The Hull-Strominger system and holomorphic string algebroids

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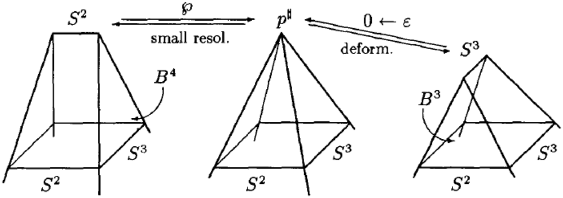
Bridging the gap between Kähler and non-Kähler complex geometry

Banff, 28th October 2019

Joint work with Rubio, Shahbazi, and Tipler arXiv:1803.01873, arXiv:1807.10329, and to appear.

The Hull-Strominger system and Reid's fantasy

In complex dimension three, a natural source of compact non-Kähler manifolds can be found via **surgeries in algebraic geometry** (*transitions* and *flops*)



Reid's Fantasy: *there could perhaps be a single moduli space of non-Kähler threefolds with trivial canonical bundle, such that the few thousand algebraic Calabi-Yau threefolds known at present arise as 'boundary phenomena' for the elements in this family*

- M. Reid, Math. Ann. **278** (1987) 329--334

X smooth projective, simply-connected, Calabi-Yau 3-fold with k embedded disjoint smooth rational curves $C = \bigcup_j C_j$, with normal bundle

$$\mathcal{O}_{\mathbb{CP}^1}(-1) \oplus \mathcal{O}_{\mathbb{CP}^1}(-1)$$

and $0 = \sum_j n_j [C_j] \in H^4(X, \mathbb{C})$, with $n_j \neq 0$.

Contracting C, we obtain a singular X_0 with double-point singularities, modelled locally on

$$\{z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0\} \subset \mathbb{C}^4.$$

There exists a *smoothing* $X_0 \rightsquigarrow X_t$ with trivial canonical bundle, which may be non-Kählerian (e.g. $\sharp_k (S^3 \times S^3)$ for any $k \geqslant 2$).

Example:

$$\begin{cases} (x_2^4 + x_4^4 - x_5^4) y_1 + (x_1^4 + x_3^4 + x_5^4) y_2 = 0 \\ x_1 y_1 + x_2 y_2 = 0 \end{cases} \to \ (x_2^4 + x_4^4 - x_5^4) x_2 - (x_1^4 + x_3^4 + x_5^4) x_1 = 0$$

$$\rightsquigarrow \ (x_2^4 + x_4^4 - x_5^4) x_2 - (x_1^4 + x_3^4 + x_5^4) x_1 = t \sum_{i=1}^{5} x_i^5 \rightsquigarrow \ \ldots \ \rightsquigarrow \ \sharp_{k \geqslant 2} (S^3 \times S^3)$$

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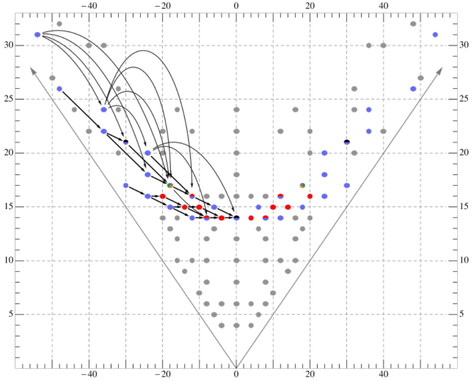
Strikingly, transitions (flops) can be regarded as a smooth process in string theory.

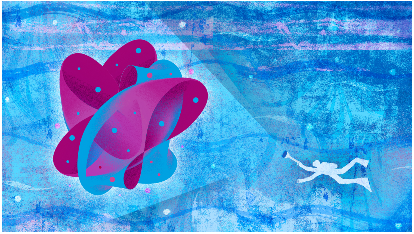
- P. Aspinwall, B. Greene, and D. Morrison, Nucl. Phys. B **416** (1994)
- B. Greene, D. Morrison, and A. Strominger, Nucl. Phys. B **451** (1995)



Question: What happens with the Calabi-Yau metric after the transition?

The exploration of this question has lead to important advances:

- V. Tosatti, J. Eur. Math. Soc. **11** (2009)
- Rong and Zhang, J. Differ. Geom. **89** (2011)
- J. Song, Commun. Math. Phys. 334 (2015)
- H. Hein, S. Sun, Publ. Math. IHES **126** (2017)



From ‘Quanta Magazine’

Nonetheless, these results use the full power of algebraic and Kähler geometry, and do not provide any understanding of the passage from Kähler to non-Kähler complex manifolds

$$z_1(z_1^4 + z_3^4 + z_5^4) - z_2(z_2^4 + z_4^4 - z_5^4) = 0 \rightsquigarrow \dots \rightsquigarrow \sharp_{k \geqslant 2}(S^3 \times S^3)$$

To solve this puzzle, S.-T. Yau has proposed the **Hull-Strominger system** of partial differential equations:

$$F \wedge \omega^2 = 0 \qquad F^{2,0} = F^{0,2} = 0$$

$$d(\|\Omega\|\omega^2) = 0 \qquad dd^c\omega + \operatorname{tr} R \wedge R - \operatorname{tr} F \wedge F = 0$$

These equations require an additional ingredient on top of our Calabi-Yau X: a holomorphic vector bundle E satisfying $c_1(E) = 0, c_2(E) = c_2(X)$.

- Very active topic of research in mathematics in the last 15 years: Yau, Li, Fu, Tseng, Fei, Fernandez, Ivanov, Ugarte, Villacampa, Fino, Vezzoni, Andreas, GF, Fei, Phong, Picard, Zhang, ...
- Two alternative approaches to the existence and uniqueness problem: *anomaly flow* and *dilaton functional*

- D.-H. Phong, S. Picard, and X. Zhang, Math. Z. (2017)
- Garcia-Fernandez, Rubio, Shahbazi, Tipler, arXiv:1803.01873 (2018)

Question: What happens with the Calabi-Yau metric after the transition?

In the context of heterotic string theory, physicists think of the Calabi-Yau metric g on X as a solution of ('standard embedding')

$$d^*\omega = d^c \log \|\Omega\|_\omega, \qquad R_g \wedge \omega^2 = 0$$
$$dd^c\omega = \operatorname{tr} R_g^2 - \operatorname{tr} R_g^2$$

After transitions, TX should produce on X_t a holomorphic bundle $V_t \to X_t$ with $c_2(X_t) = c_2(V_t)$ and a solution of the *Hull-Strominger* system:

$$F_{h_V} \wedge \omega^2 = 0, \qquad d^*\omega = d^c \log \|\Omega\|_\omega \qquad R_{h_{TX}} \wedge \omega^2 = 0$$
$$dd^c\omega = \operatorname{tr} R_{h_{TX}}^2 - \operatorname{tr} F_{h_V}^2$$

Expected: 'Hull-Strominger geometries' host some generalization of mirror symmetry, where role of Calabi-Yau manifolds is played by (naively) pairs (X, V).

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Gauge Theory and the Calabi problem

- In the 1950s, E. Calabi asked the question of whether one can prescribe the volume form of a Kähler metric g on a compact complex manifold X.

For metrics on a fixed Kähler class $[\omega_0] \in H^2(M, \mathbb{R})$, the *Calabi Problem* with smooth volume form μ reduces to the Complex Monge-Ampère equation

$$(\omega_0 + 2i\partial\bar{\partial}\varphi)^n = n!\mu$$

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In the 1970s, Yau solved the problem using the continuity method:

Theorem (Yau 1977)

Let X be a compact Kähler manifold with smooth volume μ. Then there exists a unique Kähler metric with $\omega^n = n!\mu$ in any Kähler class.

Provided that X admits a holomorphic volume form Ω

$$K_X := \Lambda^n T^*X \cong_\Omega \mathcal{O}_X,$$

the condition

$$\omega^n = (-1)^{\frac{n(n-1)}{2}} i^n \Omega \wedge \overline{\Omega}$$

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- `Fine, The Hamiltonian geometry of the space of unitary connections with symplectic curvature, J. Symp. Geom. 12, 2011.`

X compact complex manifold of dimension n, endowed with smooth hermitian line bundle (L, h). Let $\mathcal{A}^{1,1}$ the space of integrable, positive, unitary connections

$$\mathcal{A}^{1,1} = \{A \textit{ unitary s.t.} \quad \omega_A \in \Omega^{1,1}, \omega_A > 0\},$$

endowed with the Kähler structure

$$\frac{1}{(n-1)!} \int_X (\delta A_1 \wedge \delta A_2) \wedge \omega_A^{n-1}.$$

For any choice of smooth volume form μ on X, the unitary gauge group $\mathcal{G}$ of (L, h) acts in a Hamiltonian way on $\mathcal{A}^{1,1}$ with moment map $A \to \omega_A/n! - \mu$.

Solutions of the Calabi problem $\omega_A^n/n! = \mu$ are zeros of a moment map.

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From a complex point of view, the moment interpretation dictates a natural functional $F_0 \colon \mathcal{H} \to \mathbb{R}$ (the *Kempf functional*) on the space of Kähler potentials for a fixed Kähler class $[\omega_0] \in H^2(X, \mathbb{R})$

$$\mathcal{H} = \{\varphi \in C^\infty(X) \mid \omega = \omega_0 + 2i\partial\bar{\partial}\varphi > 0\}$$

whose variation is given by

$$\delta F_0 = \int_X \delta\varphi(\omega^n/n! - \mu).$$

Observation: F_0 is convex along straight lines on $\mathcal{H}$, and therefore there exists at most one solution of $\omega^n/n! = \mu$ on each Kähler class.

Theorem (Cao-Keller '11, Fang-Lai-Ma '09)

The downward gradient flow of F_0 exists for all time

$$\frac{\partial\omega}{\partial t} = \frac{i}{2\pi}\partial\bar{\partial}(\omega^n/\mu).$$

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Holomorphic string algebroids

X compact complex manifold, $A \to X$ a holomorphic Lie algebroid

Definition: A holomorphic Courant algebroid with underlying Lie algebroid A is given by data (plus axioms):

- a holomorphic sequence $0 \to T^*X \to Q \to A \to 0$,
- holomorphic metric $(\cdot, \cdot)$ on Q,
- bracket $[\cdot, \cdot]$ on $\mathcal{O}_Q$.

We are interested in a particular class of holomorphic Courant algebroids, relevant for Hull-Strominger: holomorphic string algebroids.

Let V, W holomorphic vector bundles over X with $c_1(V) = c_1(W) = 0$. Set $E = V \oplus W$ and consider the holomorphic Atiyah algebroid A_E of E:

$$0 \to End\ V \oplus End\ W \to A_E \to TX \to 0.$$

Definition: A holomorphic string algebroid with underlying bundle E is a holomorphic Courant algebroid such that $A = A_E$.

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Example: when rank $E = 0$, exact holomorphic Courant algebroid

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Motivation: in the smooth category, a string algebroid can be understood as the Atiyah Lie algebroid of a $String(r)$-principal bundle

$$String(r) \longrightarrow Spin(r) \longrightarrow SO(r) \longrightarrow O(r),$$

- Sheng, Xu, Zhu, IMRN (2016)

Observe: a solution of the Hull-Strominger system determines a *real string class*.

Idea: holomorphic string algebroids are constructed via a gluing procedure using holomorphic gauge transformations of ‘holomorphic string principal bundles’.

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Gluing

X complex manifold, G complex Lie group with quadratic Lie algebra $(\mathfrak{g}, \langle , \rangle_{\mathfrak{g}})$, and holomorphic Cartan $(3,0)$-form on G

$$\sigma^{3,0} = -\frac{1}{6}\langle \cdot, [\cdot, \cdot] \rangle_{\mathfrak{g}}.$$

Consider the holomorphic sheaf $\mathcal{S}$ of non-abelian groups ($U \subset X$ open)

$$\mathcal{S}(U) = \{(B, g) \in \Omega^{2,0}(U) \times \mathcal{O}(U, G) \text{ satisfying } dB = g^* \sigma^{3,0}\}.$$

A 1-cocycle for the sheaf $\mathcal{S}$ defines a holomorphic string algebroid Q by gluing, via its action on $TU \oplus \mathfrak{g} \oplus T^*U$ with Courant structure

$$\begin{aligned} \langle X + r + \xi, Y + r + \xi \rangle &= i_X \xi + \langle r, r \rangle_{\mathfrak{g}} \\ [X + r + \xi, Y + t + \eta] &= [X, Y] + i_X dt - i_Y dr \\ &\quad + L_X \eta - i_Y d\xi + 2\langle dr, t \rangle_{\mathfrak{g}}. \end{aligned}$$

Example: (Hull-Strominger) $G = \mathrm{SL}(3, \mathbb{C}) \times \mathrm{SL}(k, \mathbb{C})$, and

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$$\begin{aligned} \langle X + r + \xi, Y + r + \xi \rangle &= i_X \xi + \langle r, r \rangle_{\mathfrak{g}} \\ [X + r + \xi, Y + t + \eta] &= [X, Y] + i_X dt - i_Y dr \\ &\quad + L_X \eta - i_Y d\xi + 2\langle dr, t \rangle_{\mathfrak{g}}. \end{aligned}$$

Example: (Hull-Strominger) $G = \mathrm{SL}(3, \mathbb{C}) \times \mathrm{SL}(k, \mathbb{C})$, and

$$\langle , \rangle_{\mathfrak{g}} = \mathrm{tr}_{\mathfrak{sl}(n,\mathbb{C})} - \mathrm{tr}_{\mathfrak{sl}(k,\mathbb{C})}\,.$$

Classification

By definition, a string algebroid determines a holomorphic vector bundle $E = V \oplus W$ on X, such that $c_2(V) = c_2(W)$.

Denote by $\mathcal{A}$ the space of product integrable connections $\theta = \nabla \times A$ on $E = V \oplus W$ (i.e. $F_\theta^{0,2} = 0$).

Proposition (GF-Rubio-Tipler): The isomorphism classes of Q's with underlying bundle E are in bijection with

$$\{(H,\theta) \in \Omega^{3,0} \oplus \Omega^{2,1} \times \mathcal{A}_P \mid dH + \langle F_\theta \wedge F_\theta \rangle = 0\}/\sim,$$

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$$H' = H + CS(\theta) - CS(\theta') - d\langle \theta \wedge \theta' \rangle + dB. \tag{1}$$

Notation: $\langle F_\theta \wedge F_\theta \rangle = \mathrm{tr}\, R_\nabla^2 - \mathrm{tr}\, F_A^2$.

Remark: Any solution of the Hull-Strominger system on (X, E) determines a holomorphic string algebroid via $(2i\partial\omega, \nabla \times A)$.

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Bott-Chern algebroids
and Aeppli classes

Let Q be a string algebroid over X, with underlying bundle $E = V \oplus W$.

Goal: find analogue of $\partial\bar{\partial}$-closed $(1,1)$-forms for Q.

Recall: Q is described by $H \in \Omega^{3,0} \oplus \Omega^{2,1}$ and $\theta = \nabla \times A$ such that

$$dH + \langle F_\theta \wedge F_\theta \rangle = 0, \qquad F_\theta^{0,2} = 0.$$

Let $\mathcal{R}$ denote the space of product hermitian metrics on E. Define

$$B_Q = \{(\tau, h) \in \Omega^{1,1} \times \mathcal{R} \mid \textit{satisfying} * \},$$

$$* \qquad 2i\partial\tau = H + CS(\theta) - CS(\theta_h) - d\langle \theta \wedge \theta_h \rangle + dB. \qquad (2)$$

for some $B \in \Omega^{2,0}$ and θ_h the Chern connection.

Definition: Q is Bott-Chern (BC) if $B_Q \neq \emptyset$.

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Want analogue of Aeppli class for elements $(\tau, h) \in B_Q$ ... but

$$2i\partial\bar{\partial}\tau - \langle F_h \wedge F_h \rangle = 0.$$

Proposition (Donaldson '85): for $h_0, h_1 \in \mathcal{R}$, there exists

$$R(h_1, h_0) \in \Omega^{1,1} / \operatorname{Im}(\partial \oplus \bar{\partial}) \tag{4}$$

with the following properties:

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Analogue: the space of $\partial\bar{\partial}$-closed $(1,1)$-forms on X decomposes as disjoint union of sets labelled by $H_A^{1,1}(X)$.

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Exact case

Assume for a moment rank $E = 0$. Then Q is exact

$$0 \to T^*X \to Q \to TX \to 0,$$

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$$B_Q = \left\{ \tau \in \Omega^{1,1} \mid 2i\partial\tau = H + dB \text{ for } B \in \Omega^{2,0} \right\}.$$

Furthermore, the affine space of *Aeppli classes* is $\Sigma_Q = \partial_H^{-1}(0)$

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Example: $X = \mathbb{C}^2 \backslash \{0\}/\mathbb{Z}$ Hopf surface. Then $H^1(\Omega_{cl}^{2,0}) \cong \mathbb{C}$, $\Sigma_Q = \{\bullet\}$.

Remark: If X is $\partial\bar{\partial}$-manifold, $B_Q \neq \emptyset$ implies $Q \cong TX \oplus T^*X$ and $\Sigma_Q \cong H_{\bar{\partial}}^{1,1}(X)$.

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Metrics

X compact complex manifold, Q string algebroid over X, with underlying bundle E.

Definition: a hermitian metric on Q is $(\tau, h) \in B_Q$ such that

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The space of hermitian metrics on Q will be denoted by B_Q^+.

Lemma (GF-Rubio-Shahbazi-Tipler): given $(\tau_0, h_0) \in B_Q^+$ with Aeppli class σ, any other metric (τ, h) with class σ satisfies (for suitable path $h_t \in \mathcal{R}$ joining h_0 and h):

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Conjecture

Fixing (X, E), solutions of Hull-Strominger are parametrized by isomorphism classes of Bott-Chern algebroids Q and Aeppli classes in Σ_Q

$$H_A^{1,1}(X) \cong \mathrm{Ker}\,\partial \oplus \mathrm{Im}\,\partial$$

Example: X nilmanifold $\mathfrak{h}_3$. M. Fernandez, et al. found 8-parameter family of solutions of Hull-Strominger system. Normalization ($SU(n)$ condition) gives $8 - 1 = 7$.

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The Lie algebra $\mathfrak{h}_3$ admits a real basis such that

$$de_j = 0,\ j = 1, \ldots, 5, \qquad de_6 = -2(e^{12} - e^{34})$$

and a (balanced) integrable almost complex structure

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We have a 9 dimensional real Lie group of automorphisms

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$$(*) \quad (a, b) \in \mathbb{C}^2, \rho, r \geq 0, \rho \neq r, (\alpha_1, \alpha_2, \theta) \in \mathbb{R}^3$$

The action of $\mathrm{Aut}(\mathfrak{h}3, J^-)$ on a the 1-parameter family of solutions of Hull-Strominger, with hermitian form

$$\omega_t = \frac{i}{2}(\omega^{1\overline{1}} + \omega^{2\overline{2}} + t\omega^{3\overline{3}}),$$

where $\omega^j = e^j + ie^{j+1}$, gives a 7-dimensional family of solutions after $\mathrm{SU}(3)$-normalization).

Proposition (GF-Rubio-Tipler, '19): Let Q be the holomorphic string algebroid of the solution with $t = 1$. Then, the Aeppli classes on Σ_Q of the family of solutions above spans an open subset in

$$H_A^{1,1}(\mathfrak{h}_3, J^-) = \{[\omega^{1\overline{1}} + \omega^{2\overline{2}}], [\omega^{1\overline{3}}], [\omega^{3\overline{1}}], [\omega^{2\overline{1}}], [\omega^{1\overline{2}}], [\omega^{2\overline{3}}], [\omega^{3\overline{2}}]\}.$$

The dilaton functional

X compact complex manifold, Q string algebroid over X, with underlying bundle E and space of hermitian metrics B_Q^+.

Definition: Fix a smooth volume form μ on X. Given $(\tau, h) \in B_Q^+$, we define the *dilaton function* $f_\omega \in C^\infty(X)$ by

$$\omega^n/n! = e^{4f_\omega}\mu, \qquad \omega = Re\,\tau.$$

The *dilaton functional* is

$$M\colon B_Q^+ \to \mathbb{R}\colon (\omega, h) \mapsto \int_X e^{-2f_\omega}\omega^n/n!$$

Proposition (GF-Rubio-Shahbazi-Tipler): the critical points of M for metrics in Aeppli class $\sigma \in \Sigma_Q$ solve the *Calabi system*

$$F_h \wedge \omega^{n-1} = 0, \qquad d(e^{-2f_\omega}\omega^{n-1}) = 0.$$

When X admits a holomorphic volume form Ω and $\mu = (-1)^{\frac{n(n-1)}{2}} i^n \Omega \wedge \overline{\Omega}$, this is equivalent to the Hull-Strominger system.

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Lemma (GF-Rubio-Shahbazi-Tipler): the dilaton functional is concave along paths $(\omega_t, h_t) \in B_Q^+$ with fixed Aeppli class $\sigma \in \Sigma_Q$, solving

$$\Lambda_{\omega_t}(\partial\ddot{\xi}_t^{0,1} + \overline{\partial\ddot{\xi}_t^{0,1}}) = \frac{2-n}{2n}|\Lambda_{\omega_t}(ic(h_t^{-1}\dot{h}_t, F_{h_t}) + \partial\dot{\xi}_t^{0,1} + \overline{\partial\dot{\xi}_t^{0,1}})|^2 \\ - \Lambda_{\omega_t}\left(ic(h_t^{-1}\dot{h}_t, \bar{\partial}\partial^{h_t}(h_t^{-1}\dot{h}_t)) + ic(\partial_t(h_t^{-1}\dot{h}_t), F_{h_t})\right). \tag{5}$$

Analogy: *geodesic equation in Kähler geometry.*

Proposition (GF-Rubio-Shahbazi-Tipler): If (ω_0, h_0) and (ω_1, h_1) are two solutions of the Calabi system

$$F_h \wedge \omega^{n-1} = 0, \qquad d(e^{-2f_\omega}\omega^{n-1}) = 0.$$

with Aeppli class σ that can be joined by a solution (ω_t, h_t) of (5) depending analytically on t, then $\omega_1 = k\omega_0$ for some constant k, and h_1 is related to h_0 by an automorphism of E. Furthermore, when $d\omega_0 \neq 0$, we must have $k = 1$.

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Conjecture

Fixing (X, E), solutions of Hull-Strominger are parametrized by isomorphism classes of Bott-Chern algebroids Q and Aeppli classes in Σ_Q

$$H_A^{1,1}(X) \cong \operatorname{Ker} \partial \oplus \operatorname{Im} \partial$$

Theorem (GF-Rubio-Tipler, '19): Assume that the geodesic-like equation has short-time existence in any given direction in the Aeppli class at a given solution. Then, the quadratic form given by the Hessian of M is semi-negative, and the conjecture holds infinitesimally.

The moduli Kähler potential

$4D$ physical analysis

Assuming $\dim_{\mathbb{C}} X = 3$ and existence of a holomorphic volume form Ω,

$$\omega^3/6 = e^{4f_\omega} i\Omega \wedge \overline{\Omega}$$

implies that e^{2f_ω} is the 10-dimensional dilaton, and the dilaton functional is

$$M = \int_X e^{-2f_\omega}\omega^n/n! = \int_X \|\Omega\|\omega^n/n! \tag{6}$$

Remarkably, this coincides with a universal formula for the 4D dilaton in the induced effective field theory

$$e^{-2\phi_4} = \int_X e^{-2\phi_{10}} vol_6$$

- Anguelova-Quigley-Sethi, JHEP10, 2010.

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In 4-dimensional supergravity the gravitino mass can be written as

$$M_{3/2} = c_0 e^{K/2} W$$

for some universal constant $c_0 \in \mathbb{R}$. Here, W is the superpotential of the theory and e^K is the Kähler potential.

A *Gukov-type formula* for $M_{3/2}$ was derived by Lukas et al. (valid to first order in α' expansion):

$$M_{3/2} = \frac{\sqrt{8} e^{\phi^4} W}{4 \int_X \|\Omega\|_\omega \frac{\omega^3}{6}},$$

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This leads to the following formula for the moduli Kähler potential

$$K = -3\log\int_X \|\Omega\|_\omega \frac{\omega^3}{6} - 2\log c_0 - \log 2.$$

Conjecture - Physical prediction

The following formula defines the Kähler potential for a Kähler metric in the moduli space of solutions of the Hull-Strominger system

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Theorem (GF-Rubio-Tipler, '19): Assume that Conjecture holds. If $\delta\sigma$ and $\delta\mu$ are the variations of the Aeppli and balanced classes, respectively, along a non-constant path of solutions of Hull-Strominger on the Bott-Chern algebroid Q, then

$$\delta\sigma \cdot \delta\mu < \frac{1}{4M}(\delta\sigma \cdot \mu)^2.$$

Thank you!