Six-vertex and Ashkin-Teller models: order/disorder phase transition

Alexander Glazman

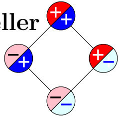
University of Fribourg

19th November 2019
– Dimers, Ising model and their interactions –

joint work with:
Ron Peled

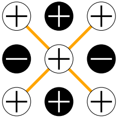
Structure of the talk

Ashkin–Teller model

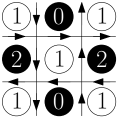


FK–Ising-type representation ξ

Six-vertex model



spins



heights

Proofs:

Input from the FK model

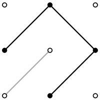
$\mathbb{T}$-circuits

Coupling $g(i,j) = 1 - f(i-1,j)$

Exp. decay of clusters in ξ^*

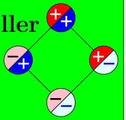
Baxter–Kelland–Wu coupling

FK model



FK model with boundary-cluster weight q_b

Ashkin–Teller model



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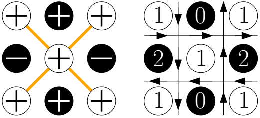

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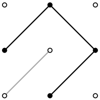

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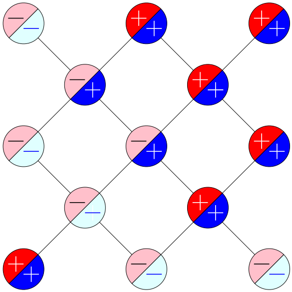
Ashkin–Teller model: definition



Finite domain $\Omega \subset \mathbb{Z}^2$ (box $N \times N$).

A pair of spin configurations: $\tau, \tau' \in \{+1, -1\}^{V(\Omega)}$.

Boundary conditions (free, $+$): $\tau = \tau'$ on $\partial\Omega$.

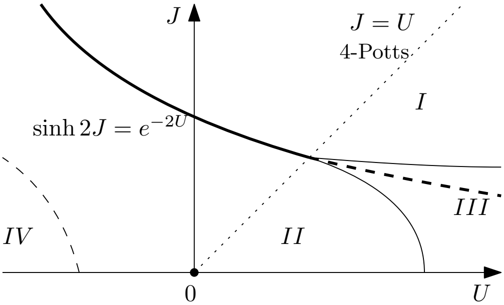
'43 Ashkin–Teller model with parameters $J, U \in \mathbb{R}$:

$$\mathsf{AT}^{\mathrm{free},+}_{\Omega,J,U} = \frac{1}{Z} \cdot \exp\left[\sum_{i\sim j} J(\tau_i\tau_j + \tau'_i\tau'_j) + U\tau_i\tau'_i\tau_j\tau'_j\right].$$

Case $U = 0$: two independent Ising models.

Question: ordering in τ, τ', $\tau\tau'$?

Ashkin–Teller model: conjectured phase diagram



$$\exp\left[\sum_{i\sim j} J(\tau_i\tau_j + \tau'_i\tau'_j) + U\tau_i\tau'_i\tau_j\tau'_j\right]$$

I: $\tau, \tau', \tau\tau'$ are ferromagnetically ordered;

II: $\tau, \tau', \tau\tau'$ are disordered;

III: τ, τ' are disordered, $\tau\tau'$ is ferromganetically ordered;

IV: τ, τ' are disordered, $\tau\tau'$ is anti-ferromganetically ordered.

Self-dual curve $\sinh \mathbf{2J} = \mathbf{e}^{-\mathbf{2U}}$ ($\rightarrow$ six-vertex model): critical only when $U < J$.

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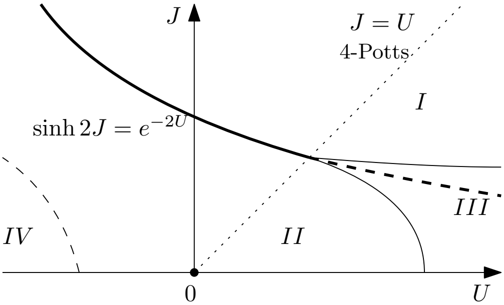

$\exp\left[\sum_{i\sim j} J(\tau_i\tau_j + \tau'_i\tau'_j) + U\tau_i\tau'_i\tau_j\tau'_j\right]$

Known results:

- three distinct regimes when $U/J >> 1$ **[Pfister '82]**
- $J > U$: sharp phase transition at the self-dual curve

[Duminil-Copin–Raoufi–Tassion '18]

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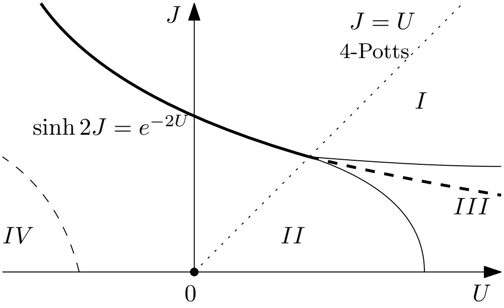

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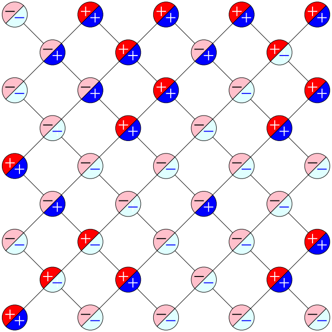
Theorem

Let $J < U$ be such that $\sinh 2J = e^{-2U}$. Then the weak limit $\mathrm{AT}^{\mathrm{free},+}_{J,U}$ under $(\mathrm{free},+)$ b.c. exists and exhibits exponential decay of correlations of τ (and τ') and ordering of the product $\tau\tau'$:

$$\mathrm{AT}^{\mathrm{free},+}_{J,U}(\tau_i\tau_j) \leq Ce^{-\alpha|i-j|} \qquad \mathrm{AT}^{\mathrm{free},+}_{J,U}(\tau_i\tau'_i\tau_j\tau'_j) \geq \delta,$$

for some $C, \alpha, \delta > 0$ depending on J, U.

Ashkin–Teller ↔ Six-vertex: coupling via duality



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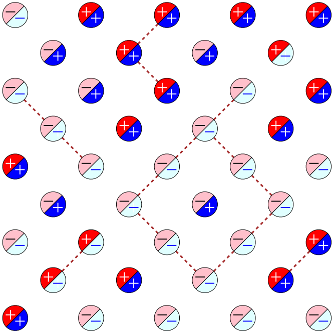

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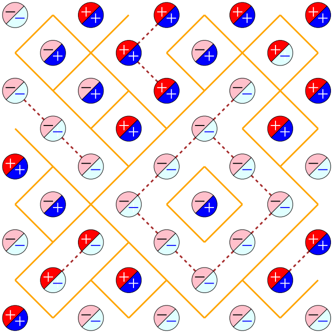

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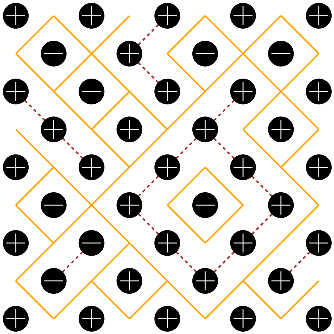

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Spin configurations $(\sigma^\bullet, \sigma^\circ)$:

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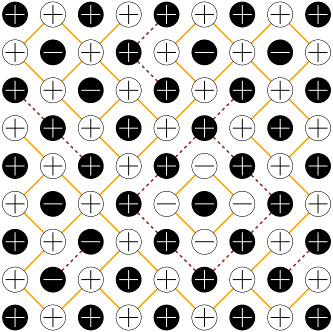

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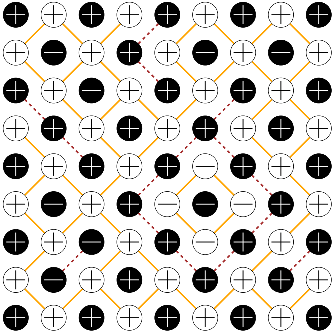

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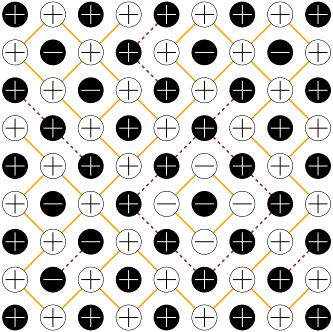

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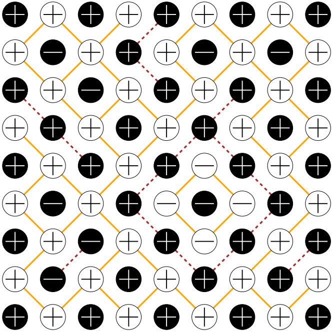

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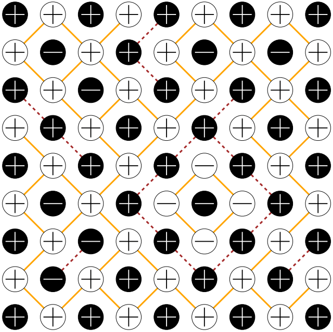

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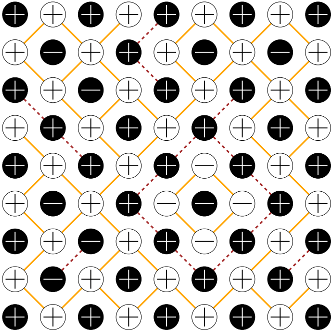

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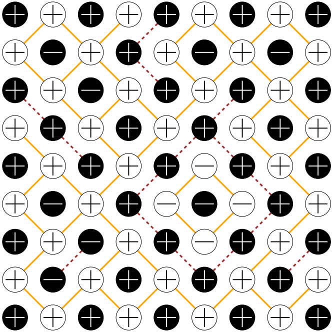

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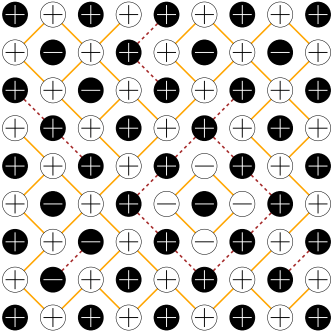

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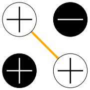

$\frac{2e^{2J-2U}}{e^{4J}-1} = 1$

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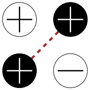

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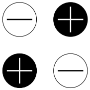
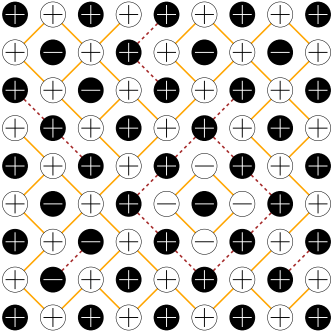
$\sigma^\bullet_i = \sigma^\bullet_j$
$\sigma^\circ_{i^*} = \sigma^\circ_{j^*}$



$\frac{e^{4J}+1}{e^{4J}-1} =: c$



$\mathrm{AT}^{\mathrm{free},+}_{\Omega,J,U} \propto \exp\left[\sum_{i\sim j} J(\tau_i\tau_j + \tau'_i\tau'_j) + U\tau_i\tau'_i\tau_j\tau'_j\right]$

FK–Ising-type representation ξ^*:

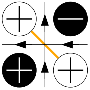
- if $\tau_i = \tau_j$ and $\tau'_i = \tau'_j$, then $ij \in \xi^*$ w.p. $1 - e^{-4J}$ and $ij \notin \xi^*$ w.p. e^{-4J}
- if $\tau_i \neq \tau_j$ or $\tau'_i \neq \tau'_j$, then $ij \notin \xi^*$;

Spin configurations $(\sigma^\bullet, \sigma^\circ)$:

- $\sigma^\bullet_i := \tau_i\tau'_i$;
- σ°(cluster of ξ) $= \pm 1$ indep. w.p. $1/2$.

$$\mathbb{P}(\sigma^\bullet, \sigma^\circ) \propto e^{(2J-2U)\#\{\sigma^\bullet_i \neq \sigma^\bullet_j\}} \sum_{\xi \perp \sigma^\bullet, \sigma^\circ} \left(\tfrac{2}{e^{4J}-1}\right)^{|\xi|}$$

$\sigma^\bullet_i \neq \sigma^\bullet_j$, $\sigma^\circ_{i^*} = \sigma^\circ_{j^*}$



$\frac{2e^{2J-2U}}{e^{4J}-1} = 1$

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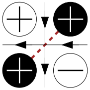

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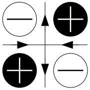

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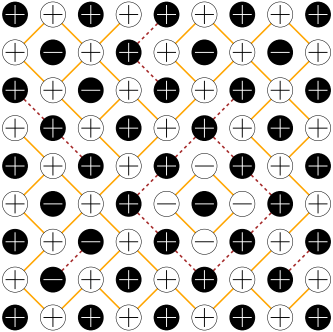
Ashkin–Teller $\leftrightarrow$ Six-vertex: coupling via duality



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$$\sum_{\xi \perp \sigma^\bullet,\sigma^\circ} \left(\frac{2}{e^{4J}-1}\right)^{|\xi|} \propto c^{\text{double-agreement}} \text{ [SixVertex]}$$

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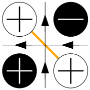

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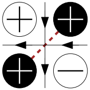

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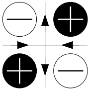
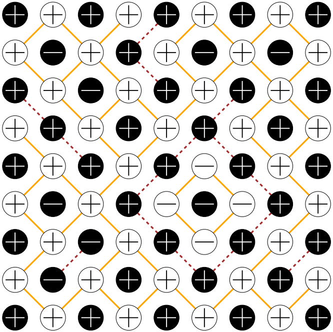


$\frac{e^{4J}+1}{e^{4J}-1} =: c$

Correlations in Ashkin–Teller via six-vertex and ξ

Fix $J < U$ such that $\sinh 2J = e^{-2U}$. Take $c := \frac{e^{4J}+1}{e^{4J}-1} > 2$.



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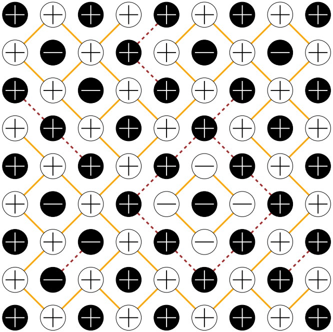

- To prove $\mathsf{AT}^{\mathrm{free},+}_{J,U}(\tau_i\tau'_i\tau_j\tau'_j) \geq \delta$:
  Since $\sigma^\bullet_i := \tau_i\tau'_i$, this is equivalent to:

$$\mathsf{SixV}_c(\sigma^\bullet_i\sigma^\bullet_j) \geq \delta.$$

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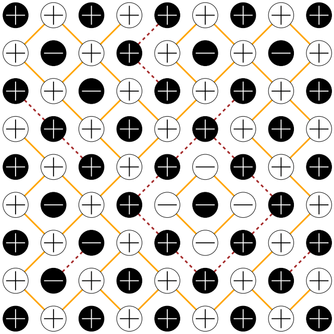

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  Enough to show

$$\mathsf{AT}^{\mathrm{free},+}_{J,U}(\tau_i\tau_j) = \mathbb{P}_c(i \stackrel{\xi^*}{\longleftrightarrow} j) \leq Ce^{-\alpha|i-j|}.$$

We saw: $\mathsf{SixV}_c(\sigma^\circ(i)\sigma^\circ(j)) = \mathbb{P}_c(i \stackrel{\xi}{\longleftrightarrow} j)$.

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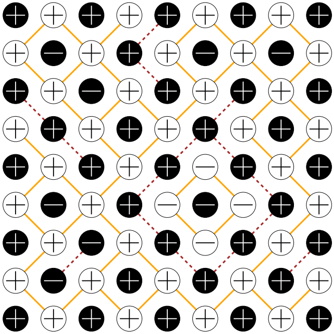

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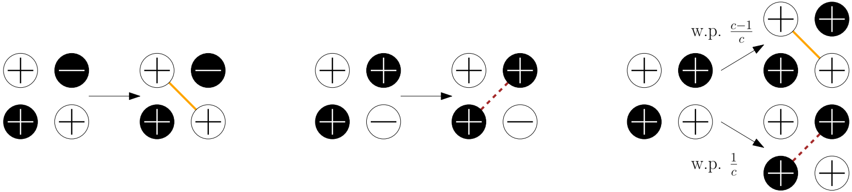
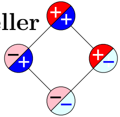
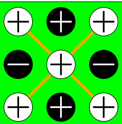


Ashkin–Teller model

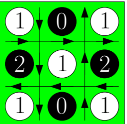


FK–Ising-type representation ξ

Six-vertex model



spins



heights

Proofs:

Input from the FK model

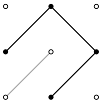
$\mathbb{T}$-circuits

Coupling $g(i, j) = 1 - f(i-1, j)$

Exp. decay of clusters in ξ^*

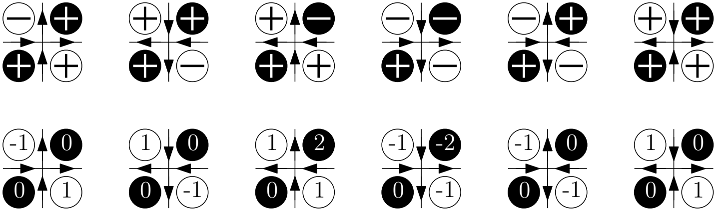
Baxter–Kelland–Wu coupling

FK model



FK model with boundary-cluster weight q_b

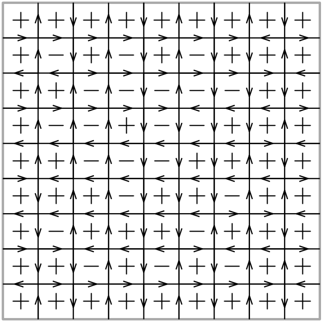
Six-vertex model

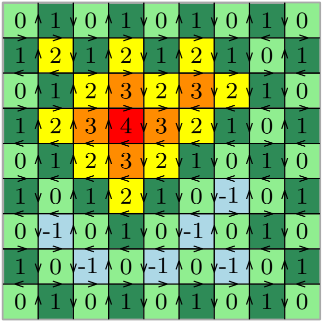


1 1 1 1 c c

Six-vertex model:

$$\propto c^{\#\left\{\substack{\text{double}\\ \text{agreement}}\right\}}$$





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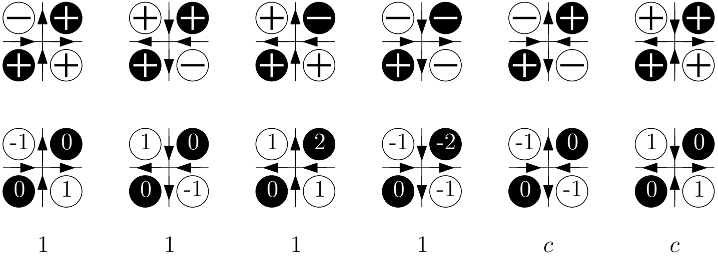

Six-vertex model:

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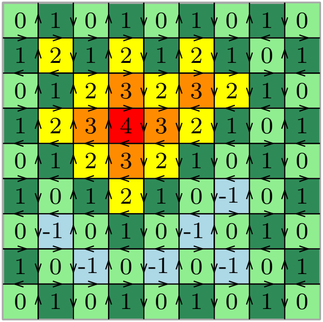
Known results:

- $h \to$ GFF : $c = \sqrt{2}$ (dimers) **[Kenyon '00]**, $c \approx \sqrt{2}$ **[Giuliani–Mastropietro–Toninelli '16]**
- log. fluctuations: $c = 1$ **[Sheffield '05]**, **[Chandgotia–Peled–Sheffield–Tassy '18]**, **[Duminil-Copin–Harel–Laslier–Raoufi–Ray '18]**
- free energy: $c > 2$ **[Duminil-Copin–Gagnebin–Harel–Manolescu–Tassion '16]**



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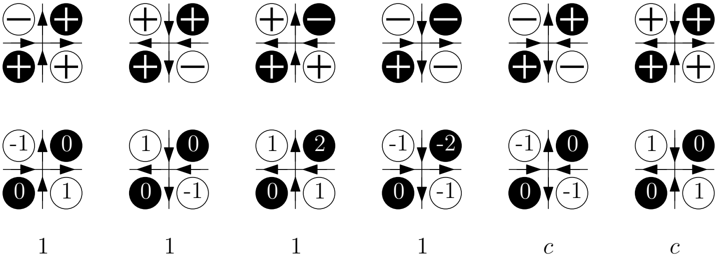


Six-vertex model:

$\propto c^{\#\left\{\substack{\text{double}\\ \text{agreement}}\right\}}$

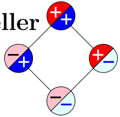
Theorem

- **Order when** $c > 2$: *convergence of measures with* $0/1$ *b.c., all extremal transl.-inv. Gibbs measures can be obtained as limits under* $n/n+1$ *b.c., for some* n; ξ *has an infinite cluster with logarithmically small holes.*
- **Disorder when** $c = 2$: *logarithmic variations of heights, no extremal transl.-inv. Gibbs measures; spin measures under* $+$ *and* $-$ *b.c. are the same.*
- **FKG**: *marginals* $\sigma^\bullet$ *and* σ° *when* $c \geq 1$; *marginal on* ξ *when* $c \geq 2$.

When $c = 2$, ξ coincides with the critical FK configuration at $q = 4$.

Part 3: Baxter–Kelland–Wu coupling six-vertex $\leftrightarrow$ FK

Ashkin–Teller model



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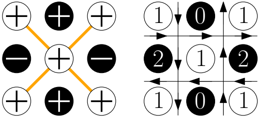

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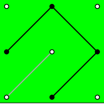
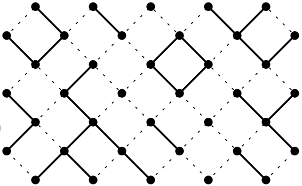


FK model with boundary-cluster weight q_b

Random-cluster (Fortuin–Kasteleyn) model

Parameters $p \in (0,1)$, $q > 0$, finite graph G (box on $\mathbb{Z}^2$), edge-config. $\omega \in \{\text{open}, \text{closed}\}^{E(G)}$:

$$\mathsf{FK}_{G,q,p}(\omega) = \frac{1}{Z} \cdot p^{\#\text{open}(\omega)} (1-p)^{\#\text{closed}(\omega)} q^{\#\text{cluster}(\omega)}$$



Wired b.c.: all boundary points are identified.
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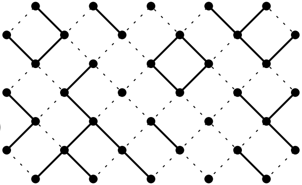

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- $q \geq 1$, phase transition at $p_c = \frac{\sqrt{q}}{\sqrt{q}+1}$: infinite cluster exists if $p > p_c$ and does not if $p < p_c$ **[Beffara–Duminil-Copin '12]**

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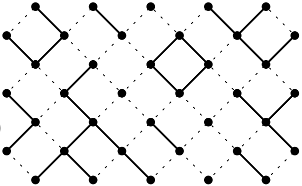

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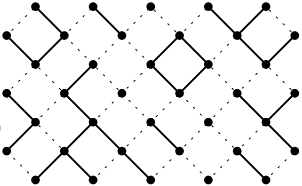

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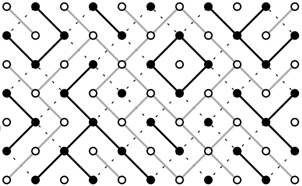

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Six-vertex $\leftrightarrow$ random-cluster: Baxter–Kelland–Wu coupling

$h \sim$ height function on $\mathcal{D}$ with $0/1$ b.c. and parameter $c = e^{\lambda/2} + e^{-\lambda/2} \geq 2$
$\eta \sim$ critical FK config. on $\mathcal{D}^\bullet$ with free b.c. and parameter $q = [e^{\lambda} + e^{-\lambda}]^2 \geq 4$

Proposition

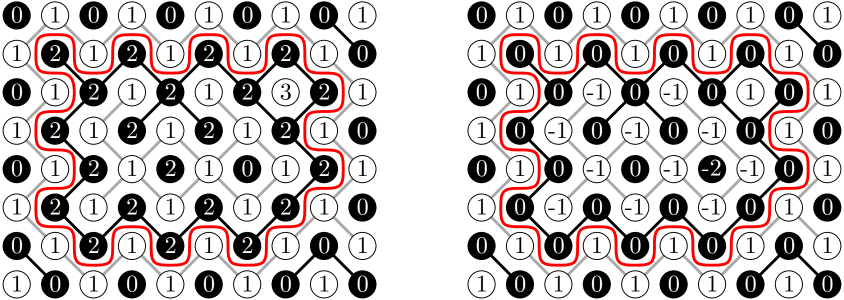
Variables h and η can be coupled in such a way that h **is constant on clusters of η and η^***. *The joint law can be written in either of the two following ways:*

$$(h,\eta) \sim \exp\left[\lambda \sum_{\mathcal{C}\sim\mathcal{C}^* \text{ clusters}} (h(\mathcal{C}^*) - h(\mathcal{C}))(-1)^{\mathbb{1}_{\mathcal{C} \text{ inside } \mathcal{C}^*}}\right], \tag{1}$$

$$(h,\eta) \sim \exp\left[\tfrac{\lambda}{4} \sum_{i\sim j \text{ black}} (h(i) + h(j) - h(i^*) - h(j^*))(-1)^{\mathbb{1}_{ij \text{ open}}}\right]. \tag{2}$$

$q = [e^{\lambda} + e^{-\lambda}]^2 \geq 4$. Coupling — h **is constant on clusters of** η **and** η^*,

$$(h, \eta) \sim \exp\left[\lambda \sum_{\mathcal{C} \sim \mathcal{C}^* \text{ clusters}} (h(\mathcal{C}^*) - h(\mathcal{C}))(-1)^{\mathbb{1}_{\mathcal{C} \text{ inside } \mathcal{C}^*}}\right] \quad (1)$$



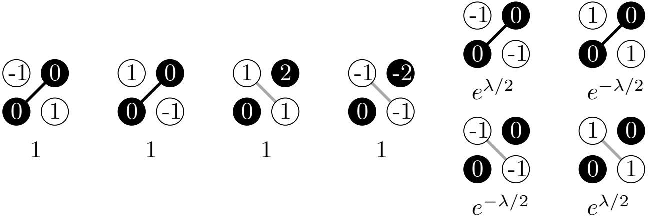
(1): clusters of η or η^* contribute $e^{\lambda} + e^{-\lambda}$; use $k(\eta^*) - k(\eta) \sim |\eta|$, $\frac{p_c}{1-p_c} = \sqrt{q}$:

$$\sum_{h \perp \eta, \eta^*} (1) \propto \sqrt{q}^{k(\eta)+k(\eta^*)} = \sqrt{q}^{k(\eta^*)-k(\eta)} \sqrt{q}^{2k(\eta)} \propto \left(\frac{p_c}{1-p_c}\right)^{\#\text{open}(\eta)} q^{k(\eta)}$$

$$\propto p_c^{\#\text{open}(\eta)} (1 - p_c)^{\#\text{closed}(\eta)} q^{\#\text{clusters}(\eta)}.$$

$c = e^{\lambda/2} + e^{-\lambda/2} \geq 2$. Coupling — h **is constant on clusters of** η **and** η^*,

$$(h, \eta) \sim \exp\left[\frac{\lambda}{4} \sum_{i \sim j \text{ black}} (h(i) + h(j) - h(i^*) - h(j^*))(-1)^{\mathbb{1}_{ij \text{ open}}}\right] \qquad (2)$$



(2): ij contributes $e^{\lambda/2} + e^{-\lambda/2}$ if $h(i) = h(j)$ and $h(i^*) = h(j^*)$ and 1, otherwise.

Six-vertex $\leftrightarrow$ random-cluster: boundary weights

$h \sim$ height function on $\mathcal{D}$ with $0/1$ b.c. and parameter $c = e^{\lambda/2} + e^{-\lambda/2} \geq 2$
$\eta \sim$ critical FK config. on $\mathcal{D}^\bullet$ with free b.c. and parameter $q = [e^{\lambda} + e^{-\lambda}]^2 \geq 4$

Proposition

Variables h and η can be coupled in such a way that h **is constant on clusters of η and η^***. *The joint law can be written in either of the two following ways:*

$$(h,\eta) \sim \exp\left[\lambda \sum_{\mathcal{C}\sim\mathcal{C}^* \text{ clusters}} (h(\mathcal{C}^*) - h(\mathcal{C}))(-1)^{\mathbb{1}_{\mathcal{C} \text{ inside } \mathcal{C}^*}}\right], \tag{1}$$

$$(h,\eta) \sim \exp\left[\frac{\lambda}{4} \sum_{i\sim j \text{ black}} (h(i) + h(j) - h(i^*) - h(j^*))(-1)^{\mathbb{1}_{ij \text{ open}}}\right]. \tag{2}$$

(1): clusters of η or η^* contribute $e^{\lambda} + e^{-\lambda}$; use $k(\eta^*) - k(\eta) \sim |\eta|$, $\frac{p_c}{1-p_c} = \sqrt{q}$
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Marginal of (1) on η is FK. Marginal of (2) on h is six-vertex. (1)$\propto$(2): by hands.

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Doesn't work on the boundary!

Six-vertex $\leftrightarrow$ random-cluster: boundary weights

$h \sim$ height f-n on $\mathcal{D}$ with 0/1 b.c., $c = e^{\lambda/2} + e^{-\lambda/2}$, $c_b = e^{\lambda/2}$ on $\partial\mathcal{D}$
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Proposition

Variables h and η can be coupled in such a way that h **is constant on clusters of η and η^***. *The joint law can be written in either of the two following ways:*

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Doesn't work on the boundary!

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$h \sim$ height f-n on $\mathcal{D}$ with 0/1 b.c., $c = e^{\lambda/2} + e^{-\lambda/2}$, $c_b = e^{\lambda/2}, c$ on $\partial\mathcal{D}$
$\eta \sim$ critical FK on $\mathcal{D}^\bullet$ with free b.c., $q = [e^\lambda + e^{-\lambda}]^2$, $q_b = 1, e^{-\lambda}\sqrt{q}$ on $\partial\mathcal{D}^\bullet$

Proposition

Variables h and η can be coupled in such a way that h **is constant on clusters of η and η^***. *The joint law can be written in either of the two following ways:*

$$(h,\eta) \sim \exp\left[\lambda \sum_{\mathcal{C}\sim\mathcal{C}^* \text{ clusters}} (h(\mathcal{C}^*) - h(\mathcal{C}))(-1)^{\mathbb{1}_{\mathcal{C}\text{ inside }\mathcal{C}^*}}\right], \quad (1)$$

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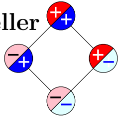
Marginal of (1) on η is FK. Marginal of (2) on h is six-vertex. (1)$\propto$(2): by hands.
Doesn't work on the boundary! $\lambda \leftrightarrow -\lambda$: **[Ray–Spinka '19]**

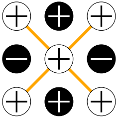
Part 4: FK model with boundary-cluster weight c_b

Ashkin–Teller model

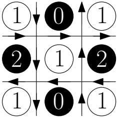


FK–Ising-type representation ξ

Six-vertex model



spins



heights

Proofs:

Input from the FK model

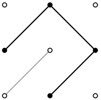
$\mathbb{T}$-circuits

Coupling $g(i,j) = 1 - f(i-1,j)$

Exp. decay of clusters in ξ^*

Baxter–Kelland–Wu coupling

FK model

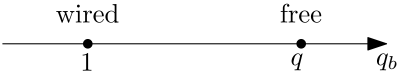


FK model with boundary-cluster weight q_b

FK model with weight $q_b \in [1, q]$ for boundary clusters

Parameters $p \in (0,1)$, $q, q_b > 0$, finite $G \subset \mathbb{Z}^2$, $\omega \in \{\text{open}, \text{closed}\}^{E(G)}$:

$$\mathsf{FK}^{q_b}_{G,q,p}(\omega) = p^{\#\text{open}(\omega)}(1-p)^{\#\text{closed}(\omega)} q^{\#\text{bulk-clusters}(\omega)} q_b^{\#\partial\text{-clusters}(\omega)}$$



Fix $q \geq 1$. Measures $\mathsf{FK}^{q_b}_{G,q,p}$ are stochastically ordered when $q_b \in [1, q]$, interpolating between wired ($q_b = 1$) and free ($q_b = q$) b.c.

FK model with weight $q_b \in [1, q]$ for boundary clusters

Parameters $p \in (0,1)$, $q, q_b > 0$, finite $G \subset \mathbb{Z}^2$, $\omega \in \{\mathrm{open}, \mathrm{closed}\}^{E(G)}$:

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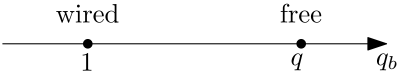

Fix $q \geq 1$. Measures $\mathsf{FK}^{q_b}_{G,q,p}$ are stochastically ordered when $q_b \in [1, q]$, interpolating between wired ($q_b = 1$) and free ($q_b = q$) b.c.

- If $p \neq p_c$, then the infinite-volume limit does not depend on q_b.
- Same at $p = p_c$ when $q \in [1, 4]$.
- Measure dual to $\mathsf{FK}^{q_b}_{G,q,p_c}$ is $\mathsf{FK}^{q_b^*}_{G^*,q,p_c}$ with $q_b^* = q/q_b$.

FK model with weight $q_b \in [1, q]$ for boundary clusters

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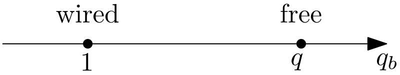

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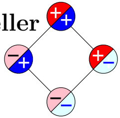
- Measure dual to $\mathsf{FK}^{q_b}_{G,q,p_c}$ is $\mathsf{FK}^{q_b^*}_{G^*,q,p_c}$ with $q_b^* = q/q_b$.

Theorem

Take $q = [e^{\lambda} + e^{-\lambda}]^2$, $\lambda > 0$. *Then* $\mathsf{FK}^{q_b}_{q,p_c} = \mathsf{FK}^{\text{wired}}_{q,p_c}$ *if* $q_b \in [1, e^{-\lambda}\sqrt{q}]$ *and* $\mathsf{FK}^{q_b}_{q,p_c} = \mathsf{FK}^{\text{free}}_{q,p_c}$ *if* $q_b \in [e^{\lambda}\sqrt{q}, q]$.

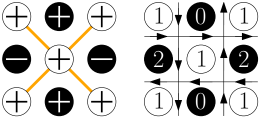
Conjecture: Phase transition at $q_b = \sqrt{q}$.

Ashkin–Teller model



FK–Ising-type representation ξ

Six-vertex model



spins

heights

Proofs:

Input from the FK model

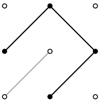
$\mathbb{T}$-circuits

Coupling $g(i,j) = 1 - f(i-1,j)$

Exp. decay of clusters in ξ^*

Baxter–Kelland–Wu coupling

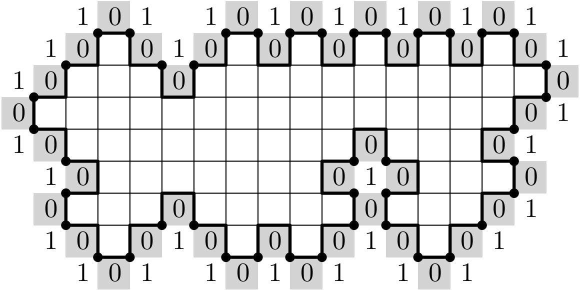
FK model



FK model with boundary-cluster weight q_b

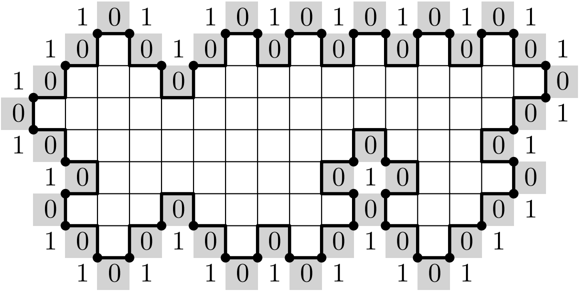
Proof, step 1: building on the FK model

- Russo–Seymour–Welsh theory at p_c when $q = 4$ implies logarithmic fluctuations of the height function at $c = 2$
- If $q > 4, p = p_c$, wired b.c. $\Rightarrow$ infinite cluster with log. small holes $\Rightarrow$
  - uniformly bdd fluctuations of height functions when $c = e^{\lambda/2} + e^{-\lambda/2}$, $\lambda > 0$
  - under $0, 1$ b.c. if $\mathcal{D}_k$ is a sequence of **even** domains, then height-function measures with $c_b = e^{\lambda/2}$ converge, the limit $\mathrm{HF}^{0,1;e^{\lambda/2}}_{c,\mathrm{even}}$ is **extremal, transl. inv., has an infinite cluster of height** 0, with logarithmically small holes
  - $\mathrm{HF}^{0,1;e^{\lambda/2}}_{c,\mathrm{odd}}$ is defined similarly, **infinite cluster of height** 1



Proof, step 2: $\mathsf{HF}_{c,\mathrm{even}}^{0,1;e^{\lambda/2}} \preceq \mathsf{HF}_{c,\mathrm{odd}}^{0,1;e^{\lambda/2}}$

$c = e^{\lambda/2} + e^{-\lambda/2}$. Let $\mathcal{D}$ be an even domain.



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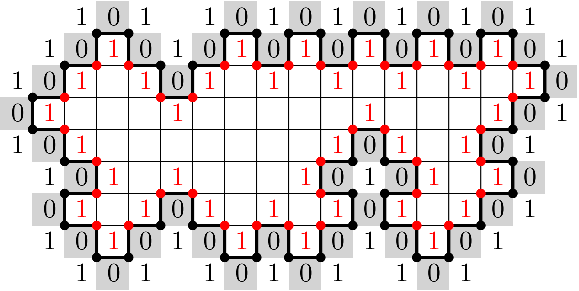
$c = e^{\lambda/2} + e^{-\lambda/2}$. Let $\mathcal{D}$ be an even domain.
Positive association of heights $\Rightarrow$ imposing height 1 on $\partial\mathcal{D}$ increases the measure:

$$\mathsf{HF}_{c,\mathcal{D}}^{0,1;e^{\lambda/2}} \preceq \mathsf{HF}_{c,\mathcal{D}\setminus\partial\mathcal{D}}^{0,1;c}$$

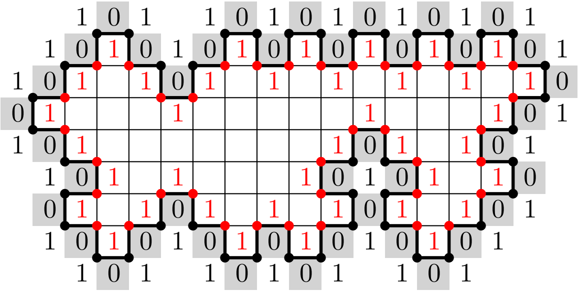


$c = e^{\lambda/2} + e^{-\lambda/2}$. Let $\mathcal{D}$ be an even domain.
Positive association of heights $\Rightarrow$ imposing height 1 on $\partial\mathcal{D}$ increases the measure:

$$\mathrm{HF}^{0,1;e^{\lambda/2}}_{c,\mathcal{D}} \preceq \mathrm{HF}^{0,1;c}_{c,\mathcal{D}\setminus\partial\mathcal{D}}$$

Domain $\mathcal{D} \setminus \partial\mathcal{D}$ is odd. Monotonicity in the boundary parameter c_b:

$$\mathrm{HF}^{0,1;c}_{c,\mathcal{D}\setminus\partial\mathcal{D}} \preceq \mathrm{HF}^{0,1;e^{\lambda/2}}_{c,\mathcal{D}\setminus\partial\mathcal{D}}$$



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Positive association of heights $\Rightarrow$ imposing height 1 on $\partial\mathcal{D}$ increases the measure:

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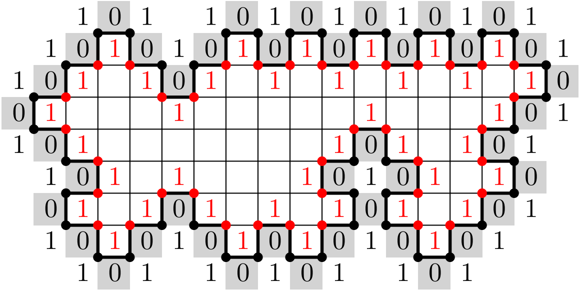
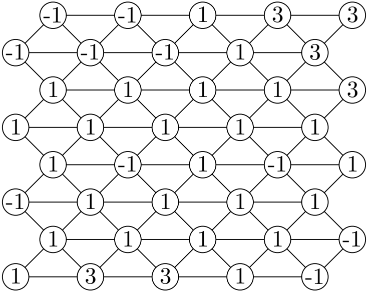


Proof, step 3.1: $\mathsf{HF}^{0,1;e^{\lambda/2}}_{c,\mathrm{even}} \succeq \mathsf{HF}^{0,1;e^{\lambda/2}}_{c,\mathrm{odd}}$

Consider $\mathbb{T}^\circ$: each odd site (i,j) is linked to $(i, j \pm 1)$, $(i \pm 1, j)$, $(i \pm 2, j)$.
This is a triangular lattice. By duality and extremality, one of the following holds:

$$\mathsf{HF}^{0,1;e^{\lambda/2}}_{c,\mathrm{even}}(\text{around every box, exists a } \mathbb{T}^\circ\text{-circuit of height} \geq 1) = 1 \tag{3}$$

$$\mathsf{HF}^{0,1;e^{\lambda/2}}_{c,\mathrm{even}}(\text{exists an infinite } \mathbb{T}^\circ\text{-cluster of height} \leq -1) = 1 \tag{4}$$



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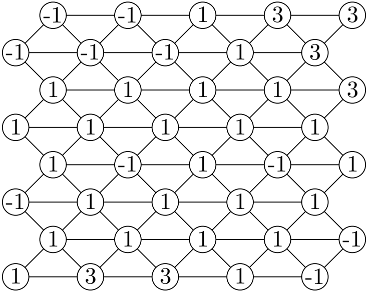
If (4) occurs, then the same holds for heights ≥ 1 (FKG for the heights).
Such coexistence is excluded **[Sheffiled '05], [Duminil-Copin–Raoufi–Tassion '18]**



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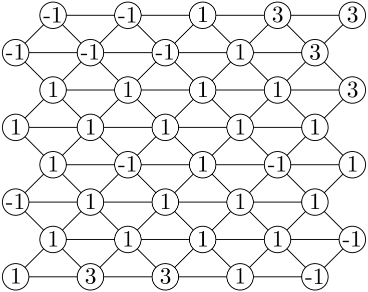
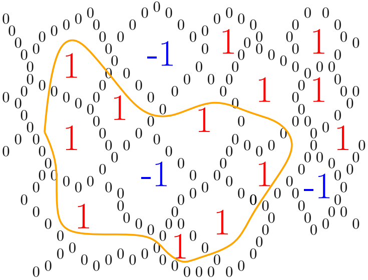
$$\mathsf{HF}^{0,1;e^{\lambda/2}}_{c,\mathrm{even}}(\text{exists an infinite } \mathbb{T}^\circ\text{-cluster of height} \leq -1) = 1 \qquad (4)$$

If (4) occurs, then the same holds for heights ≥ 1 (FKG for the heights).
Such coexistence is excluded **[Sheffiled '05], [Duminil-Copin–Raoufi–Tassion '18]**
Hence (3) occurs. Modifying locally, obtain $\mathbb{T}^\circ$-circuits of height 1.





Proof, step 3.2: $\mathsf{HF}^{0,1;e^{\lambda/2}}_{c,\mathrm{even}} \succeq \mathsf{HF}^{0,1;e^{\lambda/2}}_{c,\mathrm{odd}}$

Similarly, for $\mathsf{HF}^{0,1;e^{\lambda/2}}_{c,\mathrm{odd}}$ and $\mathbb{T}^{\bullet}$-circuits of height 0. We get:

$$\mathsf{HF}^{0,1;e^{\lambda/2}}_{c,\mathrm{even}}(\text{around every box, exists a } \mathbb{T}^{\circ}\text{-circuit of height } 1) = 1$$
$$\mathsf{HF}^{0,1;e^{\lambda/2}}_{c,\mathrm{odd}}(\text{around every box, exists a } \mathbb{T}^{\bullet}\text{-circuit of height } 0) = 1$$

Couple $f \sim \mathsf{HF}^{0,1;e^{\lambda/2}}_{c,\mathrm{even}}$ and $g \sim \mathsf{HF}^{0,1;e^{\lambda/2}}_{c,\mathrm{odd}}$ in the following way:

Proof, step 3.2: $\mathsf{HF}^{0,1;e^{\lambda/2}}_{c,\text{even}} \succeq \mathsf{HF}^{0,1;e^{\lambda/2}}_{c,\text{odd}}$

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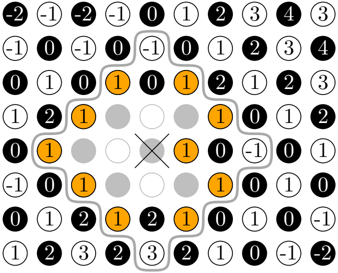
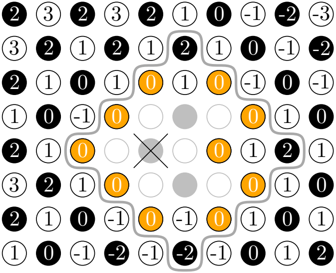
$$\mathsf{HF}^{0,1;e^{\lambda/2}}_{c,\text{odd}}(\text{around every box, exists a } \mathbb{T}^\bullet\text{-circuit of height } 0) = 1$$

Couple $f \sim \mathsf{HF}^{0,1;e^{\lambda/2}}_{c,\text{even}}$ and $g \sim \mathsf{HF}^{0,1;e^{\lambda/2}}_{c,\text{odd}}$ in the following way:

- outside of the outermost $\mathbb{T}^\bullet$-circuit of height 0 contained in a box $N \times N$:

$$g(i,j) := 1 - f(i-1,j)$$





Proof, step 3.2: $\mathsf{HF}^{0,1;e^{\lambda/2}}_{c,\mathrm{even}} \succeq \mathsf{HF}^{0,1;e^{\lambda/2}}_{c,\mathrm{odd}}$

Similarly, for $\mathsf{HF}^{0,1;e^{\lambda/2}}_{c,\mathrm{odd}}$ and $\mathbb{T}^\bullet$-circuits of height 0. We get:

$$\mathsf{HF}^{0,1;e^{\lambda/2}}_{c,\mathrm{even}}(\text{around every box, exists a } \mathbb{T}^\circ\text{-circuit of height } 1) = 1$$

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Couple $f \sim \mathsf{HF}^{0,1;e^{\lambda/2}}_{c,\mathrm{even}}$ and $g \sim \mathsf{HF}^{0,1;e^{\lambda/2}}_{c,\mathrm{odd}}$ in the following way:

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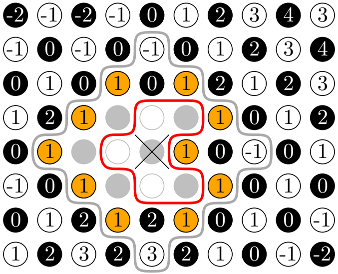
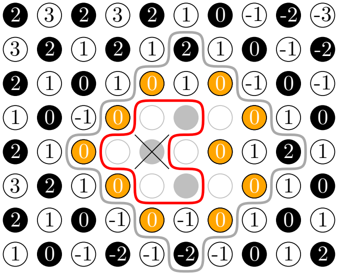
- inside of this circuit, f and g are independent.





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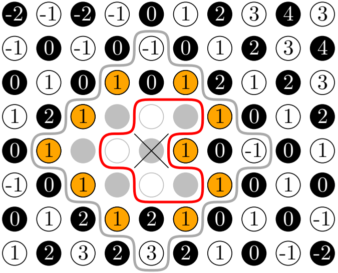
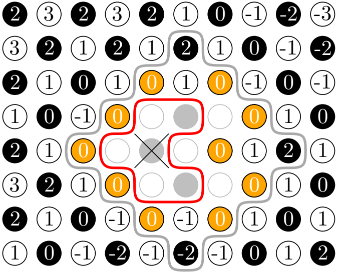


On the red domain: $f \succeq g$!



End of the proof: $\mathbb{P}(u \overset{\xi^*}{\longleftrightarrow} v) \leq e^{-\alpha|u-v|}$

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- ξ is FKG.
- If $p_n := \mathbb{P}(0 \overset{\xi^*}{\longleftrightarrow} \partial\Lambda_n)$, then $\left(\frac{p_n}{4n}\right)^{12} \leq \mathbb{P}(\Lambda_n \overset{\xi}{\not\longleftrightarrow} \infty) \leq e^{-\alpha n}$.

Open questions

- Intermediate behaviour of the Ashkin–Teller model on an interval of parameters.
- Phase transition of the FK-model in terms of boundary-cluster weight q_b.
- Phase transition of the six-vertex model in terms of the boundary weight c_b.
- Properties of the FK–Ising-type representation ξ, other b.c.