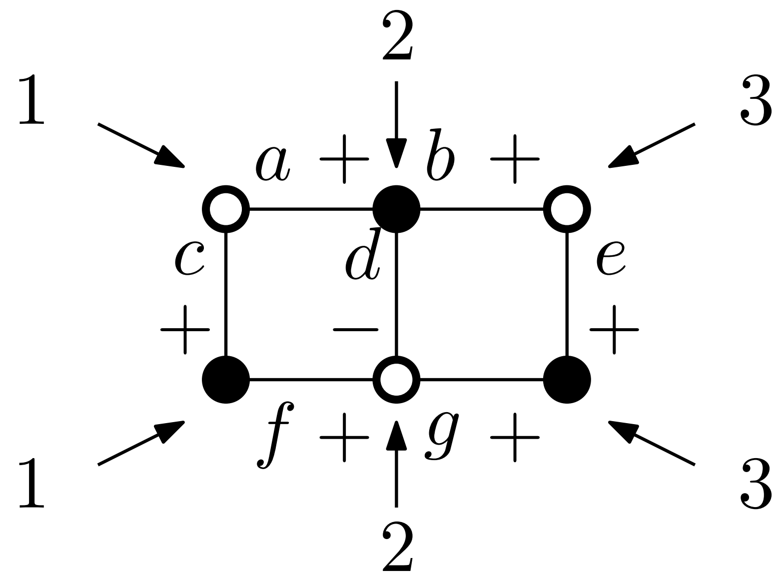

$$\begin{array}{c|ccc} & \bullet_1 & \bullet_2 & \bullet_3 \\ \hline \circ_1 & c & a & 0 \\ \circ_2 & f & -d & g \\ \circ_3 & 0 & b & e \end{array}$$

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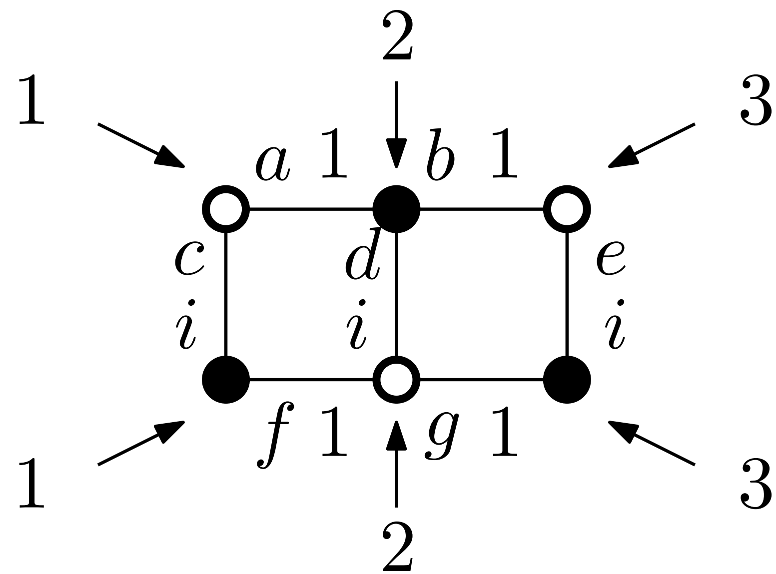


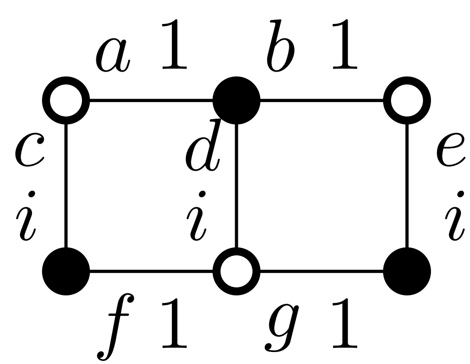
$$\begin{array}{c} \\ \circ_1 \\ \circ_2 \\ \circ_3 \end{array} \begin{array}{c} \bullet_1 \quad \bullet_2 \quad \bullet_3 \\ \begin{pmatrix} ic & a & 0 \\ f & id & g \\ 0 & b & ie \end{pmatrix} \end{array}$$

- *Complex Kasteleyn signs*: assign a unit complex number to each edge such that the alternating product of these numbers around a face of degree 2 mod 4 (resp. 0 mod 4) is 1 (resp. -1).

- K: weighted signed adjacency matrix with rows (resp. columns) indexed by white (resp. black) vertices.

- The partition function (sum of the weights of all dimer coverings) is $|\det K|$. The dimer correlations are given by minors of K^{-1} (Kasteleyn, Temperley-Fisher).

- Merge the complex Kasteleyn signs with the positive edge weights to get complex edge weights (entries of K).



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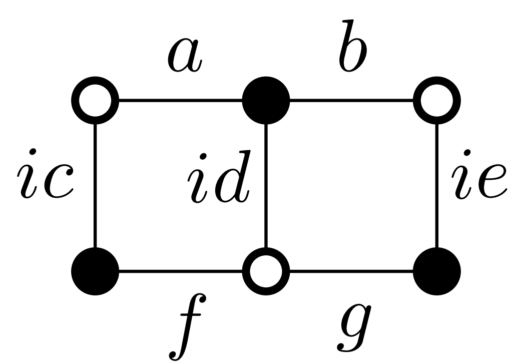

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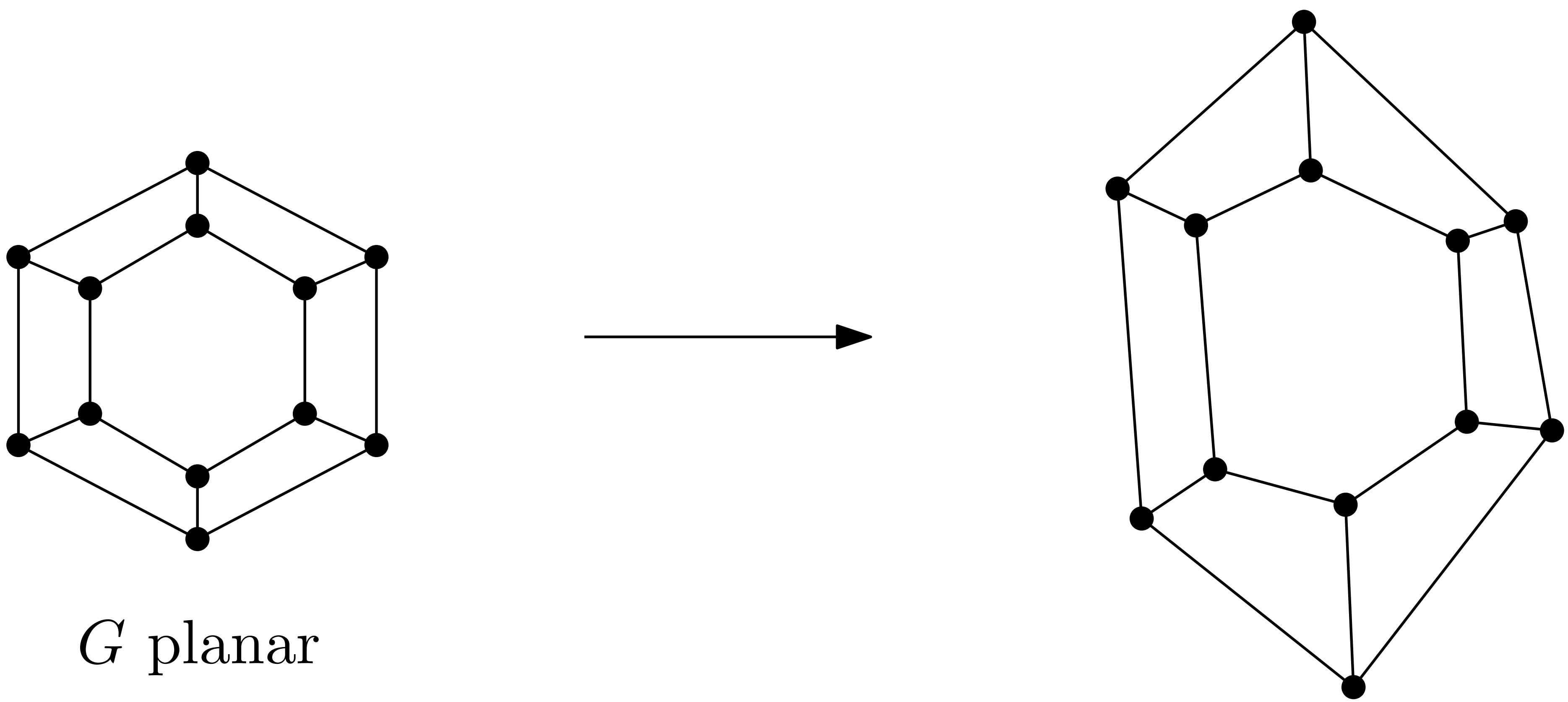
- The alternating product of complex edge weights is real positive (resp. real negative) around a face of degree 2 mod 4 (resp. 0 mod 4).

2 Circle patterns and circle centers

- *Circle pattern for G*: map from the vertex set of G to $\mathbb{R}^2$ sending all the vertices around any bounded face to concyclic points.



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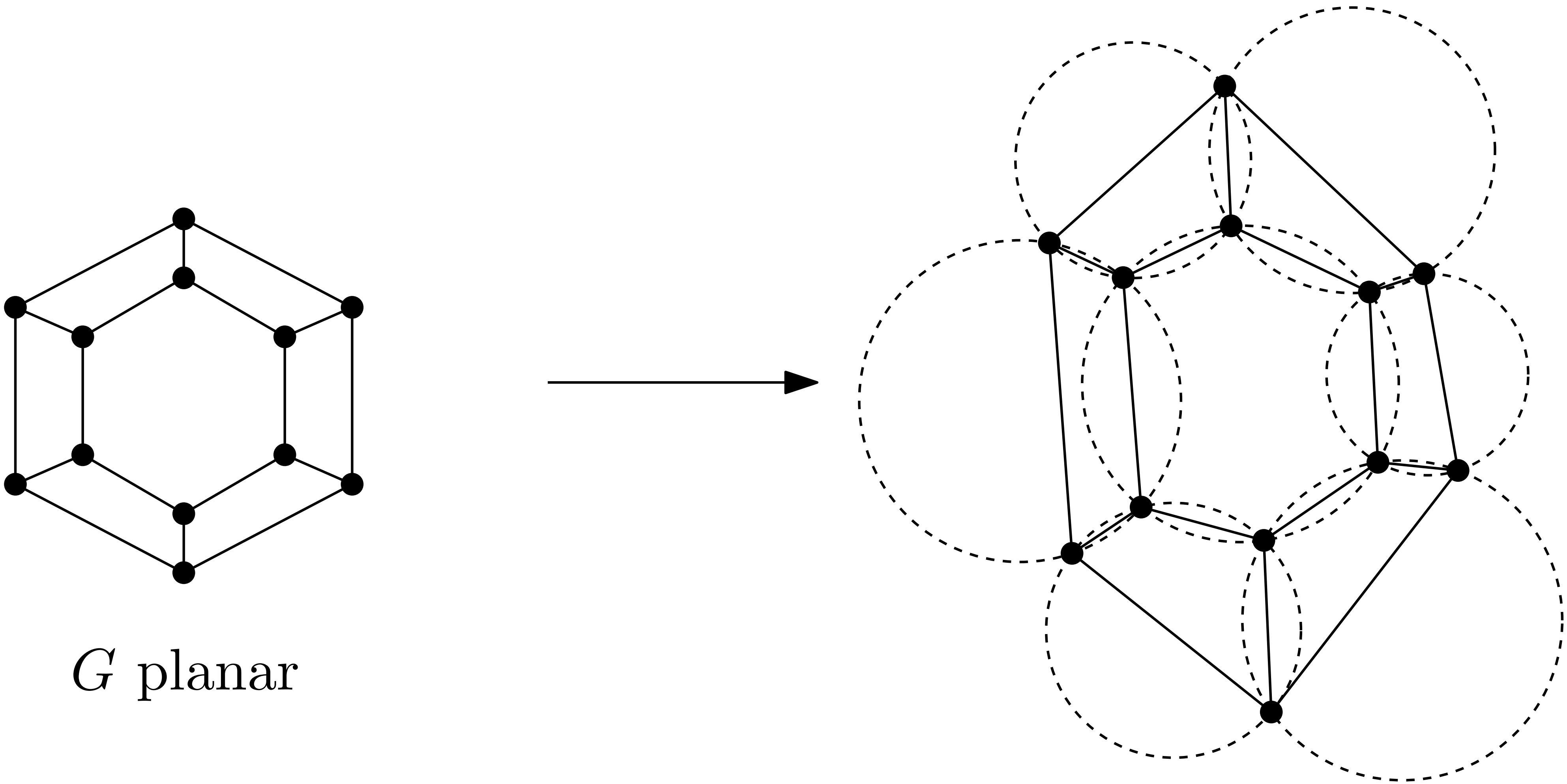

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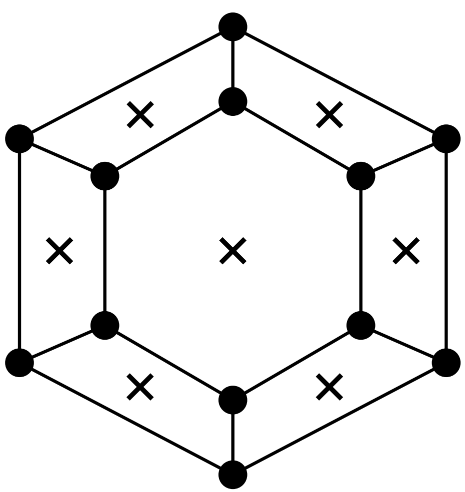
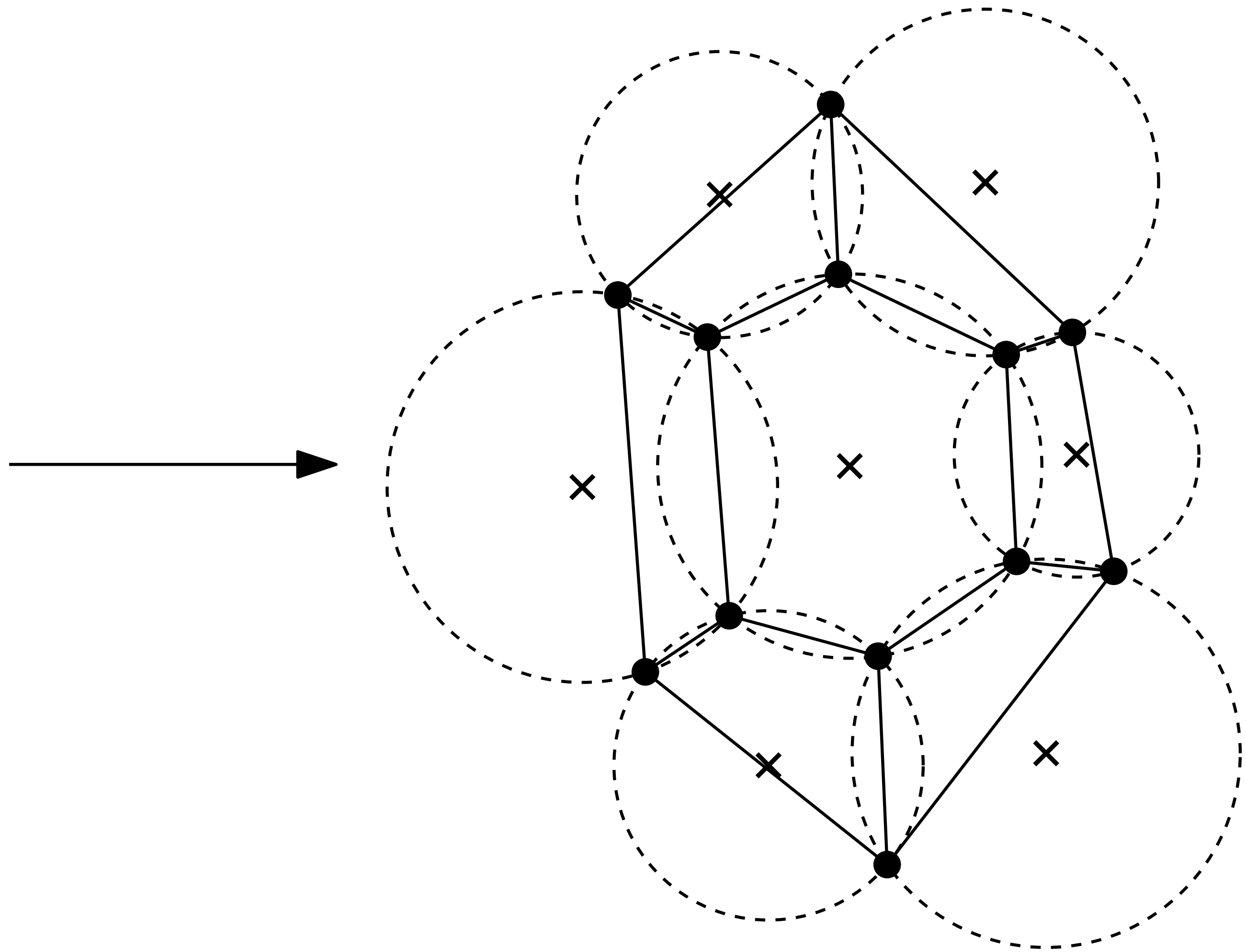


G planar



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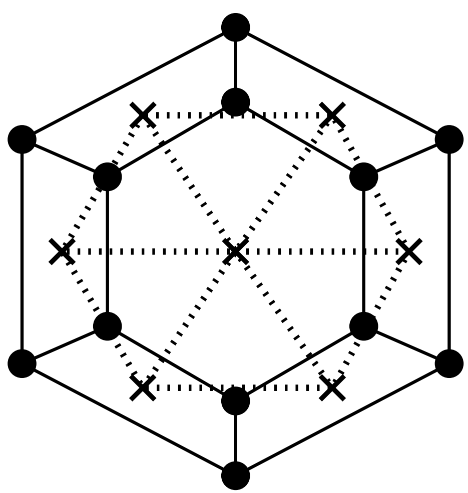

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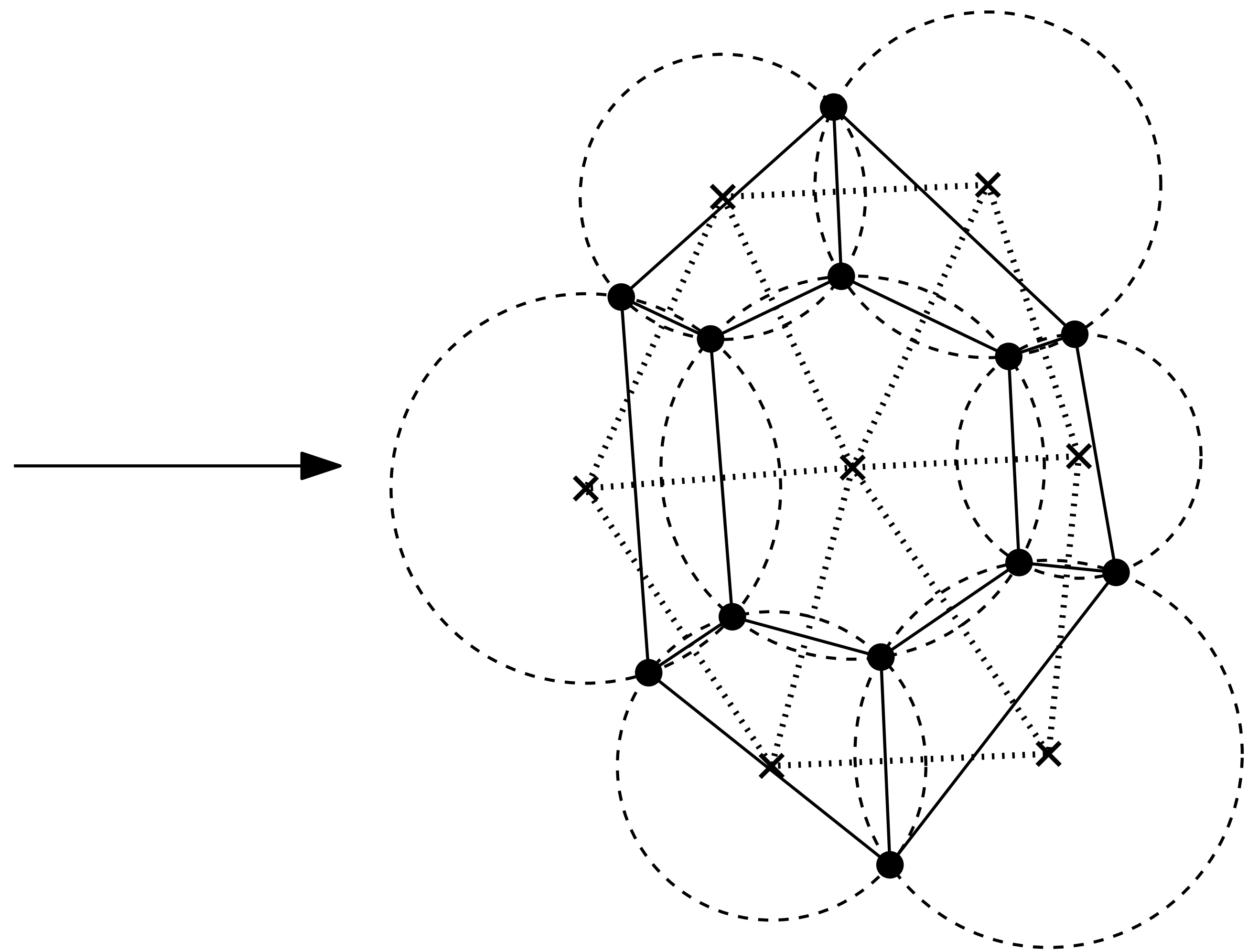
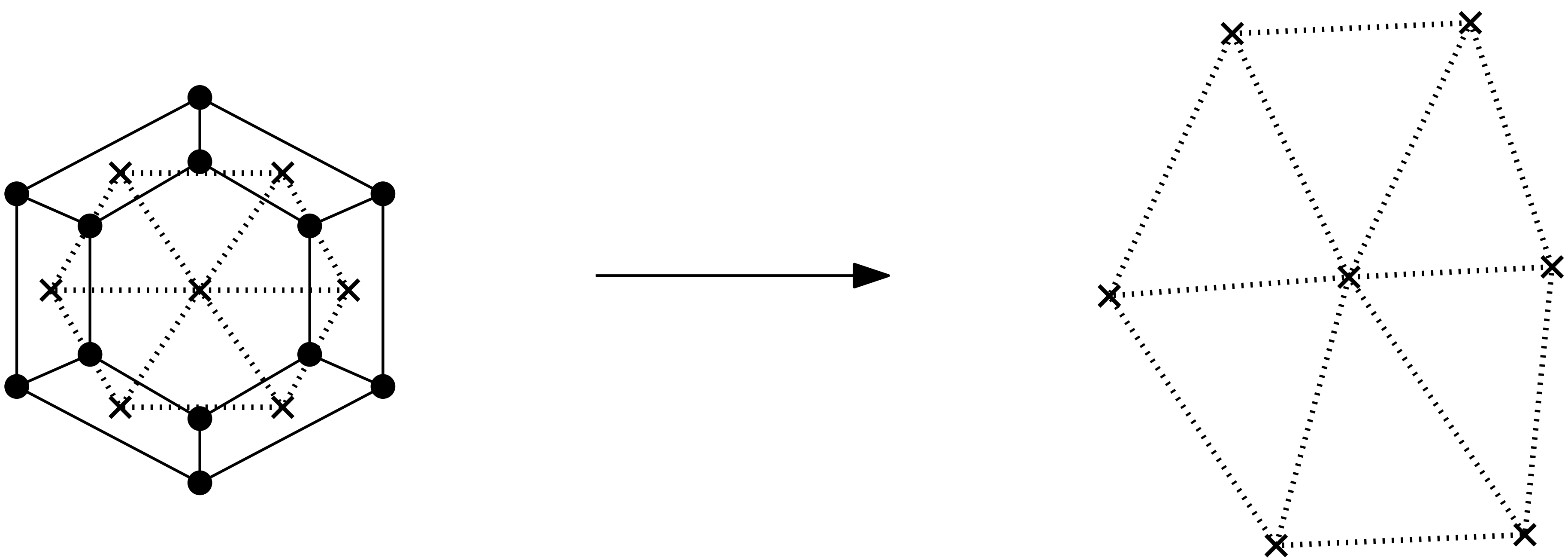


- *Circle centers for G*: drawing of the dual graph of G arising as centers of some circle pattern for G.



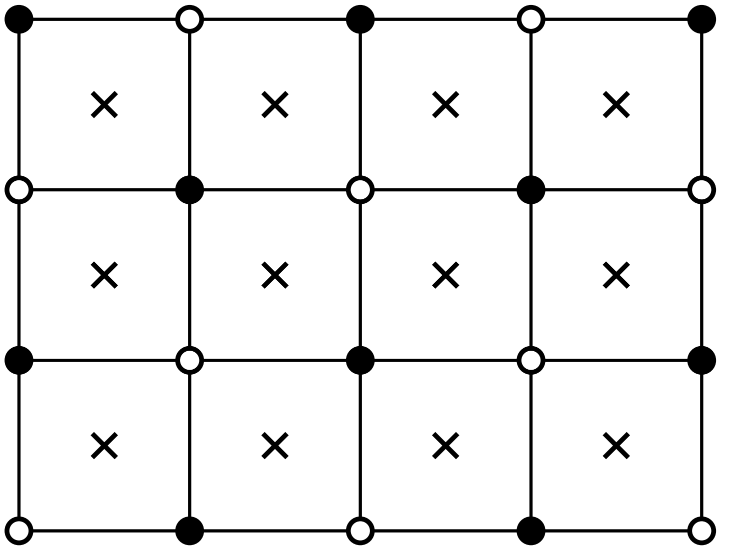
G planar

- Recover the circle pattern from the circle centers ?

  How many circle patterns have the same centers ?

- Given a drawing of the dual graph of G, how to see if it corresponds to the centers of a circle pattern for G ?

- Answers in the case when G is bipartite.



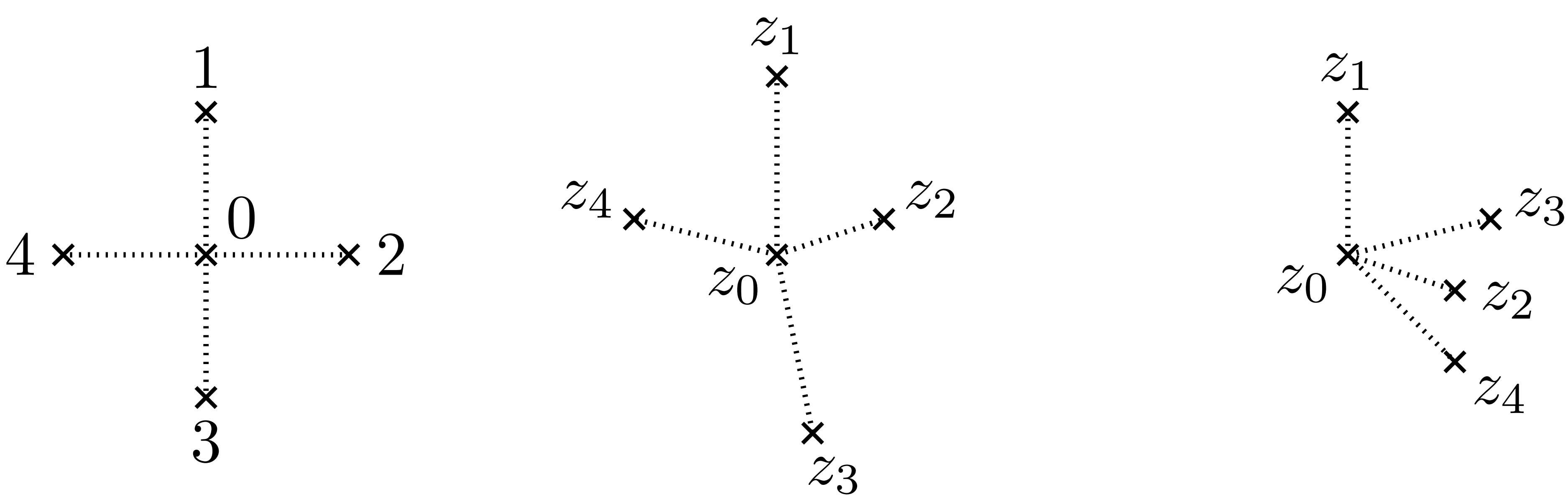
[Geogebra]

- From now on G is bipartite.

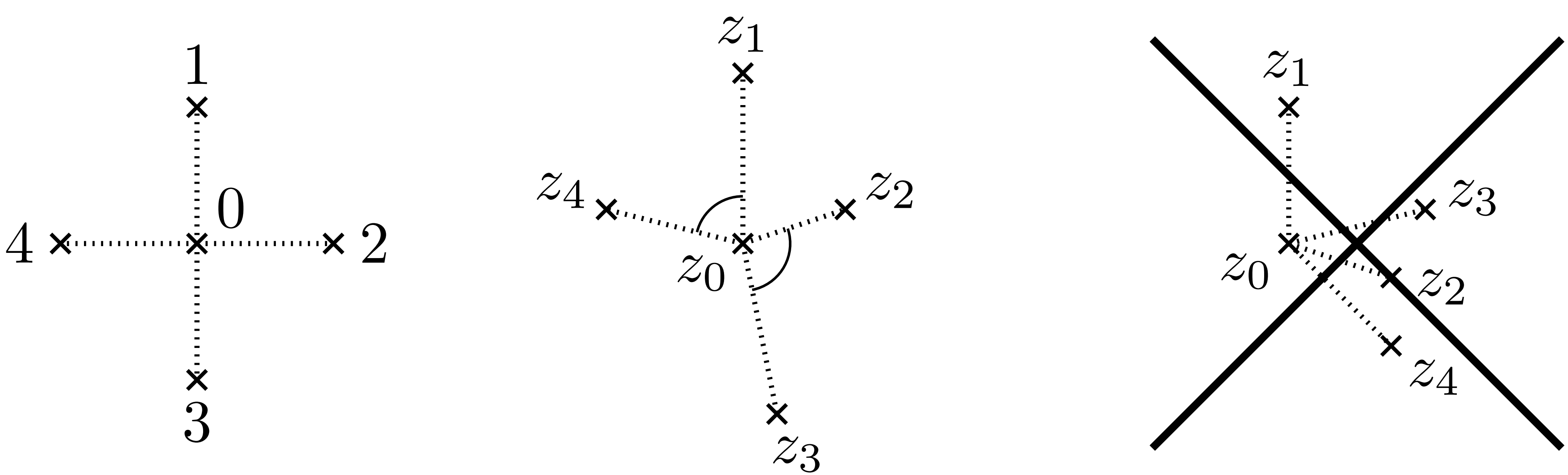
- 2-parameter family of patterns with the same centers.

- A drawing of the dual graph of G corresponds to circle centers for G if and only if around each dual vertex the sum of every other angle is 0 mod π.

- An *embedding* of the dual graph of G corresponds to circle centers for G if and only if around each dual vertex the sum of every other angle is equal to π.



dual of G embedding not embedding



dual of G embedding not embedding

$$\arg \frac{z_4-z_0}{z_1-z_0} + \arg \frac{z_2-z_0}{z_3-z_0} = \pi \Leftrightarrow \frac{(z_2-z_0)(z_4-z_0)}{(z_1-z_0)(z_3-z_0)} \in \mathbb{R}_{<0}$$

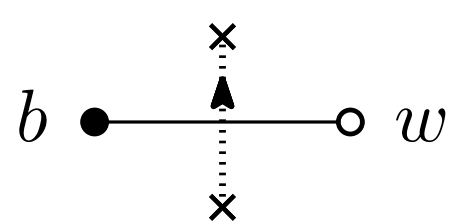
3 Dimer models and circle centers

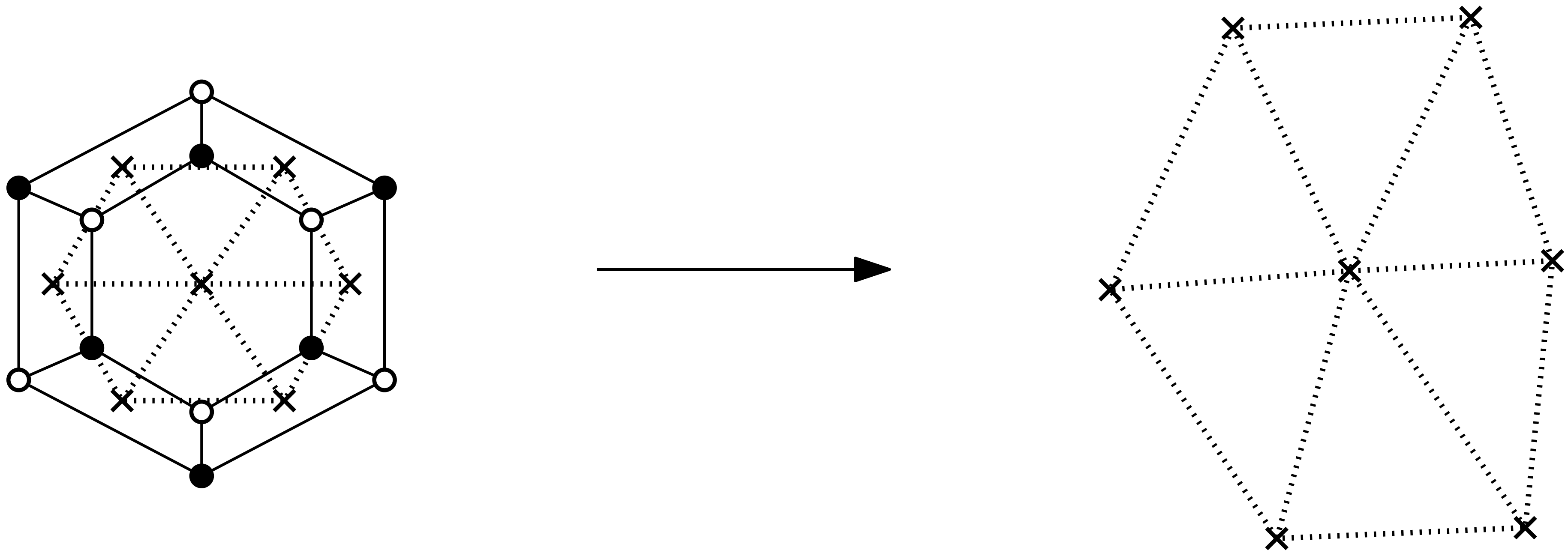
From circle centers to dimer weights

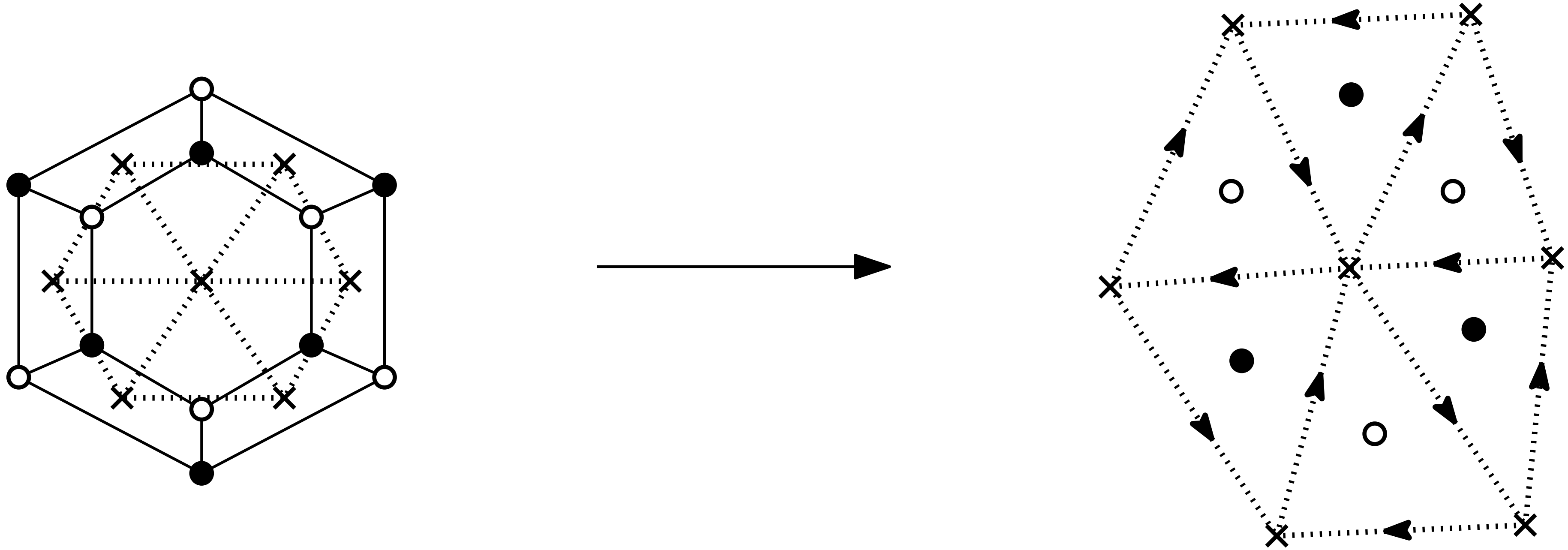
- Fix G a planar unweighted bipartite graph. Start with an embedding of the dual of G as circle centers (a.k.a. t-embedding for G).

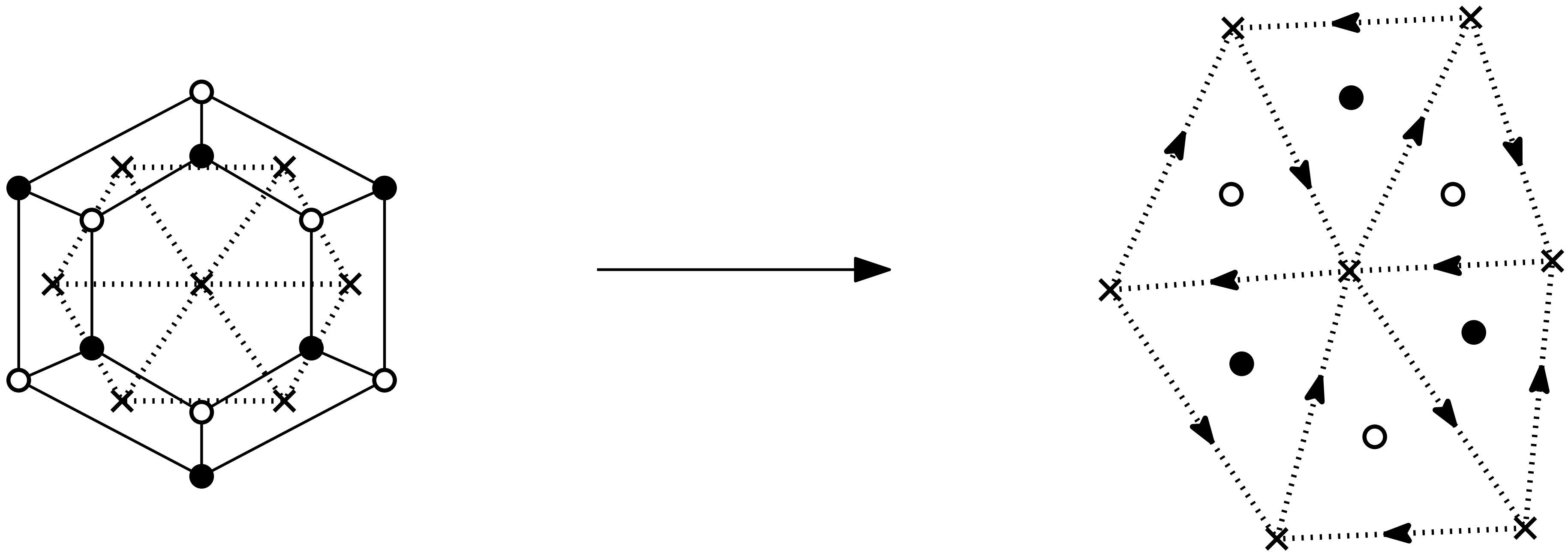
- Construct complex edge weights for G associated to that embedding which satisfy the Kasteleyn condition.

- For an edge in G between b and w, the weight is the vector (complex number) of its corresponding dual edge, oriented so that b lies to its left.

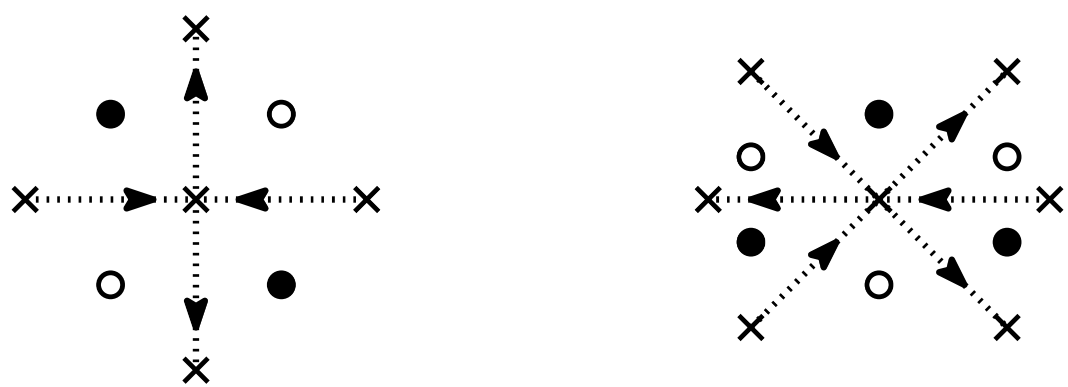


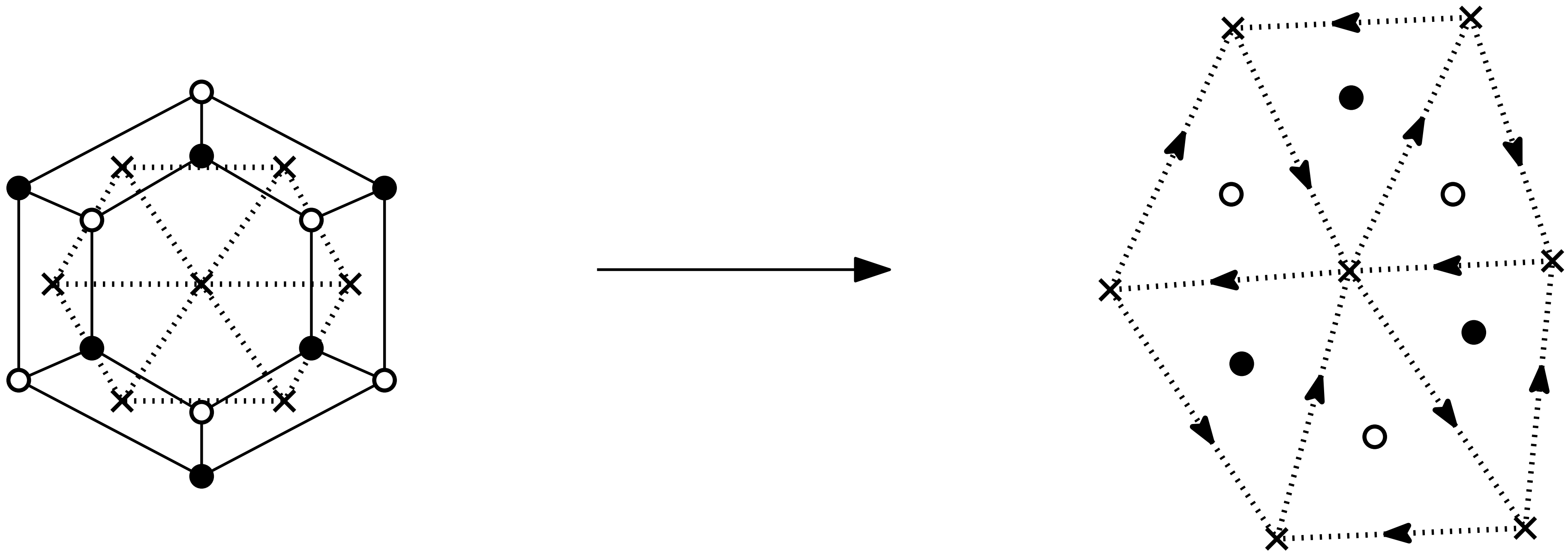






- The complex edge weights satisfy the Kasteleyn condition: the alternating product around of a face of degree 2 mod 4 (resp. 0 mod 4) is positive (resp. negative).





- The complex edge weights satisfy the Kasteleyn condition: the alternating product around of a face of degree 2 mod 4 (resp. 0 mod 4) is positive (resp. negative).

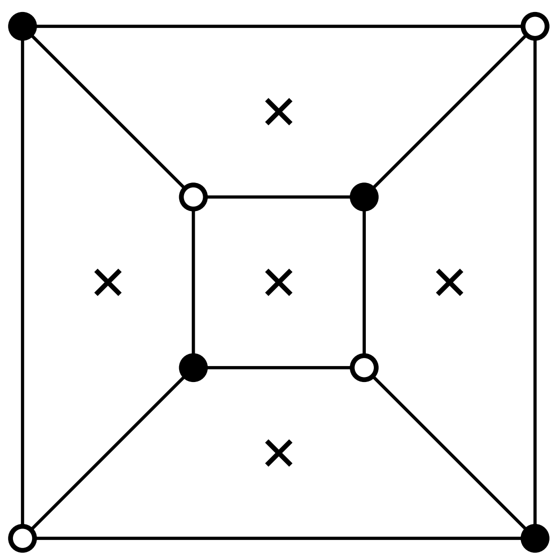
- Around every vertex, the sum of the complex edge weights is zero, i.e. the edge weights have zero divergence.

For a bipartite graph, the geometric local condition
"being centers of a circle pattern with embedded dual"
implies the local condition
"being Kasteleyn edge weights with zero divergence"
(Kenyon-Lam-R.-Russkikh, 2018)

- The fact that circle center embeddings satisfy the Kasteleyn condition was also observed by Affolter (2018).
- Positive edge weights are obtained from circle centers as distances between adjacent centers.
- Generalizes the construction from the isoradial case (Kenyon 2002).

From dimer weights to circle centers

- Given a bipartite graph with positive edge weights, find gauge equivalent weights coming from circle centers. → *Coulomb gauge*



From dimer weights to circle centers

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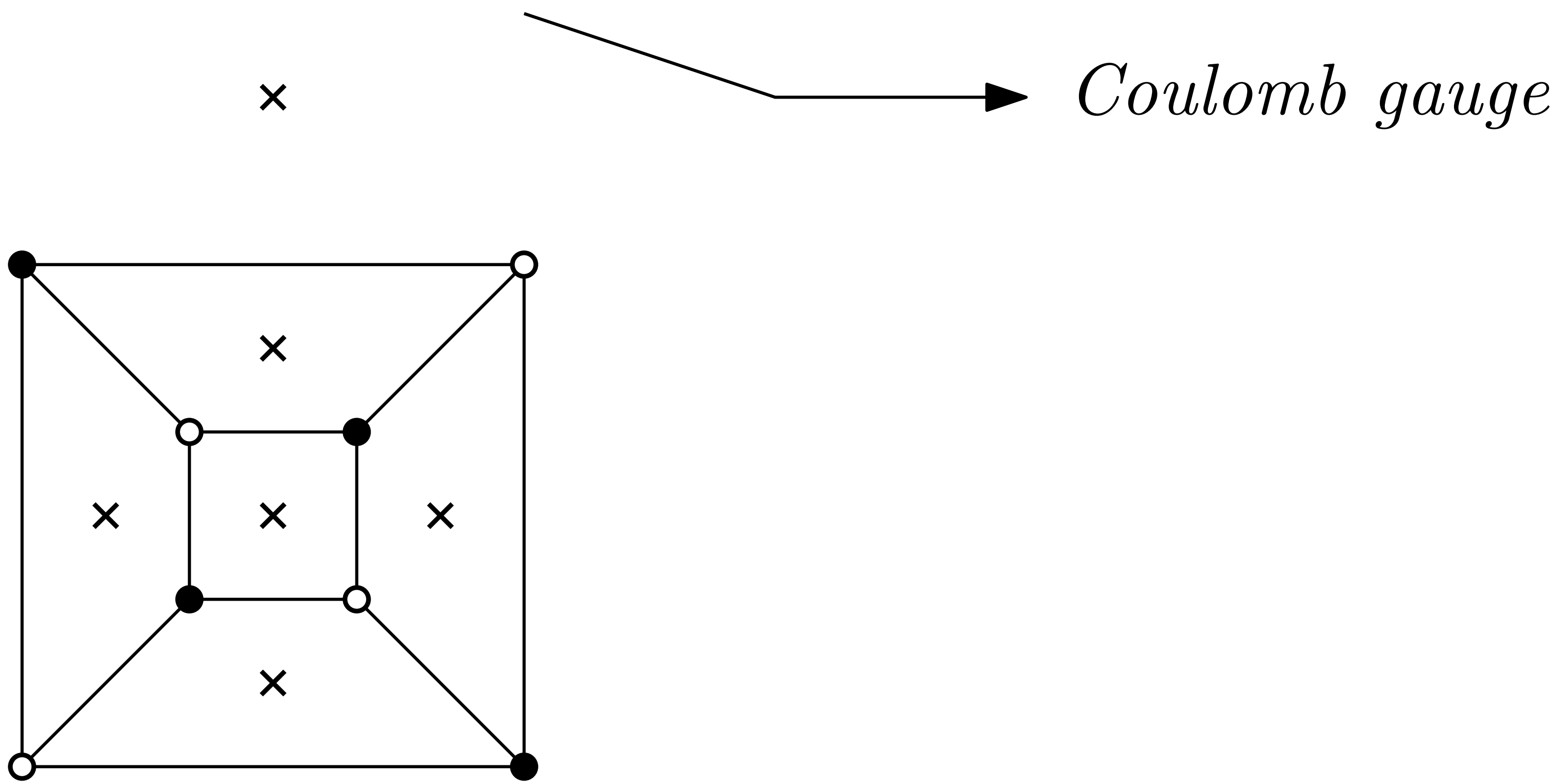

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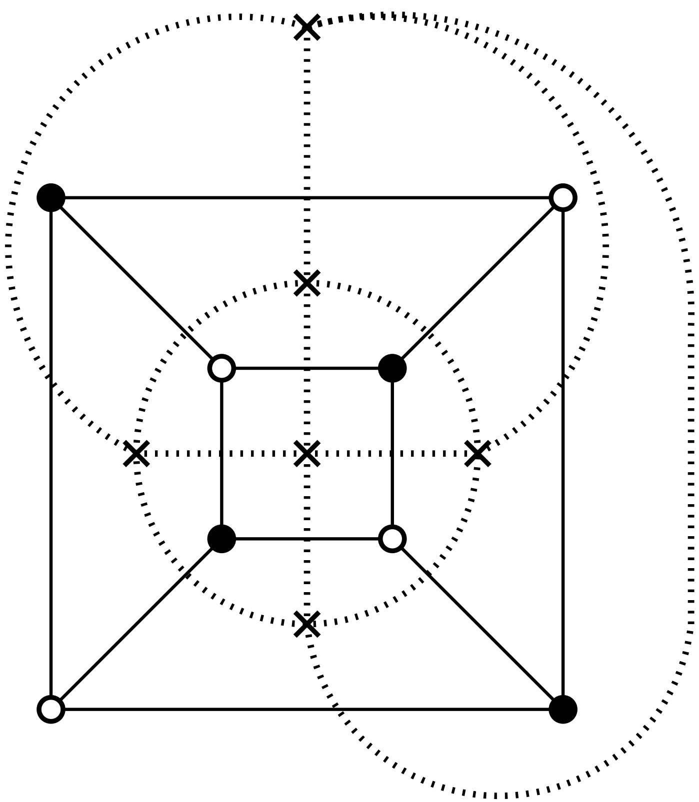
Coulomb gauge



dual

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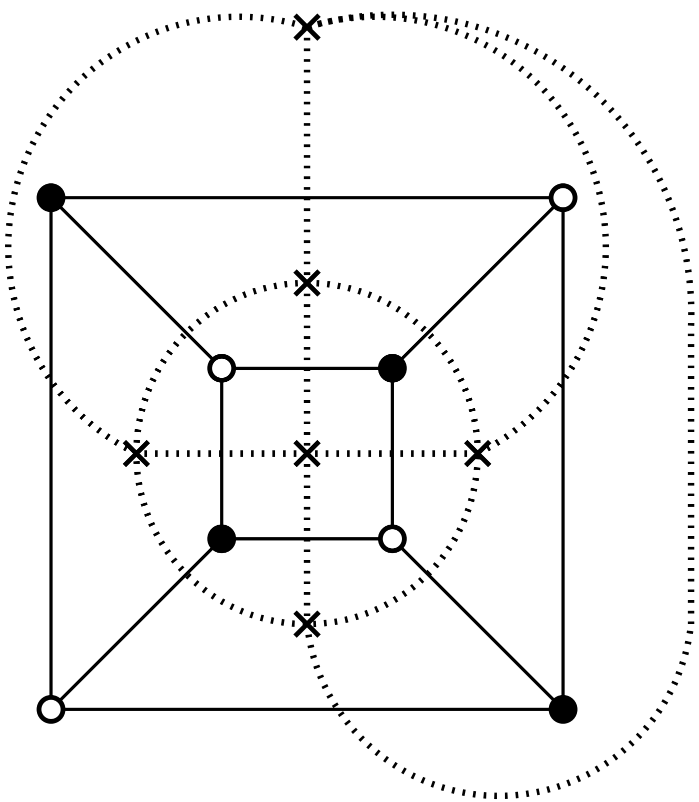

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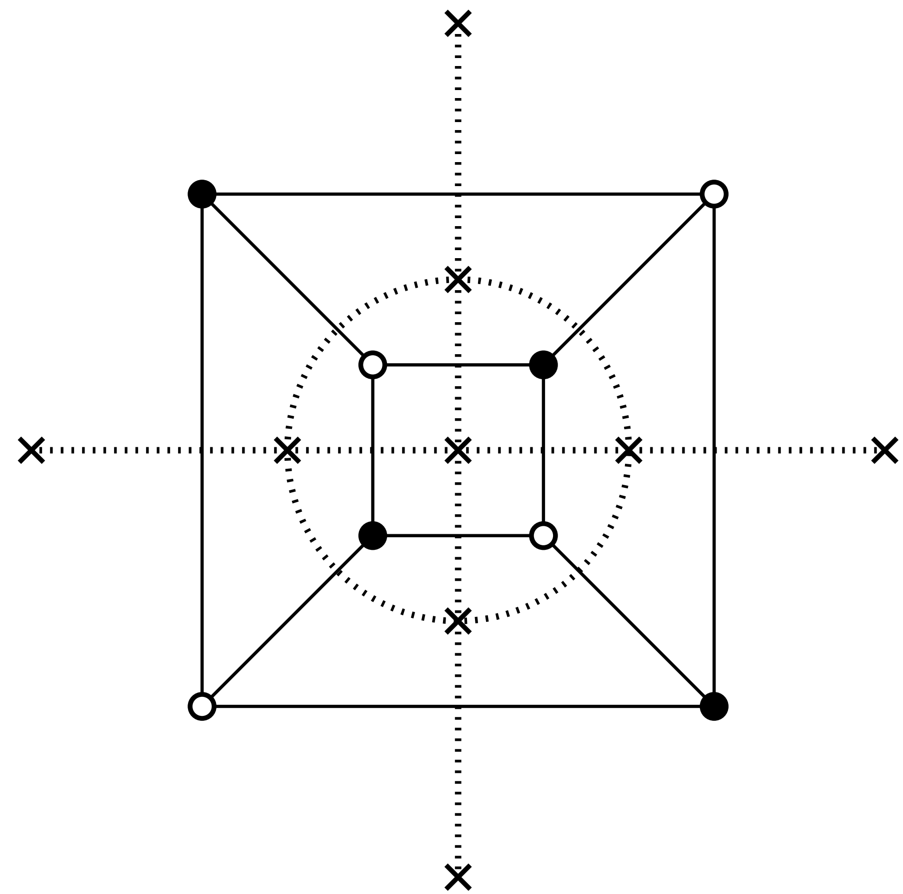

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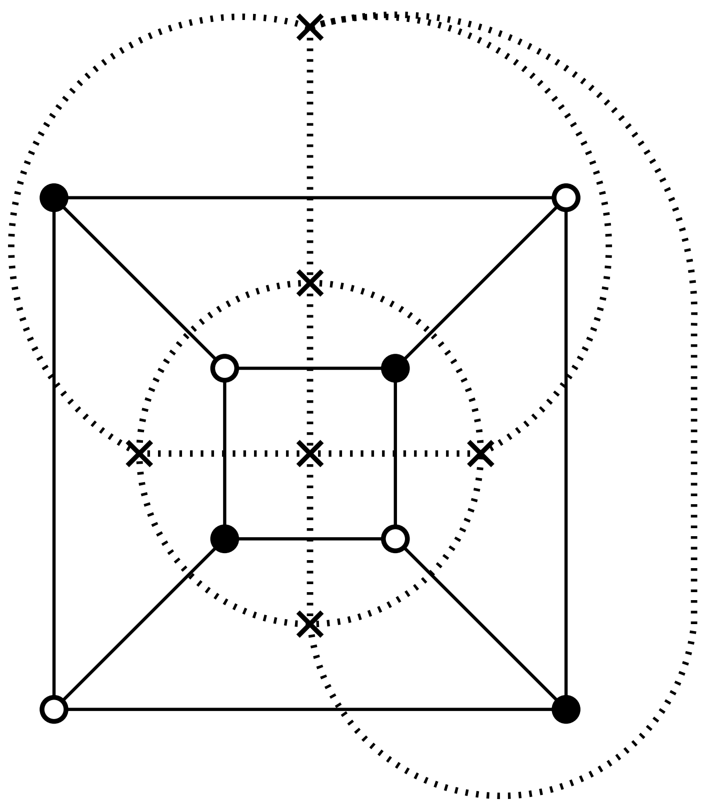

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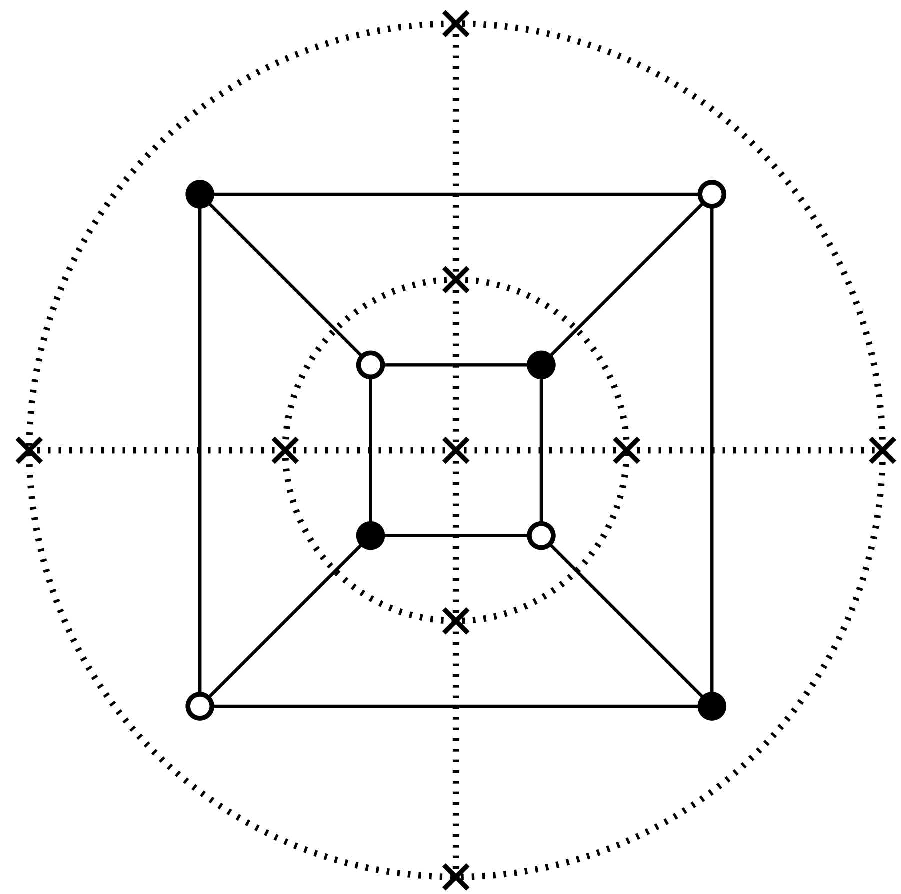
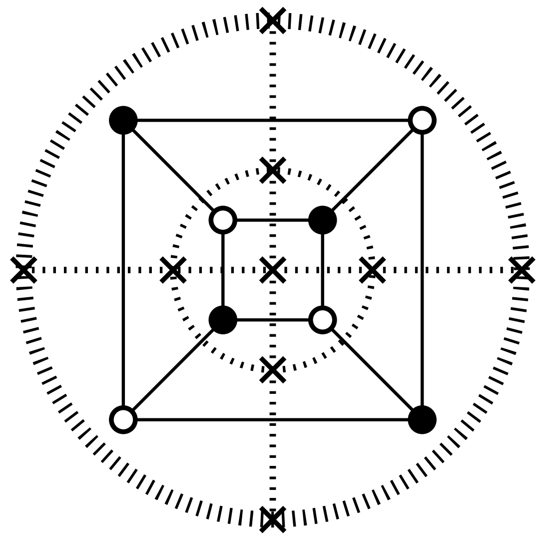


augmented dual



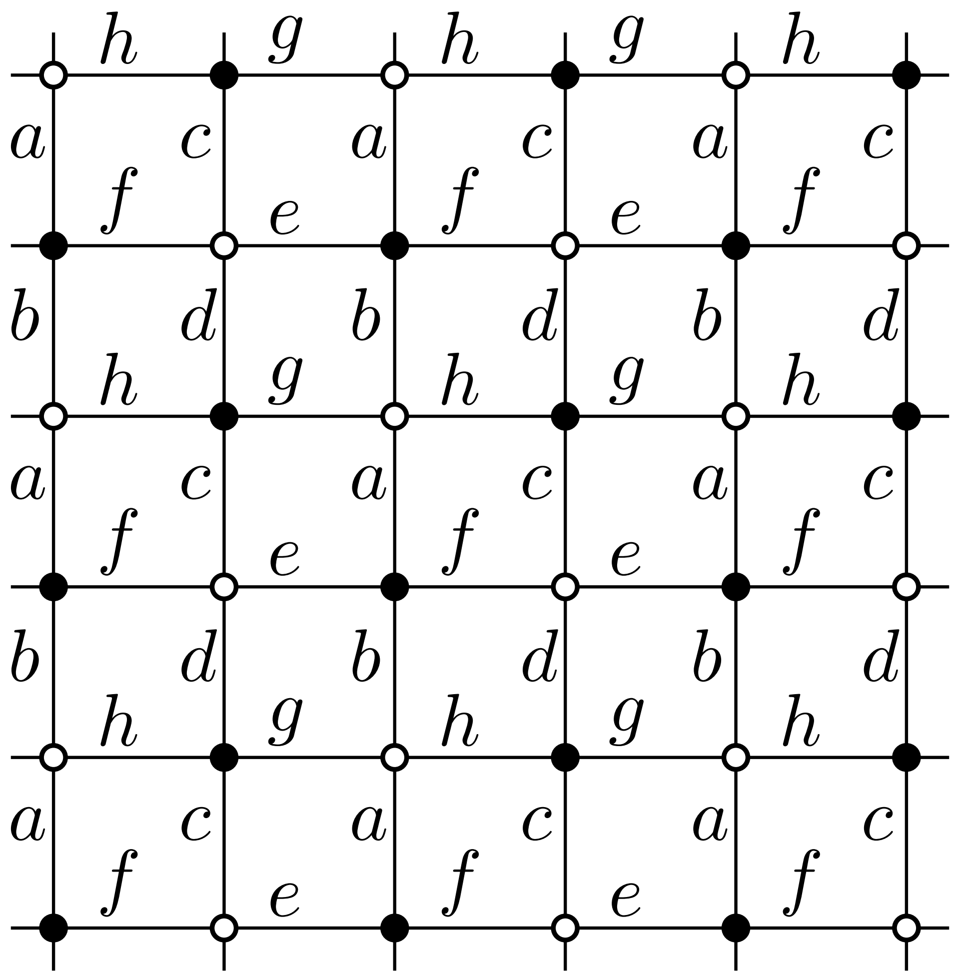
Theorem (Kenyon-Lam-R.-Russkikh 2018). *Let G be a planar bipartite weighted graph with outer face of degree 4. Fix a convex quadrilateral P.*
There are two circle center embeddings of the augmented dual of G which produce weights that are gauge equivalent to the original weights and such that the four outer dual vertices are mapped to the vertices of P.

- Given
  - an unweighted bipartite planar graph G with boundary of length 4
  - a convex quadrilateral (boundary condition)

  there is a 2-to-1 correspondence between embeddings of the augmented dual of G as circle centers and dimer Boltzmann measures on G.

- Expected to hold in some form for other boundary lengths.

- Other setting: infinite planar bipartite graphs, periodic in two directions with edge weights also periodic.



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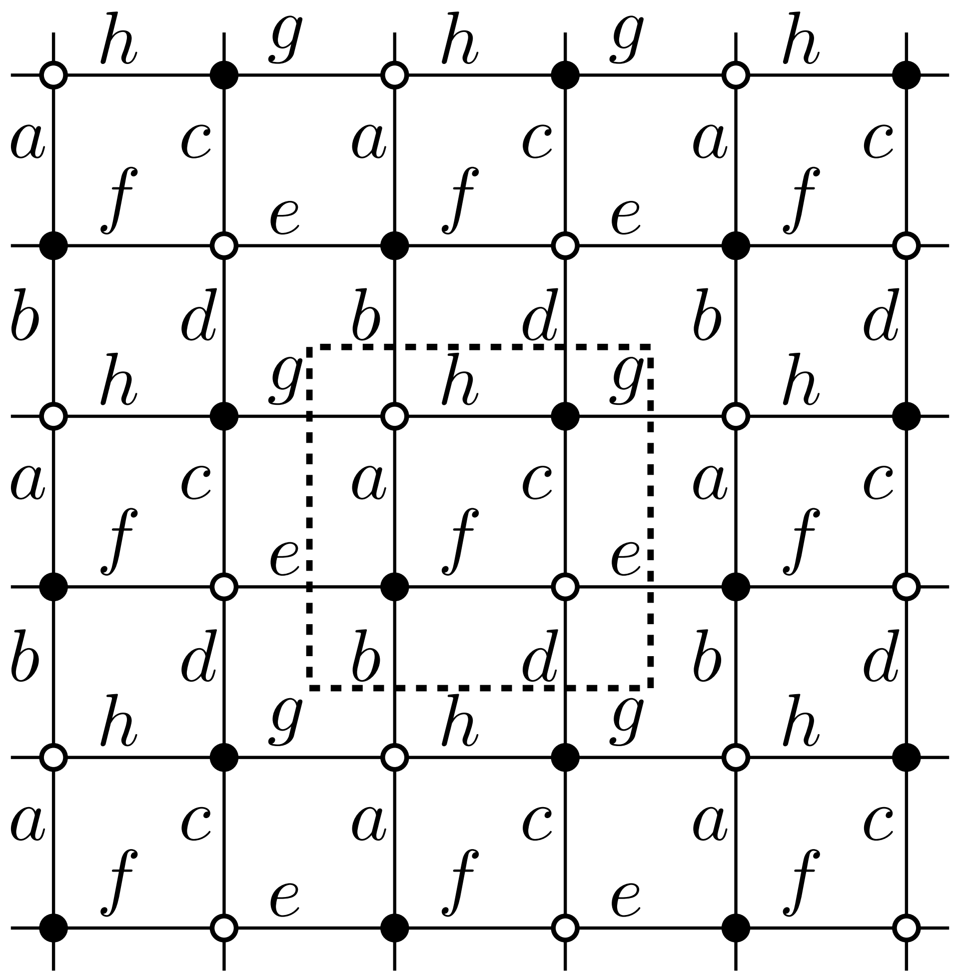


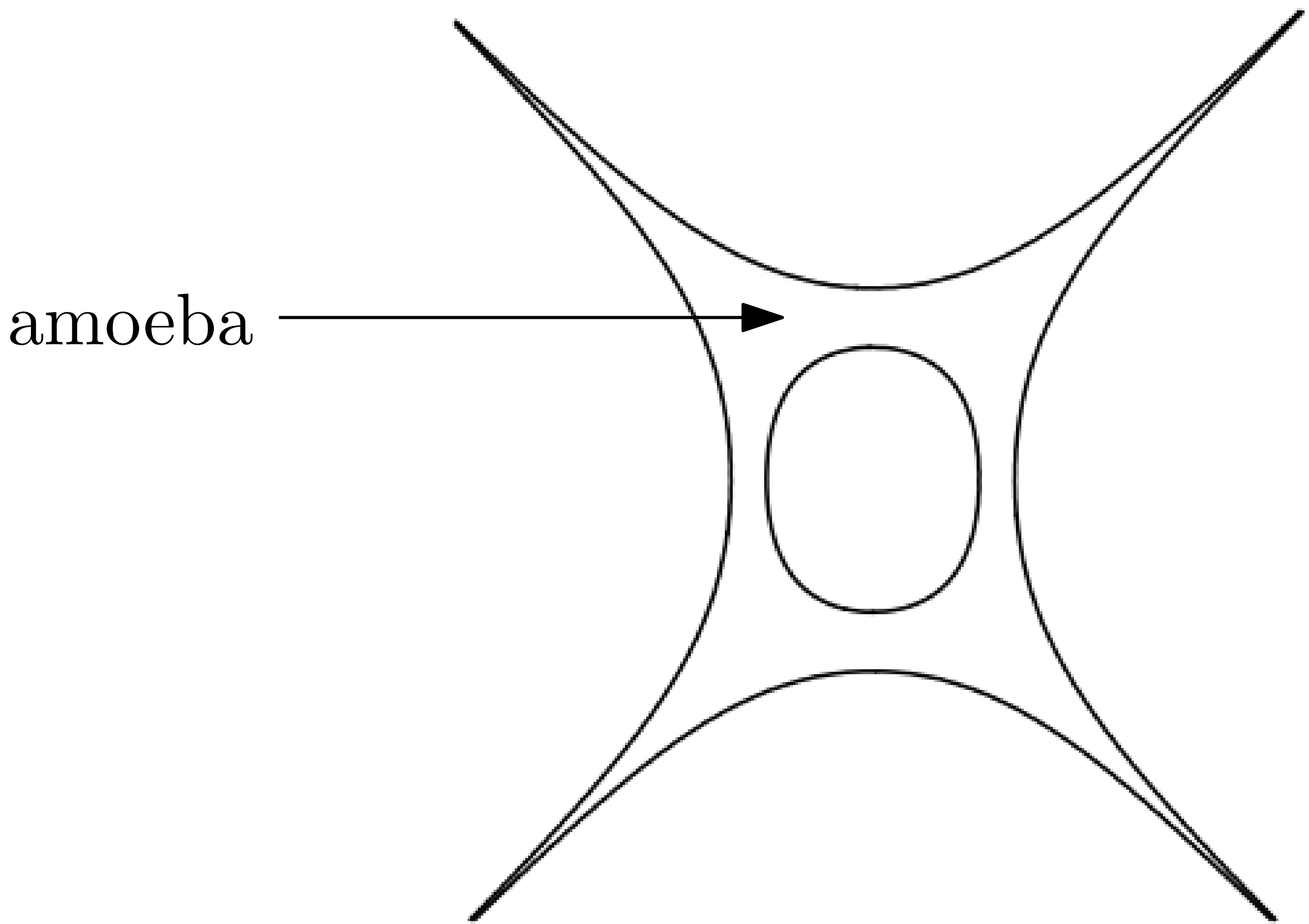
- Let G be an infinite periodic weighted graph.

- *Gibbs measure*: probability measure on the dimer coverings of G, whose restriction to finite subgraphs are Boltzmann measures induced by the edge weights.

- *Ergodic Gibbs measure*: not a convex combination of other Gibbs measures.

- *Liquid*: correlations decay polynomially.

- The interior points of the amoeba (log-log representation of the spectral curve of G) parametrize the liquid ergodic Gibbs measures on G (Kenyon-Okounkov-Sheffield 2006).

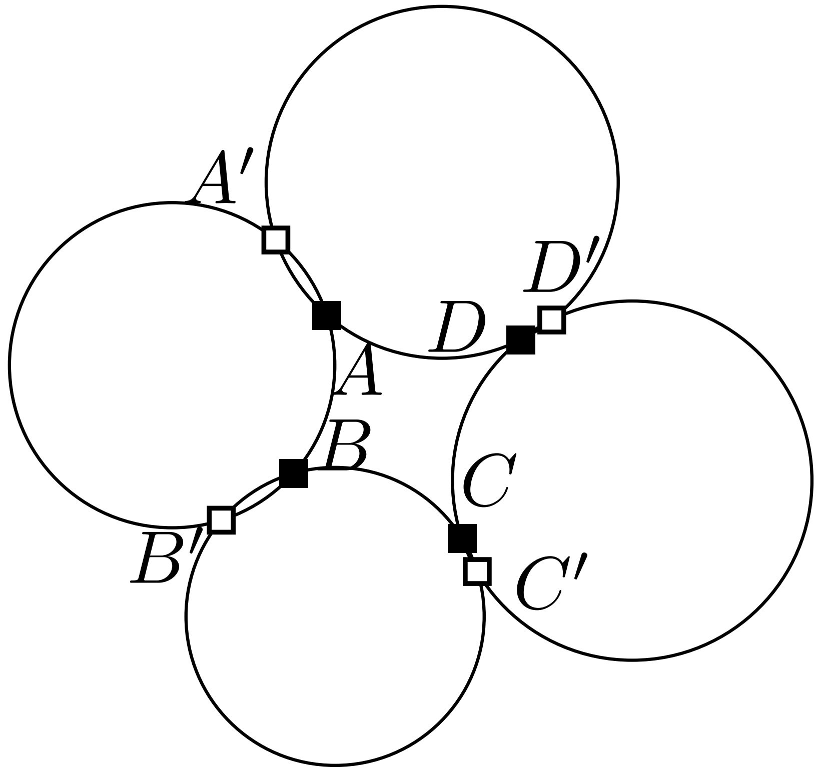

amoeba

Theorem (Kenyon-Lam-R.-Russkikh 2018). *Let G be an infinite weighted bipartite graph, periodic in two directions. Periodic circle center embeddings of the dual of G producing edge weights that are gauge equivalent to the original ones are in bijection with liquid ergodic Gibbs measures on G.*

- In both the finite and the infinite case, the construction of a circle center embedding associated with a weighted planar graph G depends globally (not locally) on G.

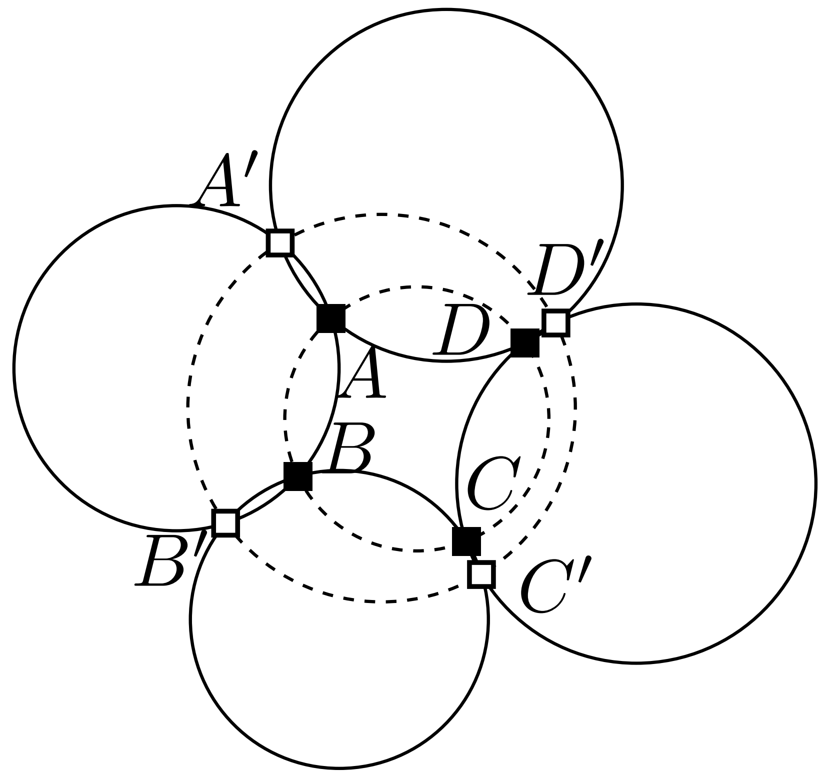
4 Local moves and scaling limits

Miquel's theorem



Theorem (Miquel, 1838). *In this setting,*
A, B, C, D *concyclic* $\Leftrightarrow$ A', B', C', D' *concyclic.*

Miquel's theorem



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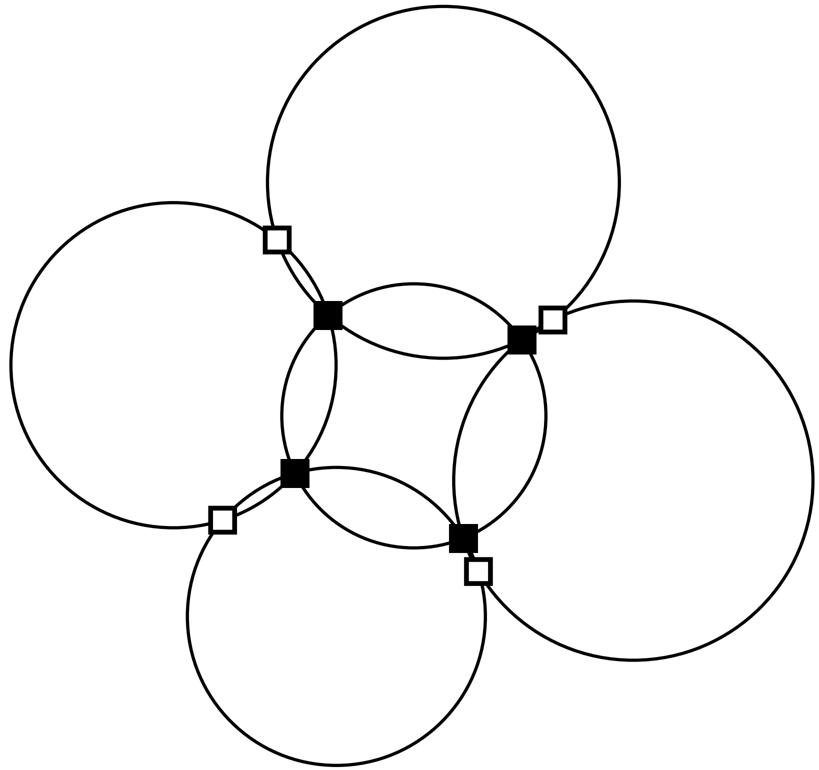

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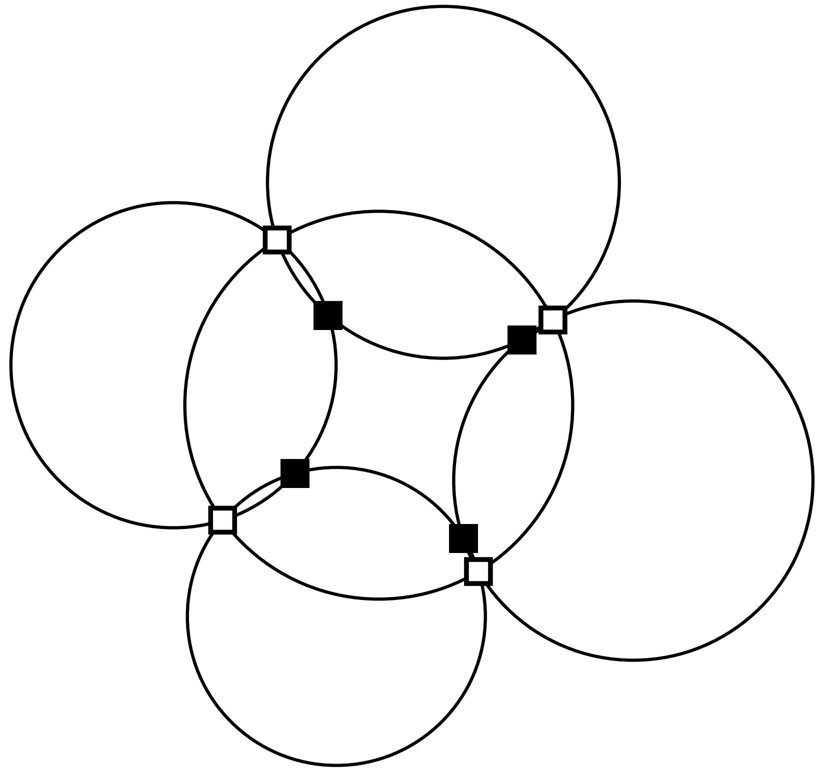

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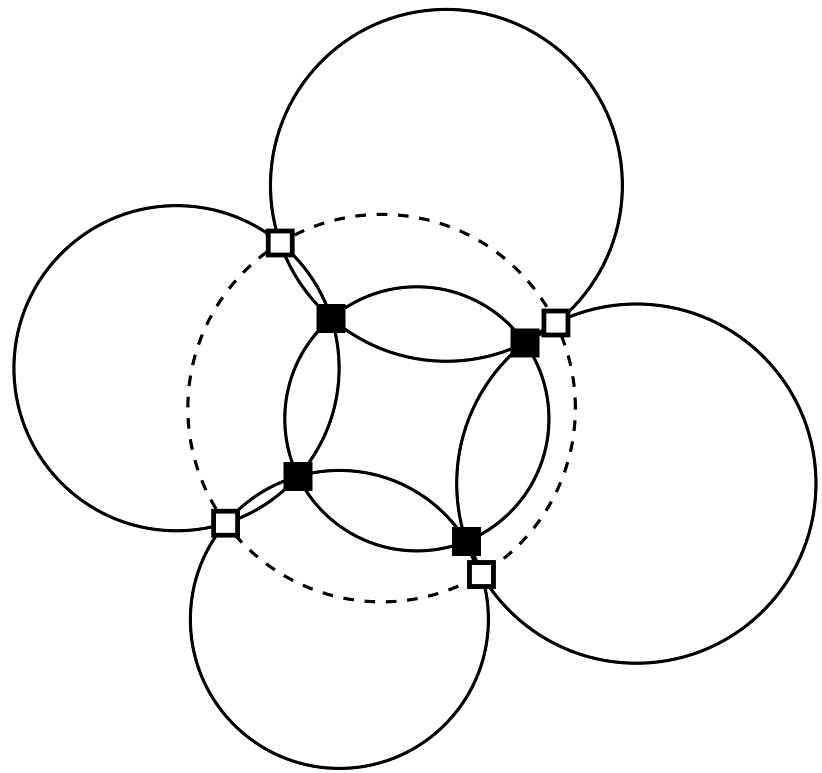
Miquel's theorem revisited



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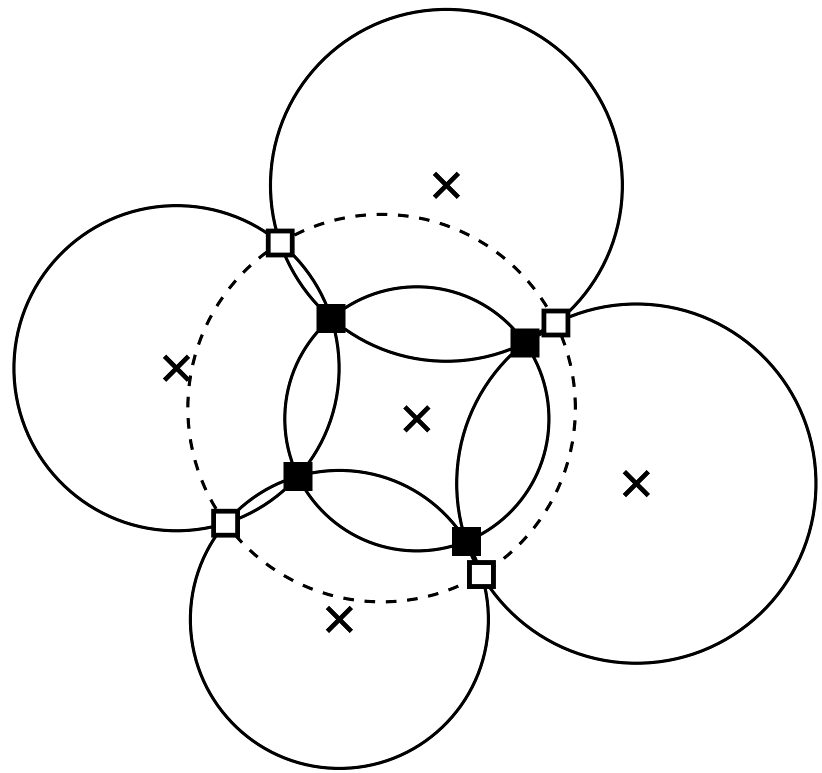

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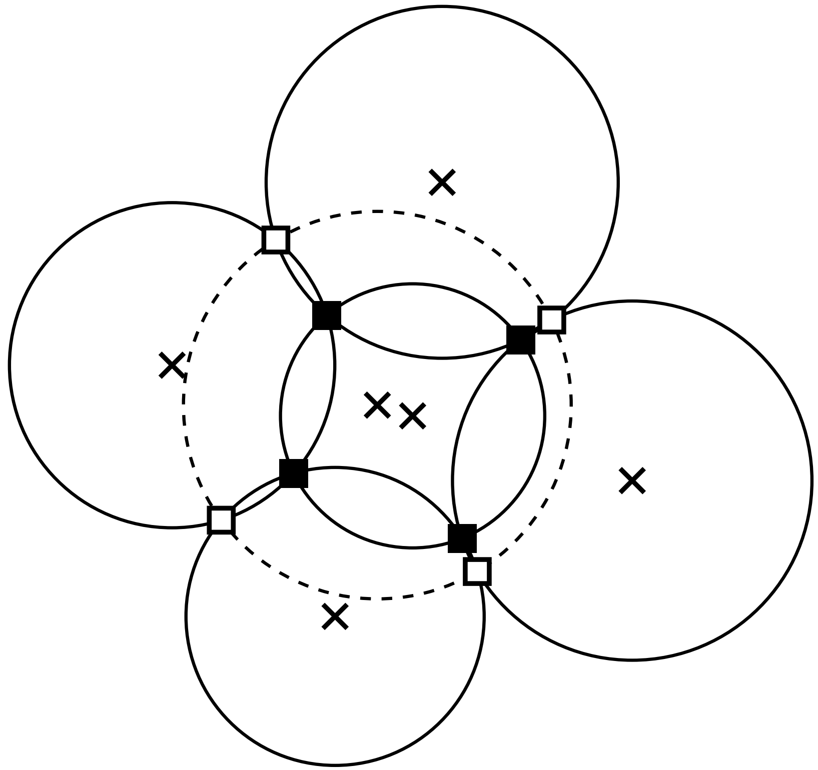

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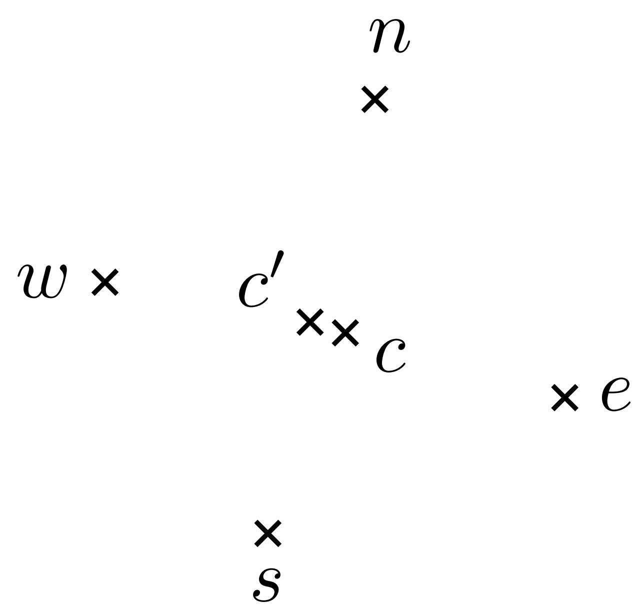


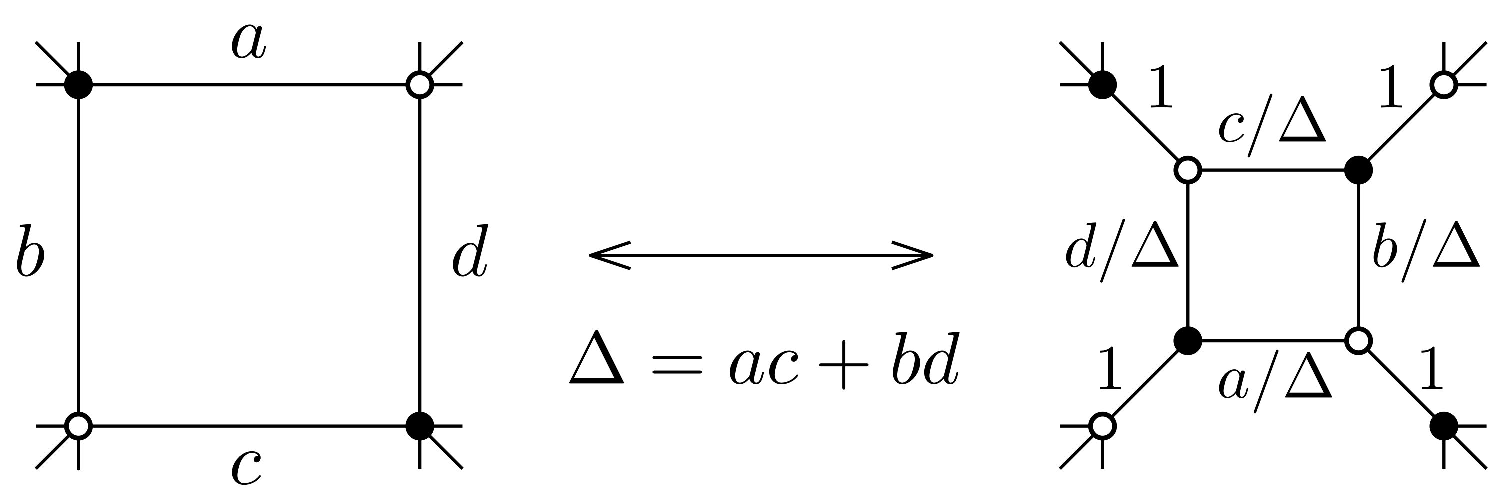
$n, w, s, e, c, c' \in \mathbb{C}$

Theorem (Affolter 2018, Kenyon-Lam-R.-Russkikh 2018).

$$\frac{(c-w)(s-c')(e-n)}{(w-s)(c'-e)(n-c)} = -1$$

Discrete Schwarzian KP equation

Urban renewal



Theorem (Affolter 2018, Kenyon-Lam-R.-Russkikh, 2018). *The Miquel move for circle centers corresponds to the urban renewal for dimer models.*

Miquel dynamics

- Miquel dynamics defined as a discrete-time dynamics on the space of square-grid circle patterns: alternate Miquel moves on all the white faces then on all the black faces.

- Its integrability follows from the identification with the Goncharov-Kenyon dimer dynamics.

- The evolution is governed by cluster algebras mutations.

Embeddings in statistical mechanics

- Consider an infinite planar graph periodic in two directions on which we study a statistical mechanical model (random walk, dimers, Ising,...).

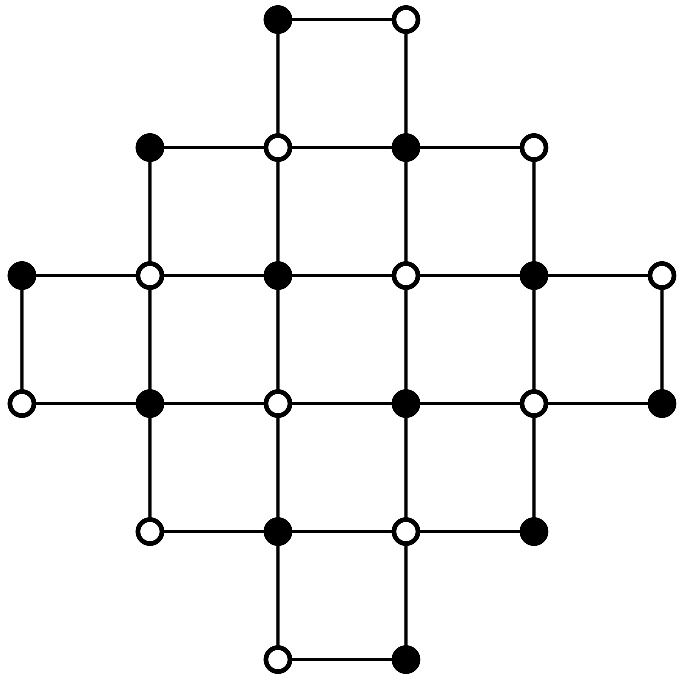
- Find an embedding of it such that universal conformally invariant objects appear in the scaling limit.

- Same issue for formulating the convergence to Liouville quantum gravity of random planar maps decorated with some statistical mechanical model.

Embeddings in statistical mechanics

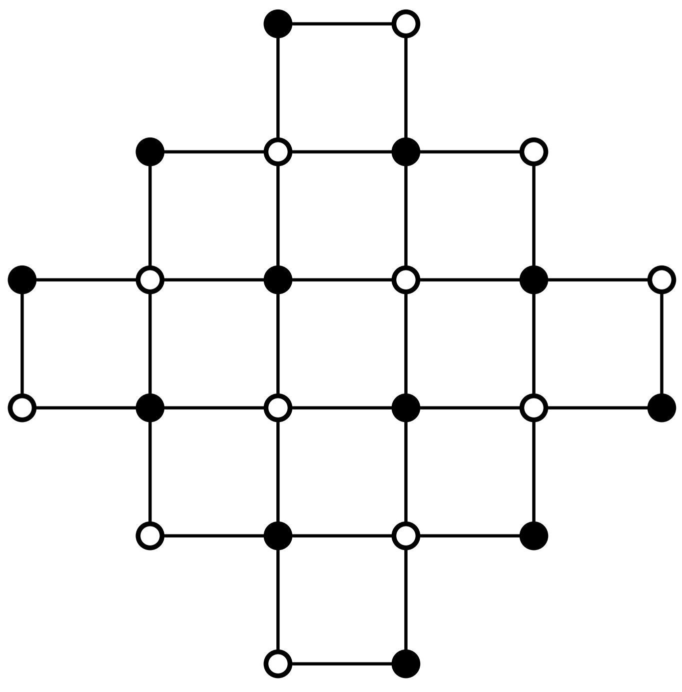
Theorem (Kenyon-Lam-R.-Russkikh, 2018)**.** *Circle center embeddings for dimers generalize the Tutte embedding adapted to spanning trees and the s-embeddings adapted to the Ising model.*

The Aztec diamond limit shape

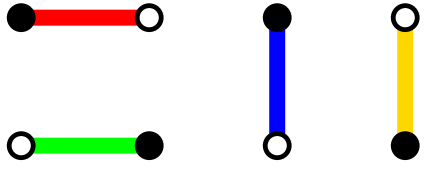


Aztec diamond of size 3

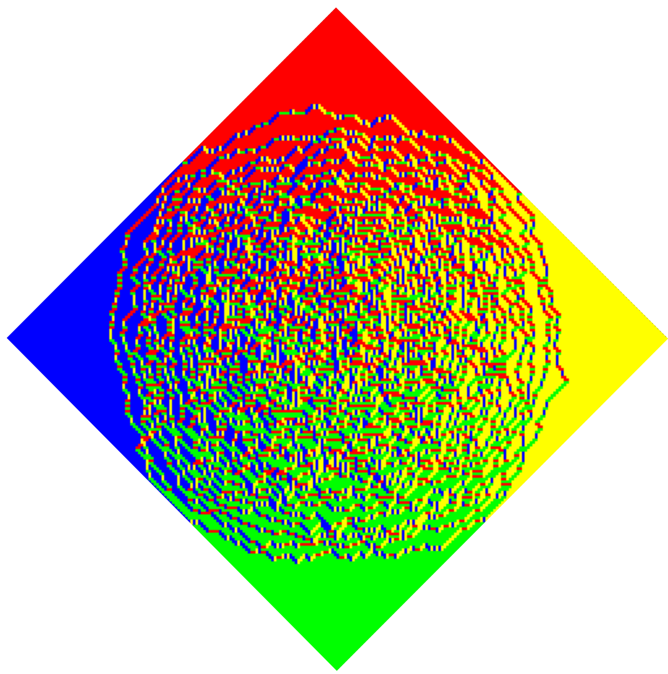
The Aztec diamond limit shape



Aztec diamond of size 3

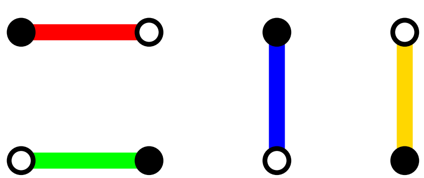


The Aztec diamond limit shape



picture by Cris Moore

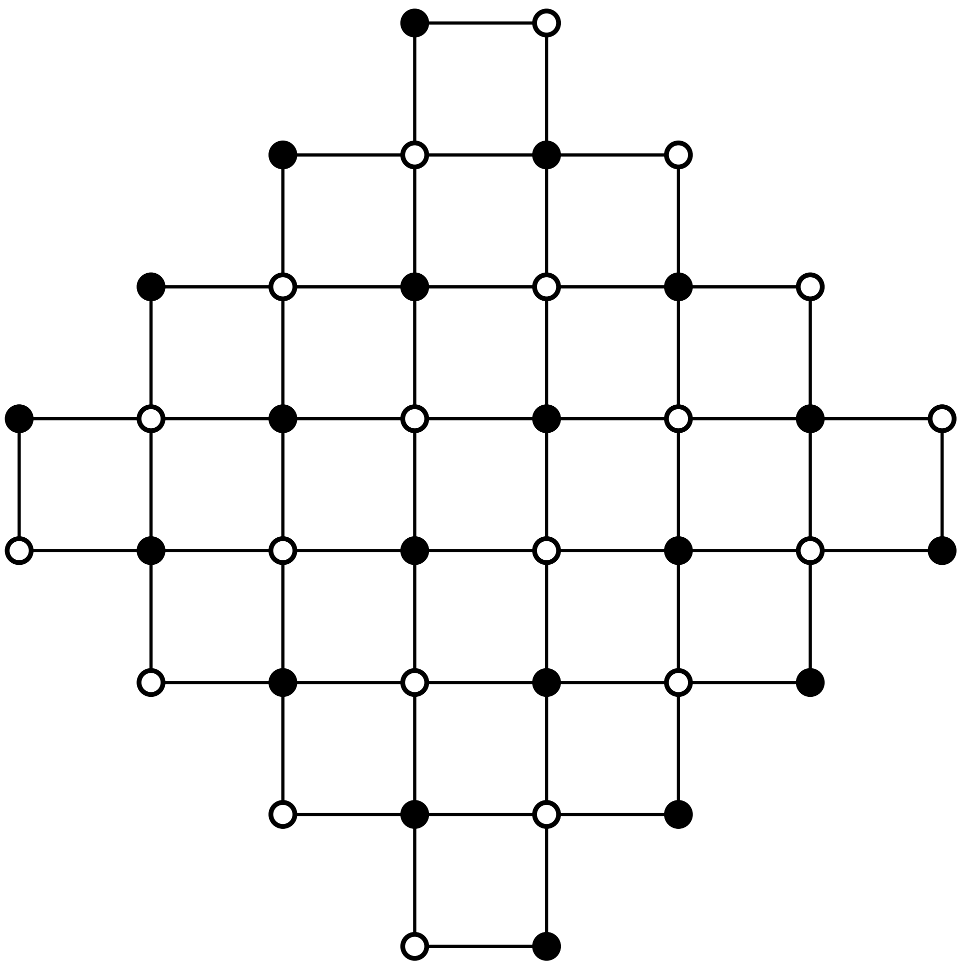
tuvalu.santafe.edu/
~moore/aztec256.gif



- Frozen corners
- Gaussian free field (GFF) fluctuations around the limit shape
- Conformal structure for this GFF not given by the Euclidean metric

(Jockush-Propp-Shor 1998,
Cohn-Kenyon-Propp 2001,
Chhita-Johansson-Young 2015,
Bufetov-Gorin 2018,...)

- After contraction of degree 2 vertices and merging of parallel edges, the outer face has degree 4.



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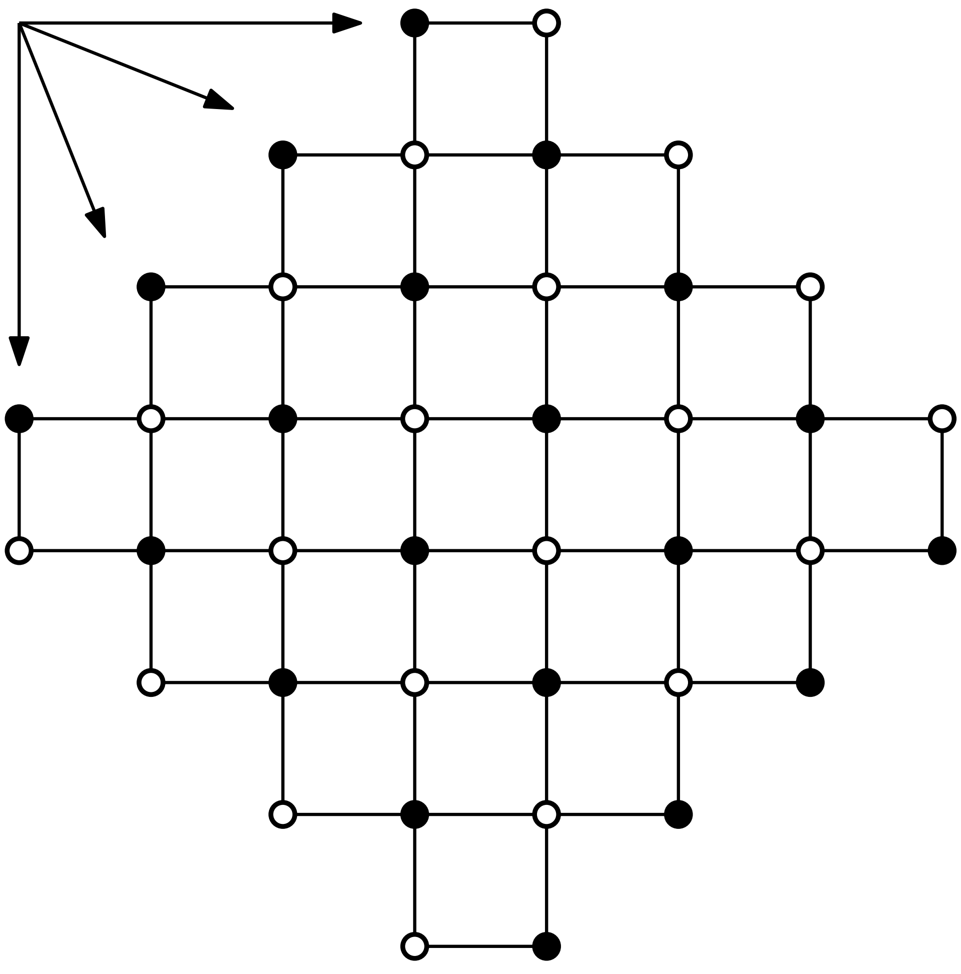

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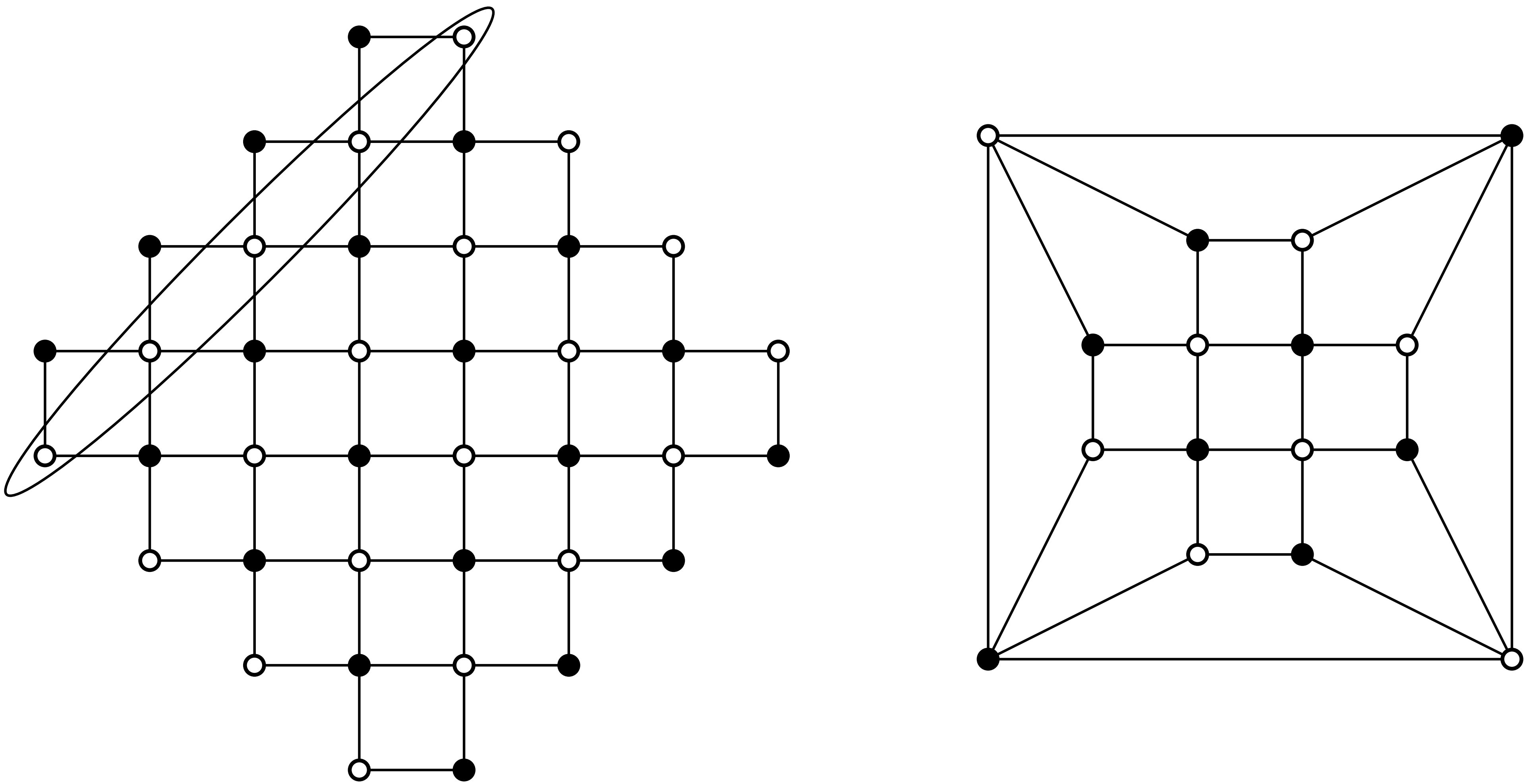
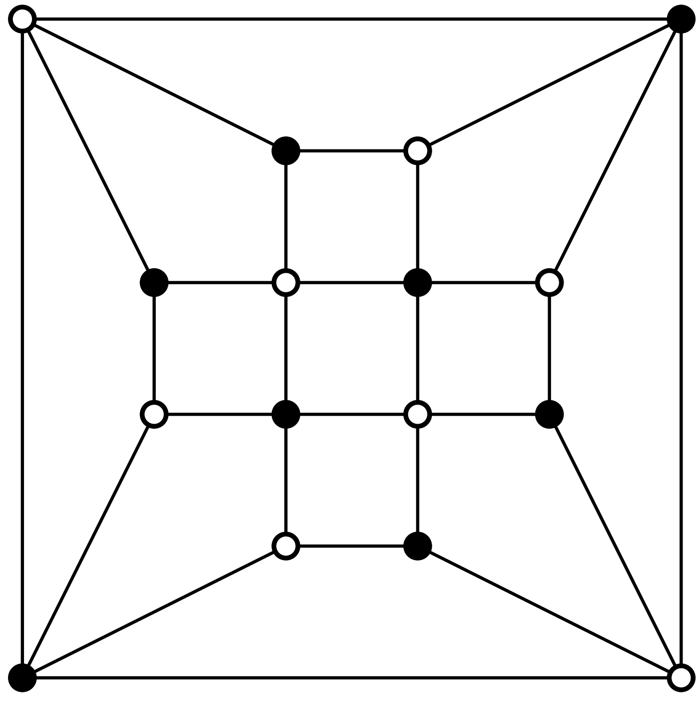
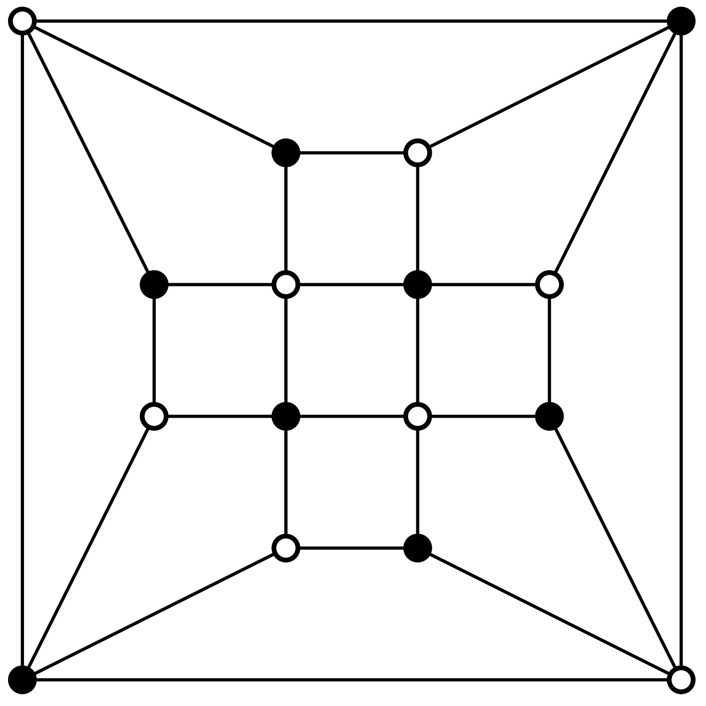
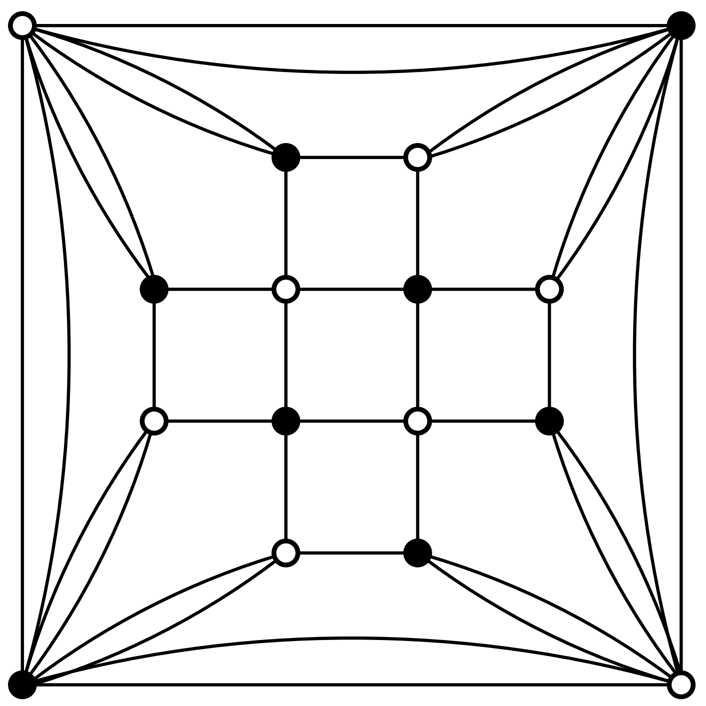
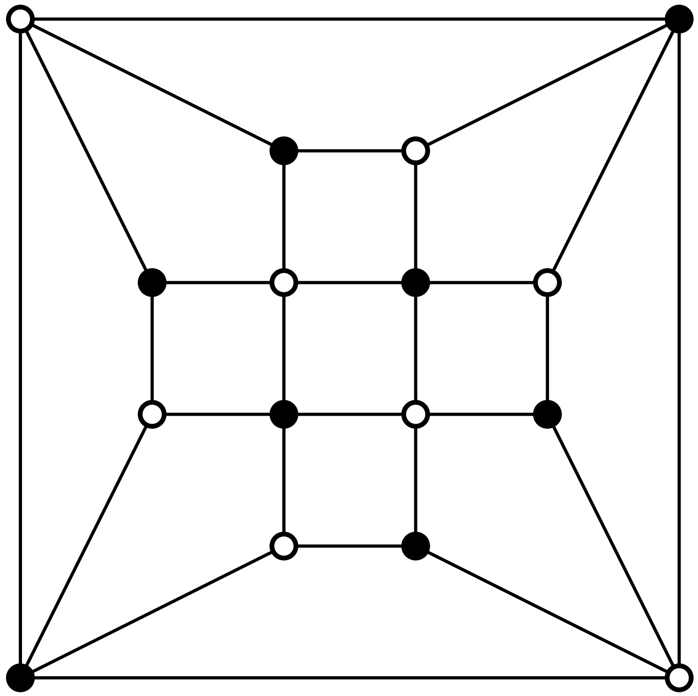
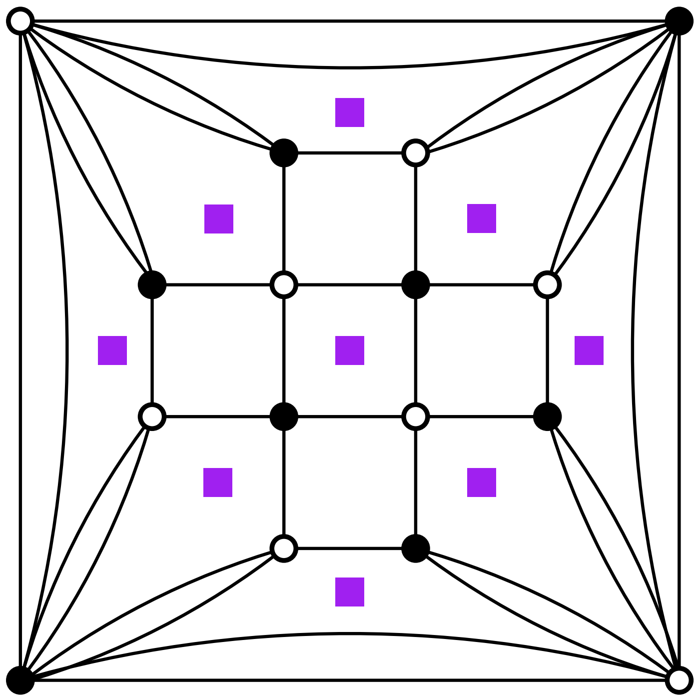


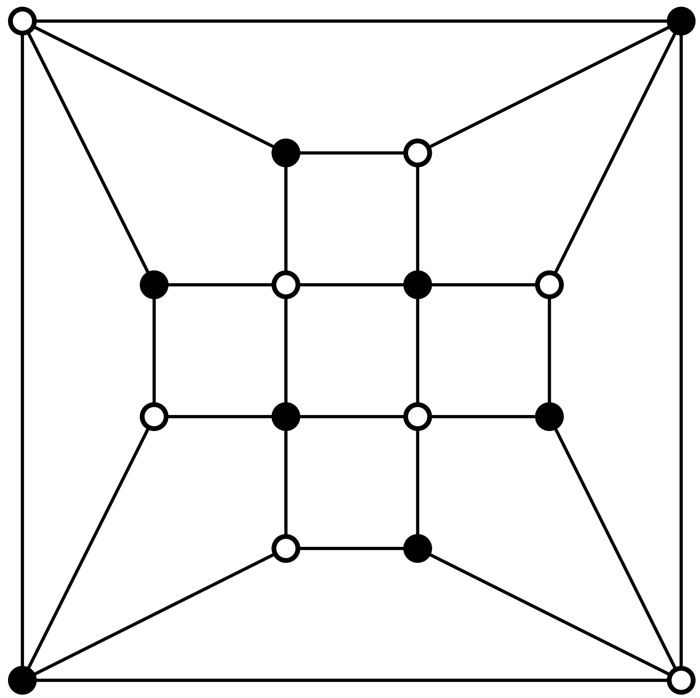


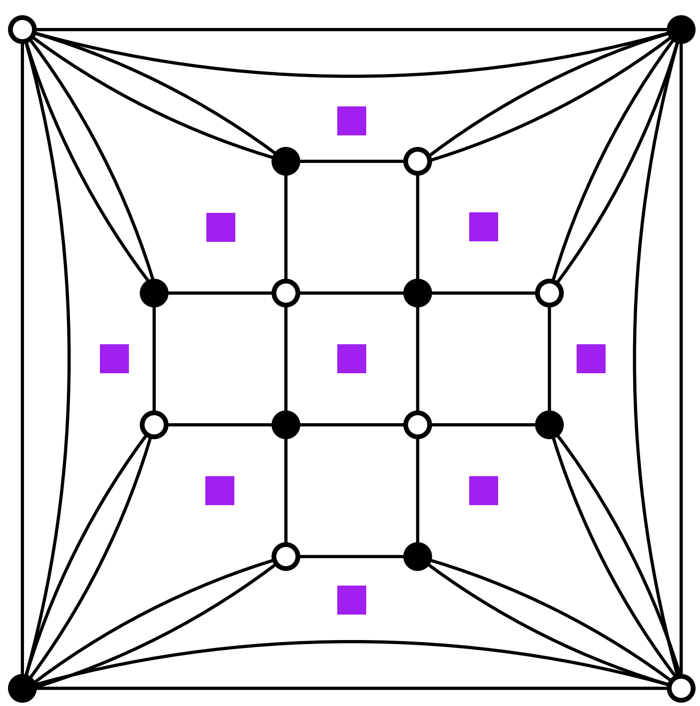


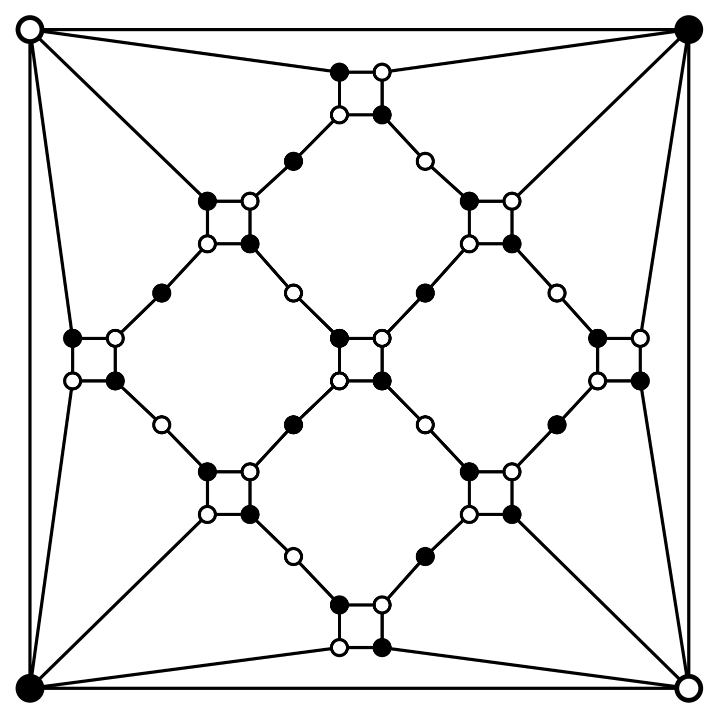


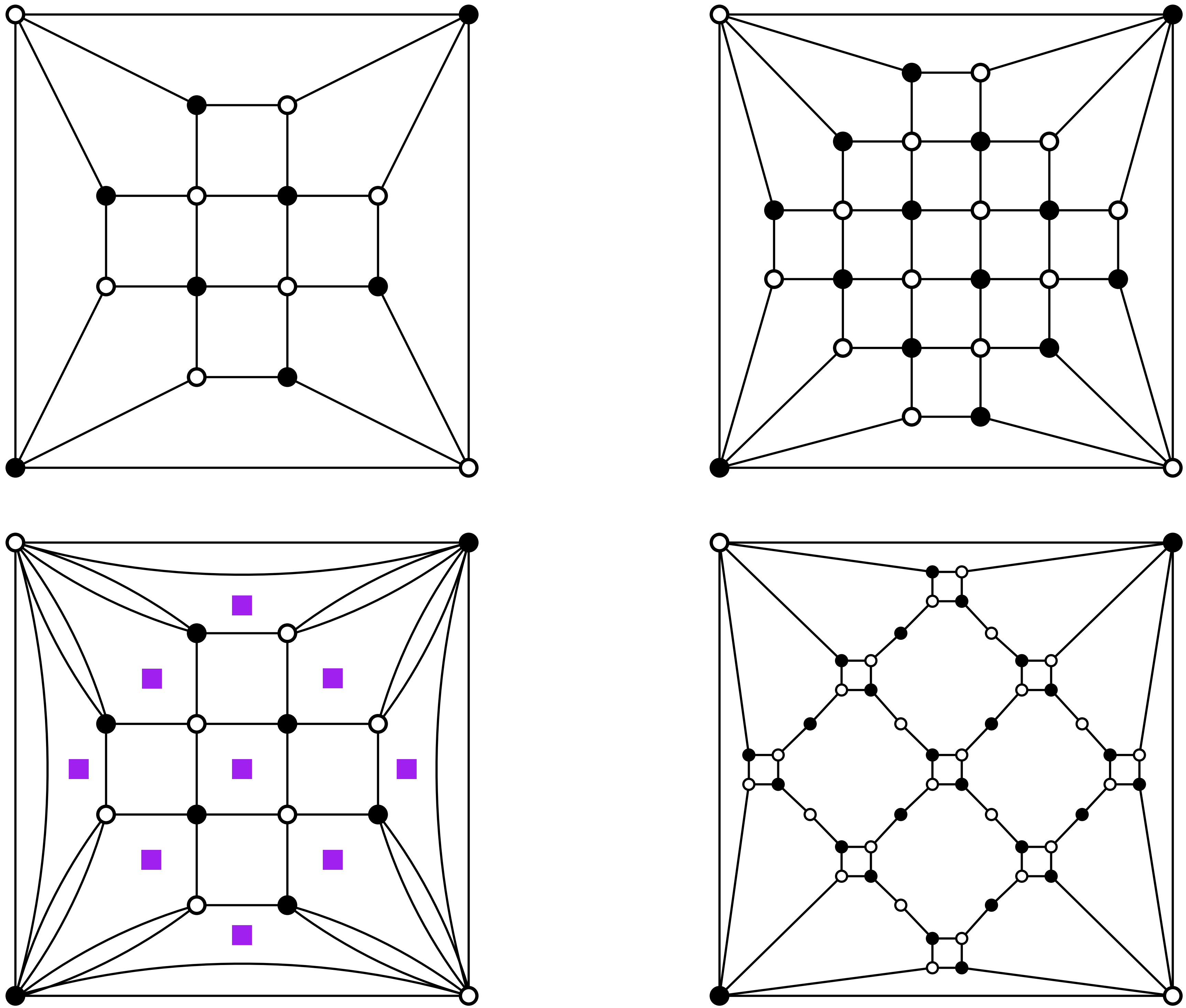


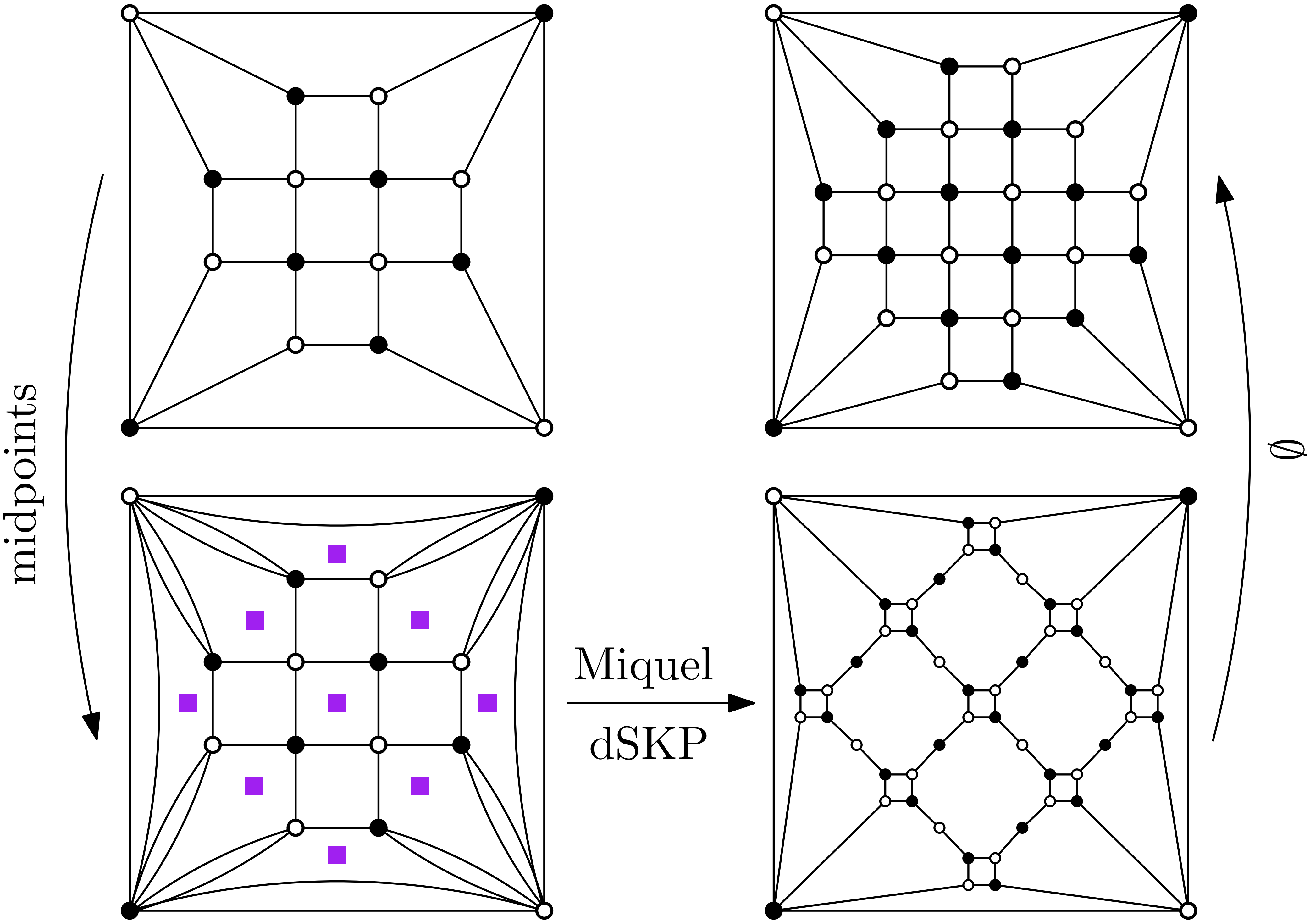


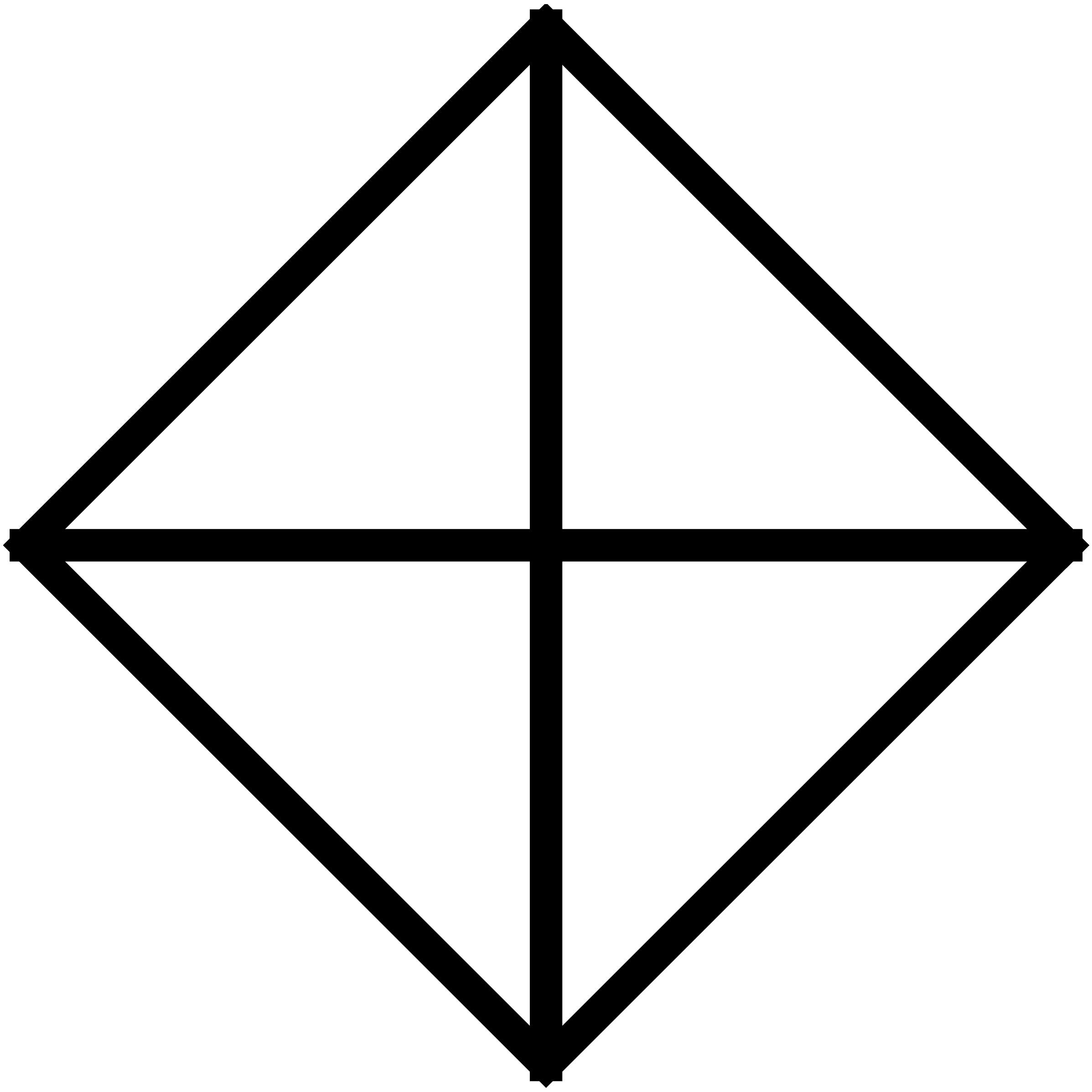




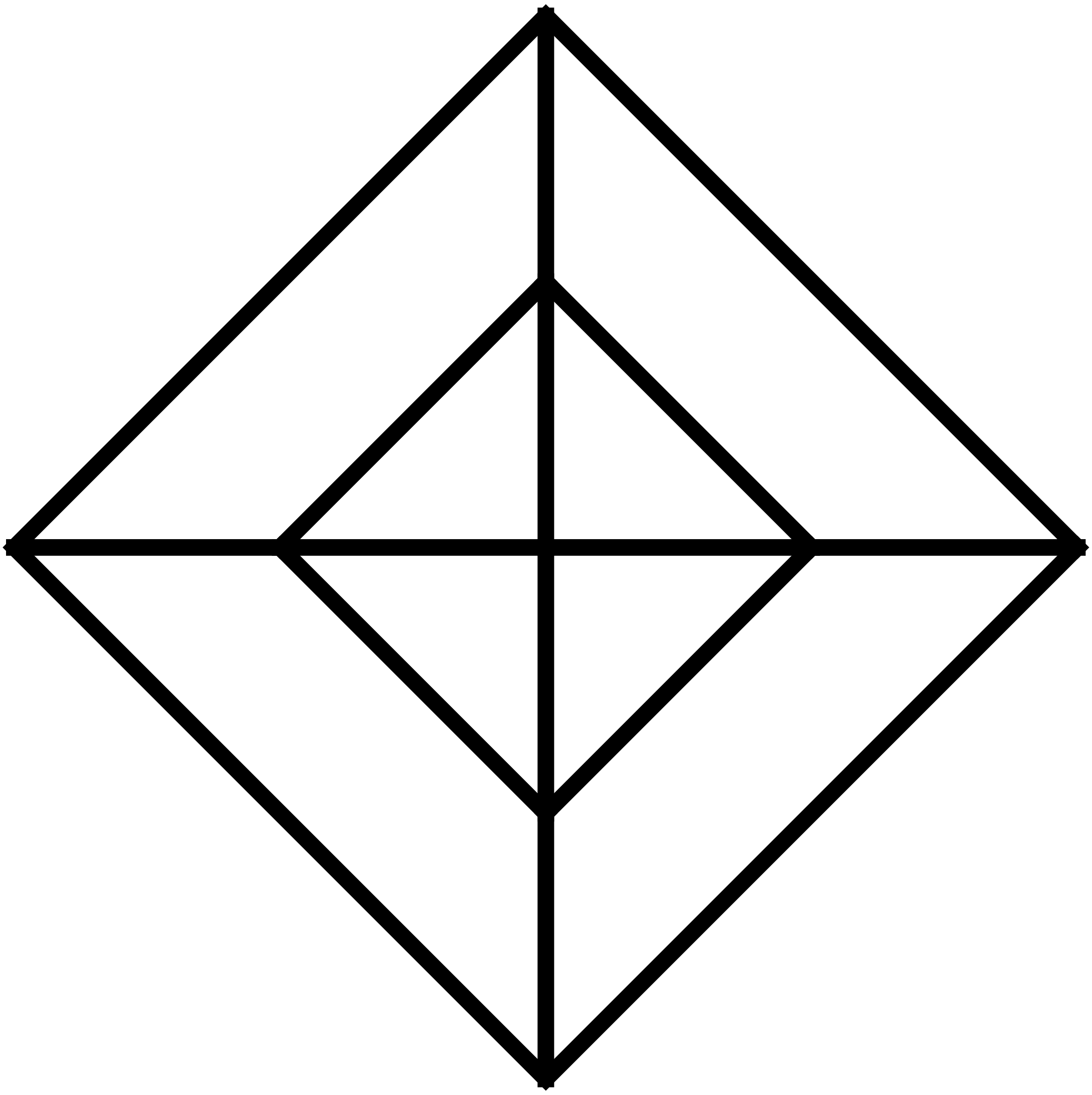




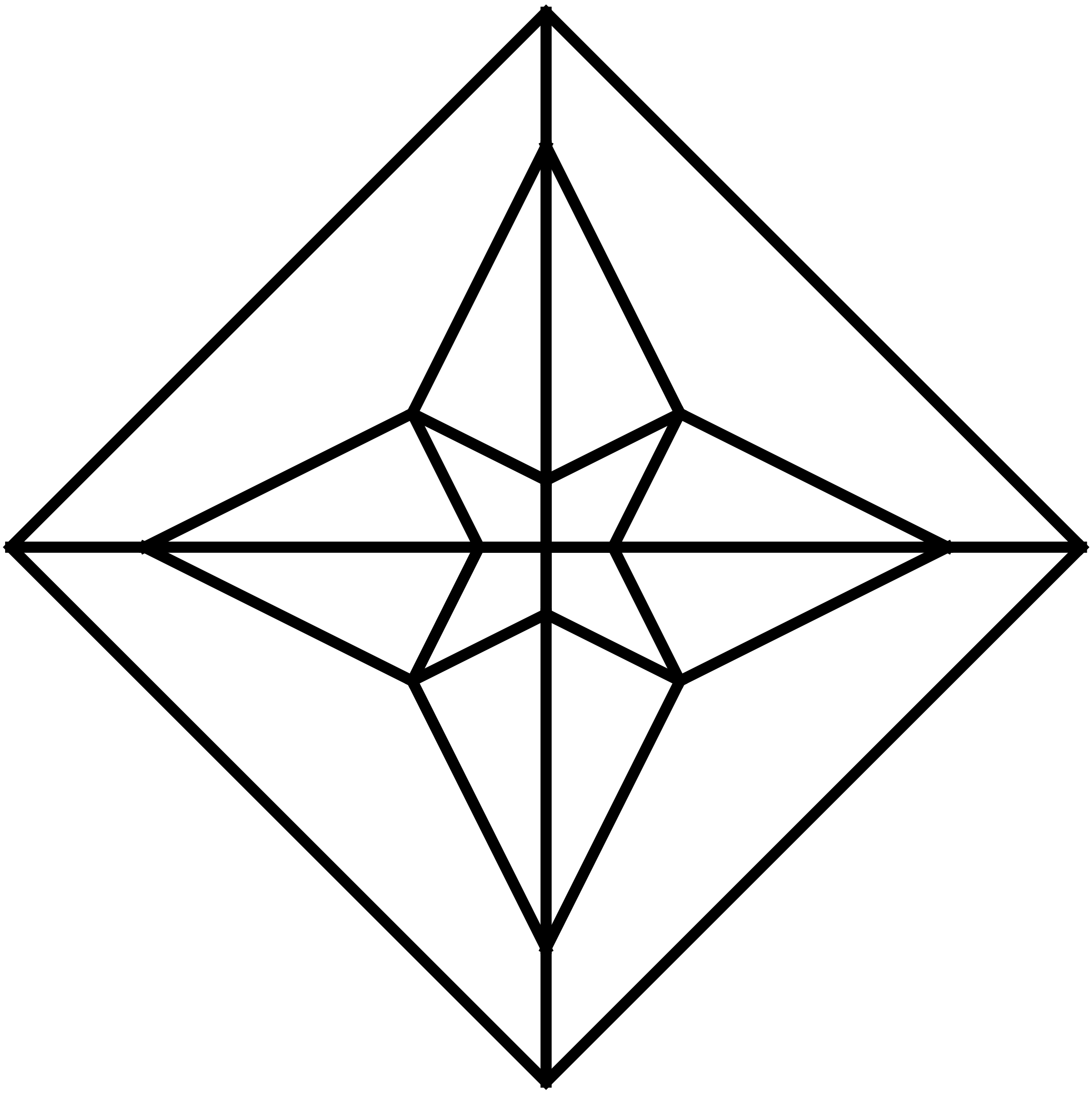

midpoints
Miquel
dSKP
∅



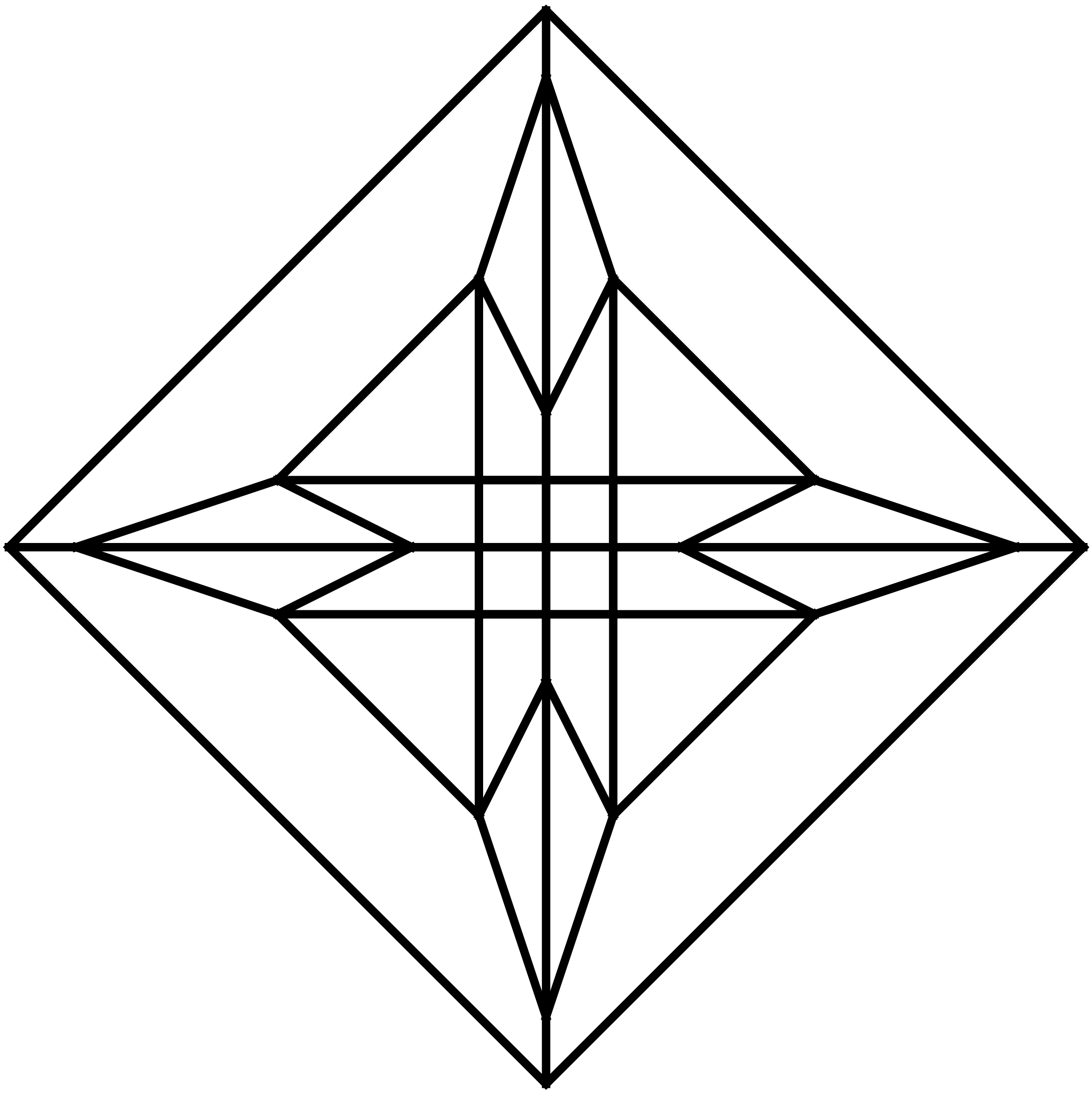
Size 2



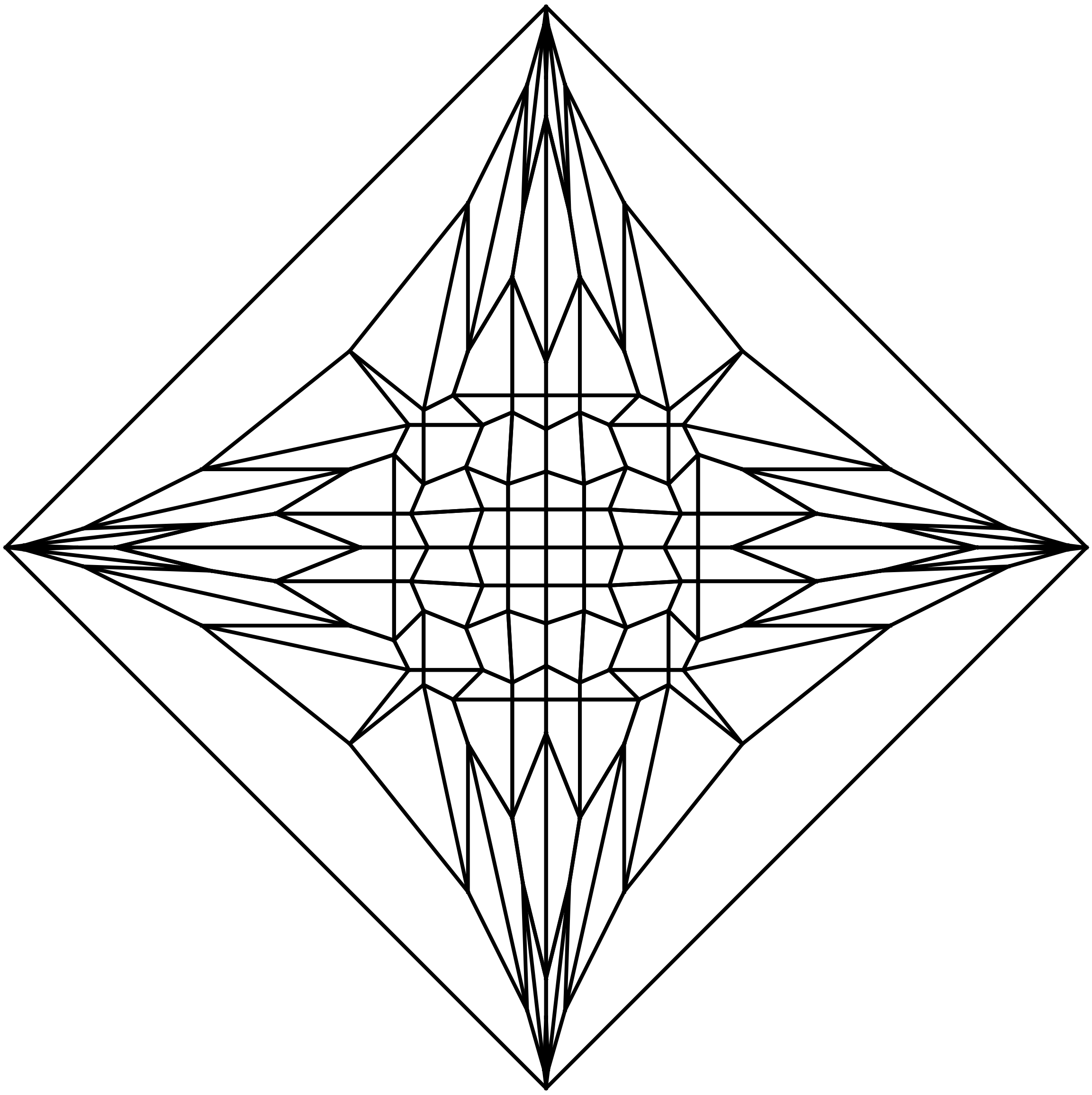
Size 3



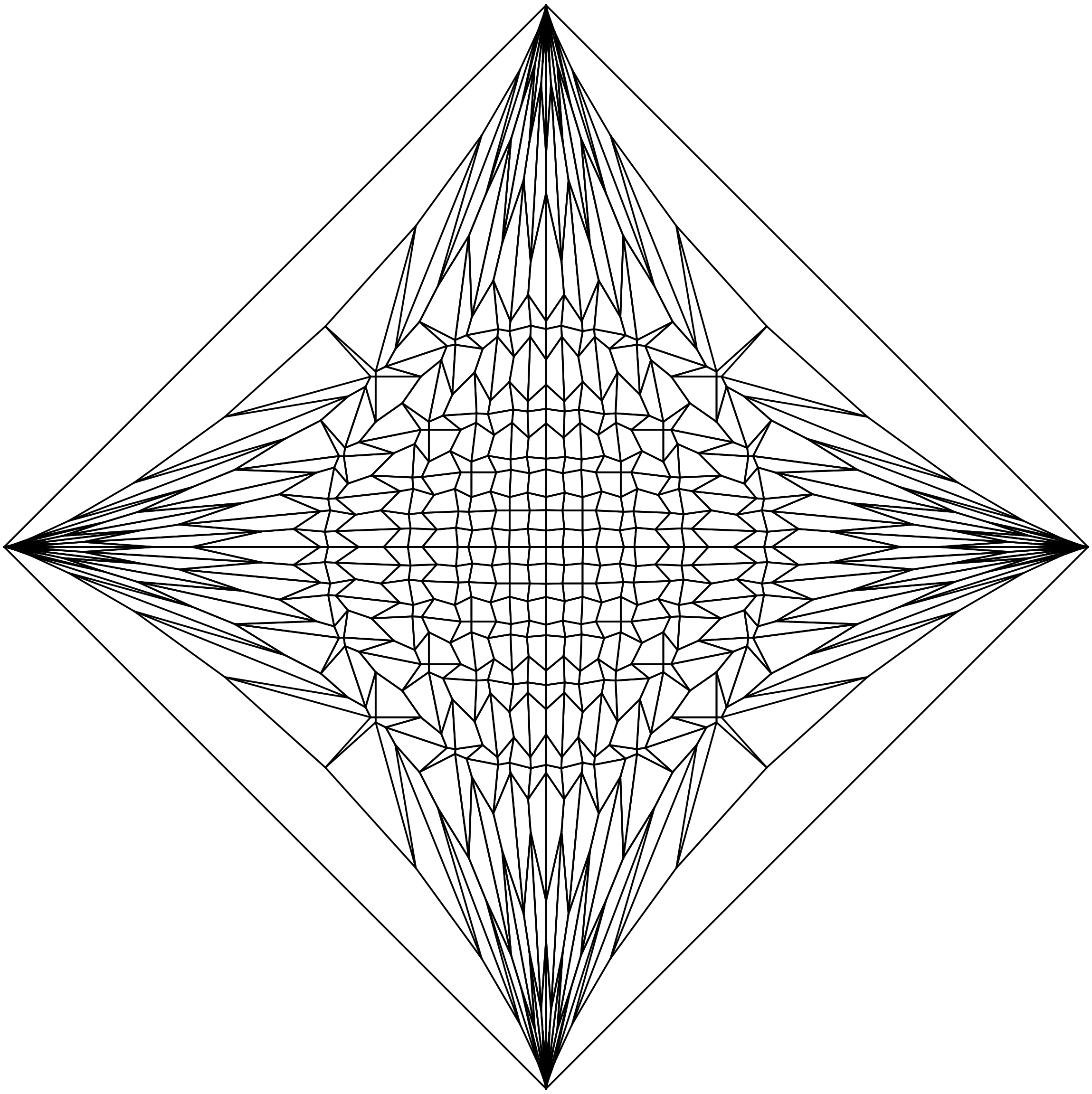
Size 4



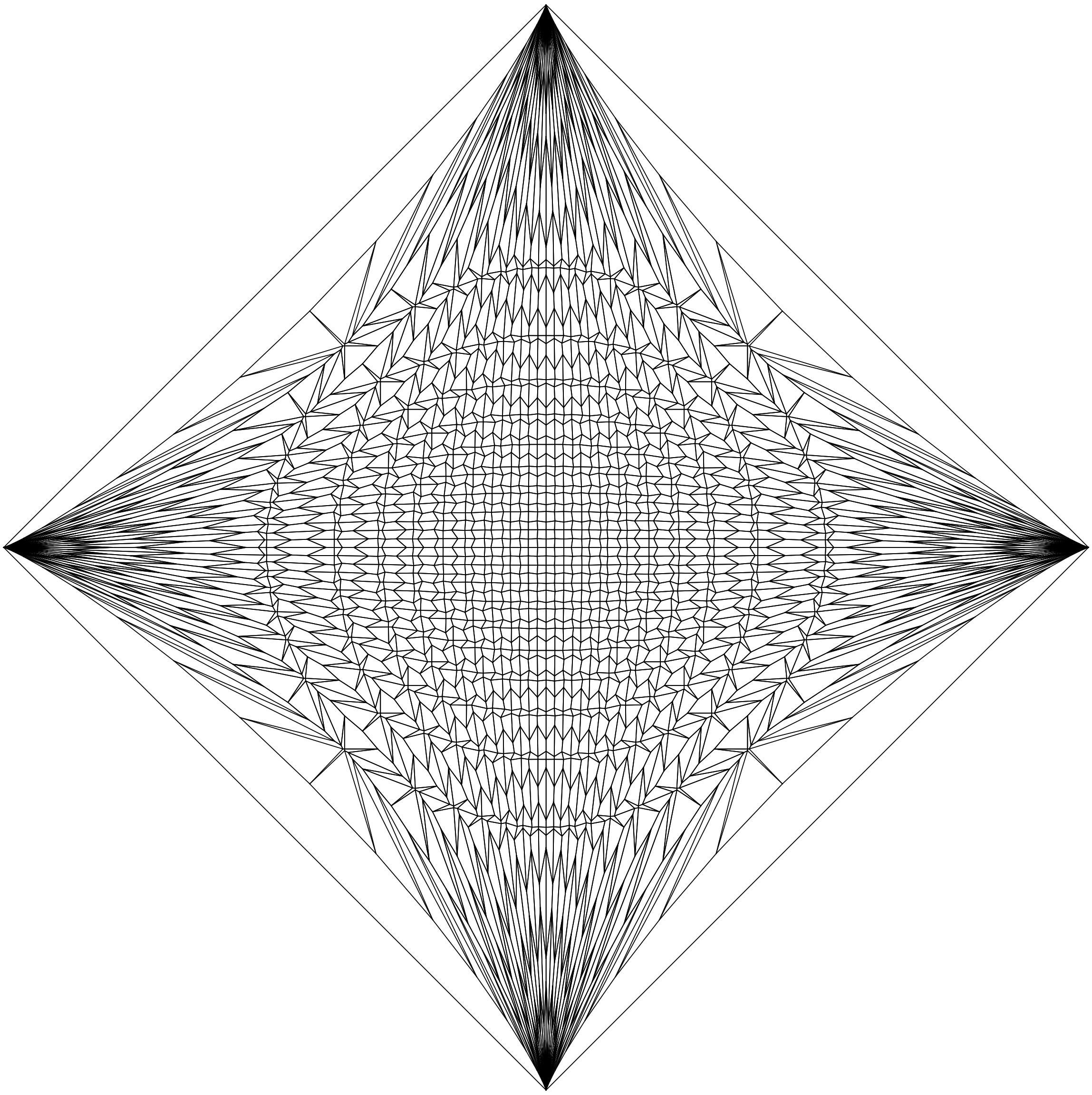
Size 5



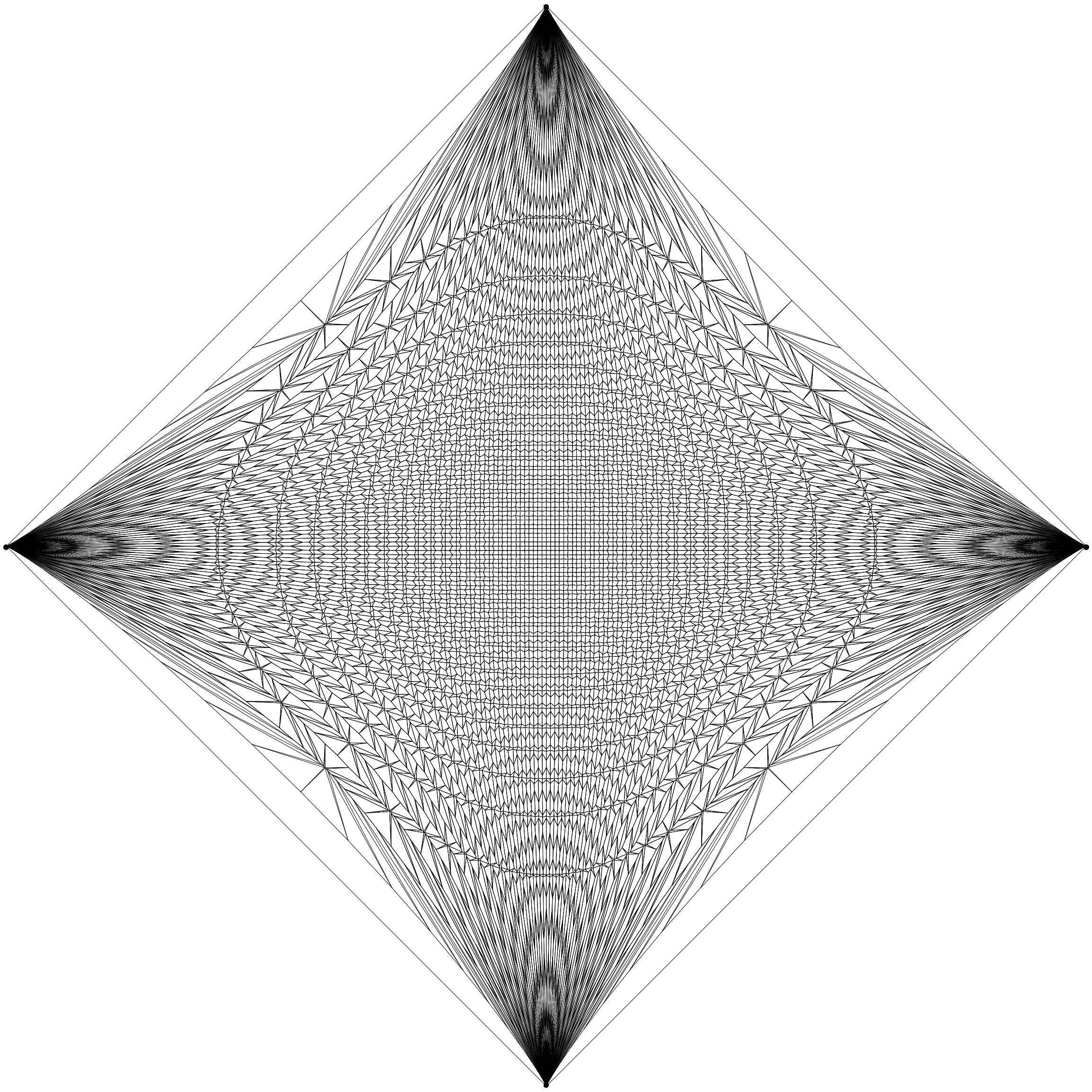
Size 10



Size 20

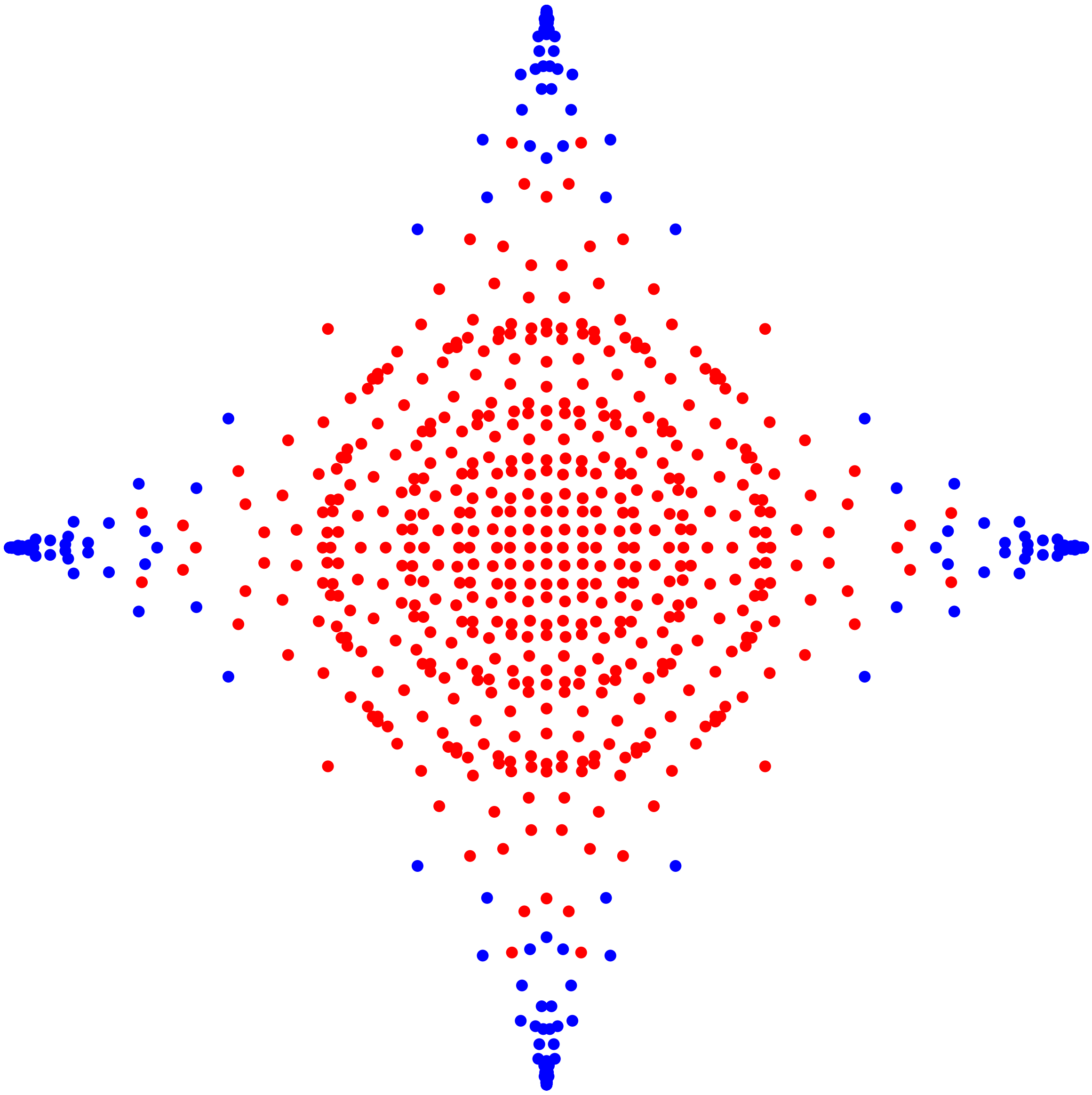


Size 40



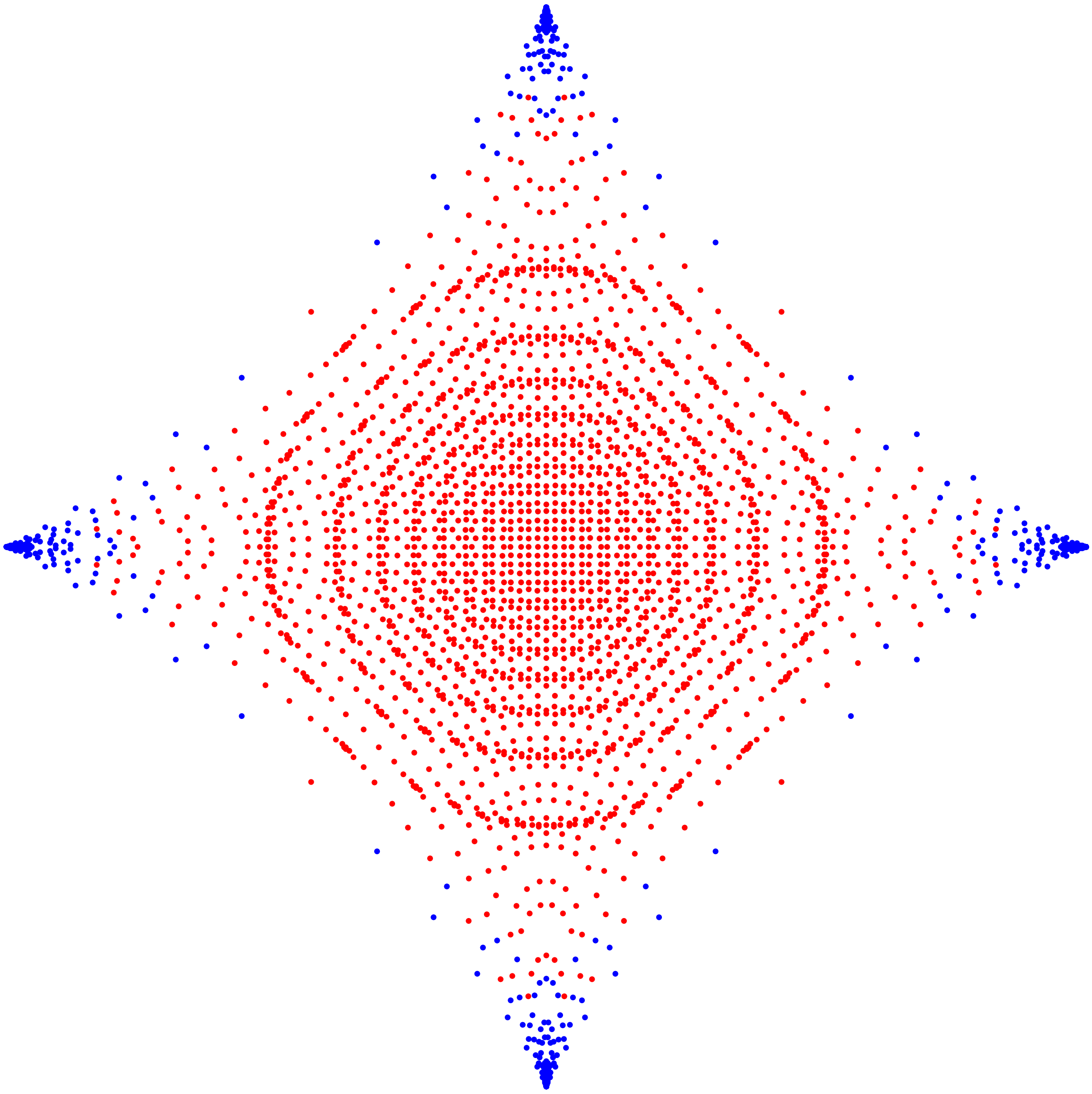
Size 80

The image of a dual vertex inside (resp. outside) the arctic circle is red (resp. blue).



Size 20

The image of a dual vertex
inside (resp. outside) the
arctic circle is red (resp. blue).



Size 40

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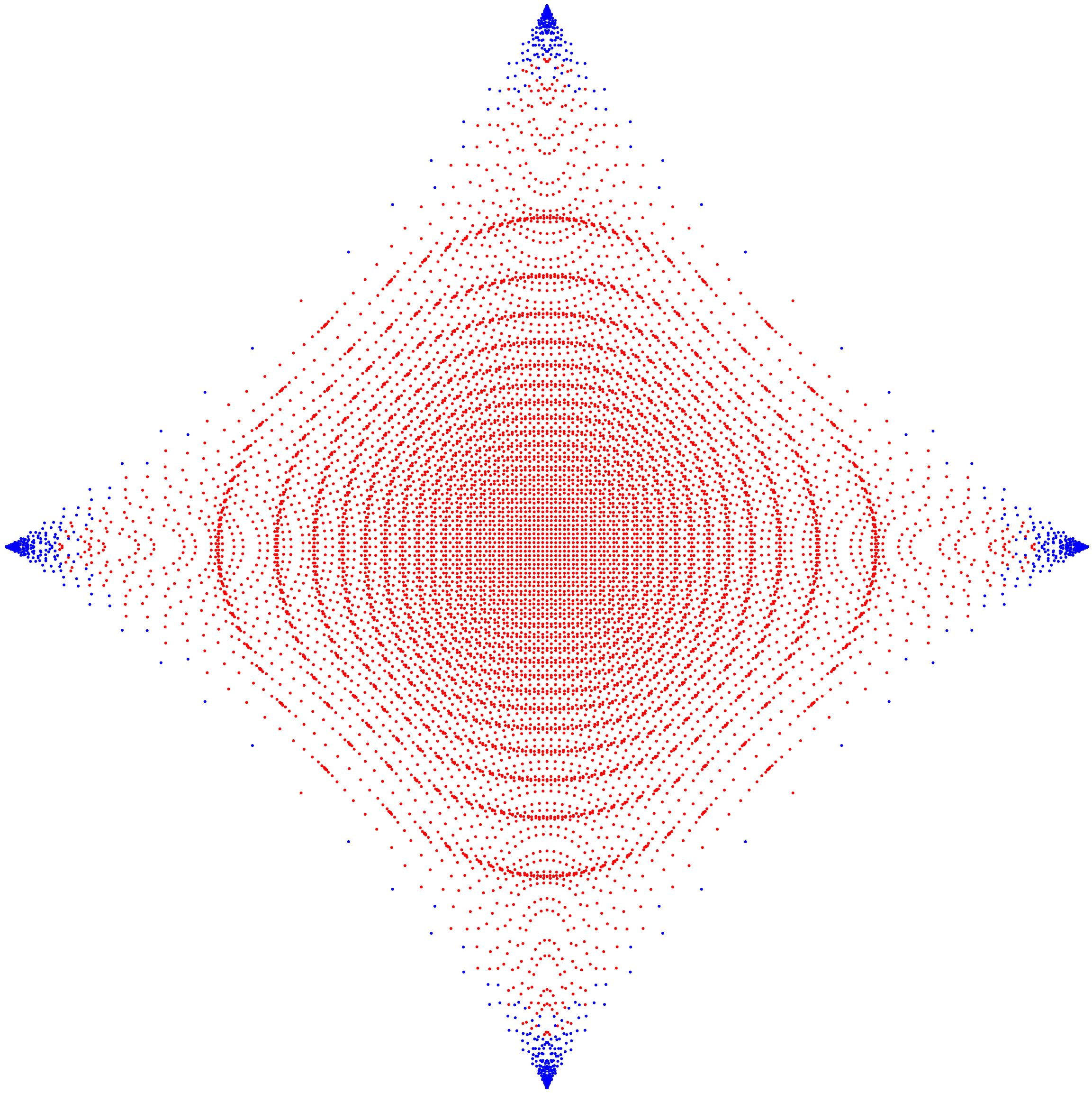
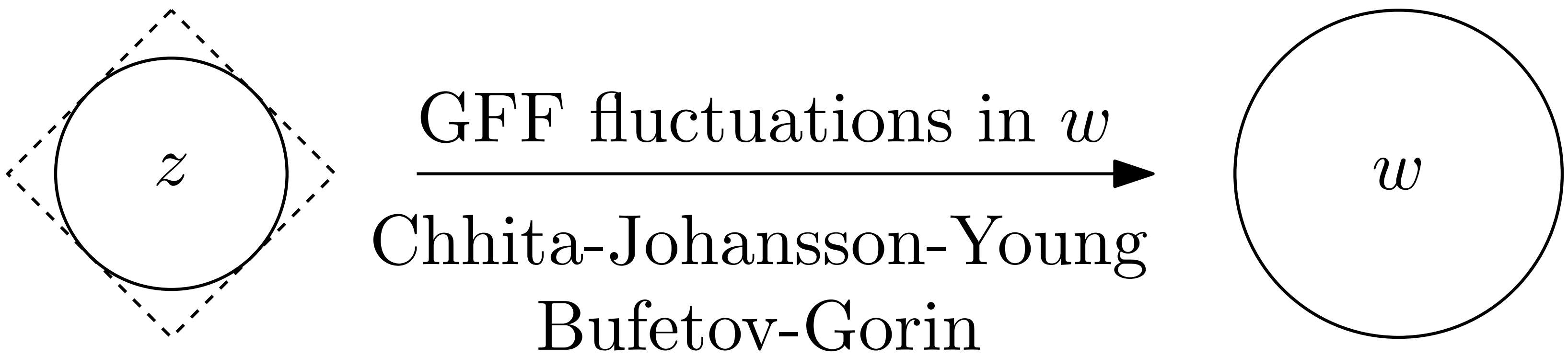


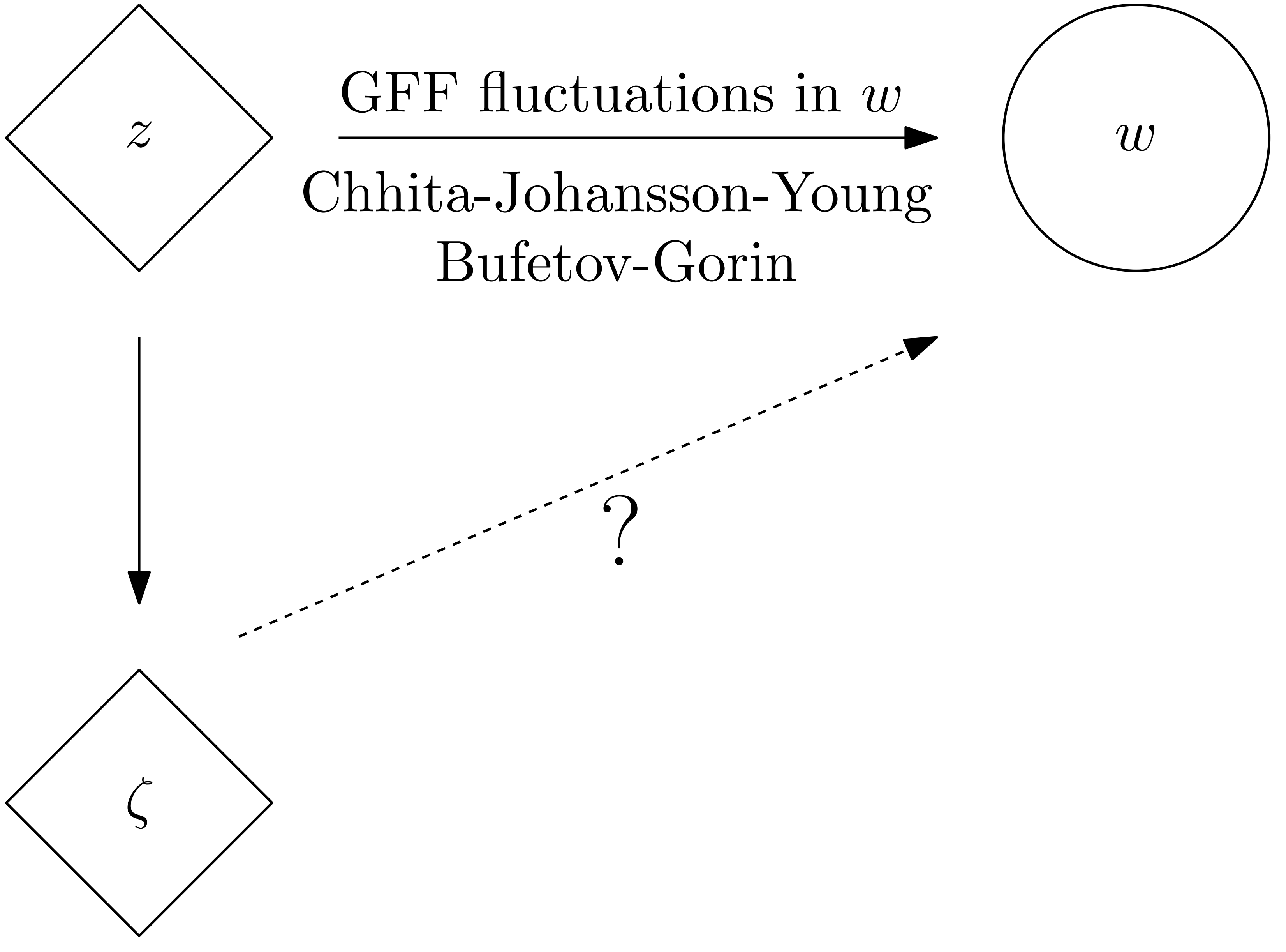
Size 80

- Expect convergence to a continuous map $z \mapsto \zeta$ from the unit square to itself.

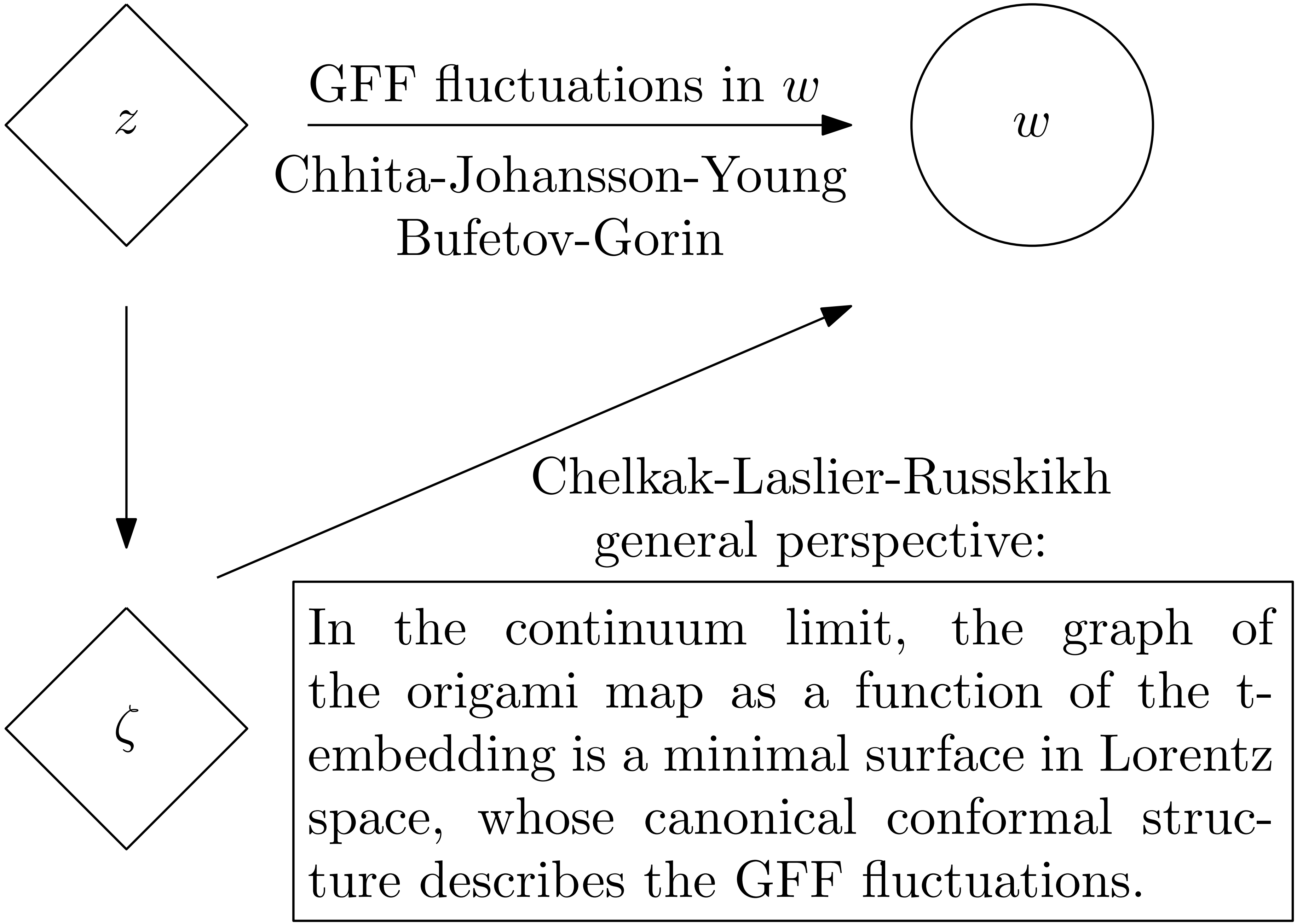
- Each frozen region is collapsed to a vertex of the square.

- The map ζ does <u>not</u> directly give the right conformal structure to describe the GFF fluctuations.


z
GFF fluctuations in w
Chhita-Johansson-Young
Bufetov-Gorin
w



z
GFF fluctuations in w
Chhita-Johansson-Young
Bufetov-Gorin
w
?
ζ


z
GFF fluctuations in w
w
Chhita-Johansson-Young
Bufetov-Gorin
ζ
Chelkak-Laslier-Russkikh
general perspective:
In the continuum limit, the graph of the origami map as a function of the t-embedding is a minimal surface in Lorentz space, whose canonical conformal structure describes the GFF fluctuations.

Conclusion

- Circle center embeddings (a.k.a. t-embeddings) for bipartite graphs are a particular choice of gauge.

- The angle condition for circle centers implies the Kasteleyn condition.

- Correspondence between embeddings as circle centers and Boltzmann/Gibbs measures for the dimer model for planar graphs with outer face of degree 4 and graphs periodic in two directions.

- Expected to provide the right geometric setting to study the scaling limit of the bipartite dimer model.

R. Kenyon, W. Y. Lam, S. Ramassamy and M. Russkikh, Dimers and circle patterns, arXiv:1810.05616 (2018).

N. Affolter, Miquel dynamics, Clifford lattices and the dimer model, arXiv:1808.04227 (2018).

D. Chelkak and S. Ramassamy, Limit theorems for the Aztec diamond via t-embeddings, in preparation.

D. Chelkak, B. Laslier and M. Russkikh, Dimer model and holomorphic functions on t-embeddings of planar graphs, in preparation.

THANK YOU !