# Recent Progress in Anomaly Flow

Teng Fei

Rutgers University - Newark

Bridging the Gap between Kähler and non-Kähler Complex Geometry

BIRS, Oct. 29, 2019

# Overview

Based on joint work with Zhijie Huang, Duong H. Phong and Sebastien Picard

1. Backgrounds
2. Anomaly flow on Riemann surfaces
3. Unification with Kähler-Ricci flow when $\alpha' = 0$

# Superstrings

Heterotic superstring theory is a 10d theory with following ingredients:

- A 10-dimensional Lorentzian manifold $M^{10}$
- A gauge bundle $E$ over $M^{10}$

They satisfy

- SUSY constraint
- anomaly cancellation equation

which can be described as equations involving curvature of $M^{10}$ and $E$, and other additional fields.

# Compactifications

Observed physical world: 4-dimensional spacetime $M^4$ (e.g. $\mathbb{R}^{1,3}$)
Compactification: $M^{10} = M^4 \times X$, $X$ is a 6d compact space, tiny in scale

Consider $M^{10} = \mathbb{R}^{1,3} \times X$ and wish to reduce all the equations required by superstring theory to geometry of $X$ (and $E$).

- (CHSW 85') fluxfree compactification: $M^{10} = \mathbb{R}^{1,3} \times X$ equipped with a product metric, "embed the gauge into spin connection"($E = TX$) $\rightsquigarrow$ $X$ must be a Calabi-Yau 3-fold with Kähler-Ricci-flat metric (solved by Yau'77)
- (Hull'86, Strominger'86) compactification with flux: $M^{10} = \mathbb{R}^{1,3} \times X$ equipped with a warped product metric
  $\rightsquigarrow$ Hull-Strominger system, in particular $X$ is a Calabi-Yau 3-fold ($K_X \cong \mathcal{O}$, not necessarily Kähler).

# Hull-Strominger system

- $(X, \omega, \Omega)$: Hermitian 3-fold with canonical bundle globally trivialized by $\Omega$
- $(E, H) \to X$: holomorphic Hermitian vector bundle
- $R, F$: curvature forms of $T^{1,0}X$ and $E$
- $\alpha'$ (positive) coupling constant

The Hull-Strominger system consists of three equations

$$\begin{aligned} F \wedge \omega^2 &= 0, \\ i\partial\overline{\partial}\omega &= \frac{\alpha'}{4}\left(\mathrm{Tr}(R \wedge R) - \mathrm{Tr}(F \wedge F)\right), \\ \mathrm{d}\left(\|\Omega\|_\omega \cdot \omega^2\right) &= 0. \end{aligned}$$

The system is a generalization of Ricci-flat metrics on non-Kähler Calabi-Yau 3-folds coupled with Hermitian-Yang-Mills equation!

# Known solutions on compact $X$

$$\begin{aligned} F \wedge \omega^2 &= 0, \\ i\partial\overline{\partial}\omega &= \frac{\alpha'}{4}(\mathrm{Tr}(R \wedge R) - \mathrm{Tr}(F \wedge F)), \\ \mathrm{d}(\|\Omega\|_\omega \cdot \omega^2) &= 0. \end{aligned}$$

- Kähler-Ricci-flat solutions (Yau'77) and their perturbations (Strominger'86, Li-Yau'05)
- Fu-Yau solution (Fu-Yau'08), orbifolded version (Fino-Grantcharov-Vezzoni'18)
- Quotients of Lie groups (various authors from '09 onward)
- F.-Huang-Picard'17: generalized Calabi-Gray manifolds of genus $\geq 3$.

# Difficulties

- Weaker cohomological condition $\rightsquigarrow$ lack of $\partial\bar{\partial}$-lemma $\rightsquigarrow$ no effective way to parameterized the space of metrics in a fixed cohomology class
- Not a scalar PDE, but a coupled system
- quadratic curvature terms

To overcome the difficulty of lacking $\partial\bar{\partial}$-lemma, Phong-Picard-Zhang'15 initiated the Anomaly flow program. The idea is that a natural parabolic equation would guide us to a natural choice of path in the space of metrics with given cohomology class.

# Anomaly flow

The idea is to evolve the conformally balanced metric term by the anomaly term and couple it with Donaldson's heat flow:

$$\begin{aligned} \partial_t(\|\Omega\|_\omega \cdot \omega^2) &= i\partial\bar{\partial}\omega - \frac{\alpha'}{4}\left(\mathrm{Tr}(R \wedge R) - \mathrm{Tr}(F \wedge F)\right), \\ H^{-1}\partial_t H &= -\Lambda F. \end{aligned}$$

It is clear from the expression that if one starts with a conformally balanced metric, then the conformally balanced class is preserved under the flow. Moreover the stationary points of this flow are solutions to the Hull-Strominger system.

# Anomaly flow cont'd

Coupled flow is hard to study. Suppose we have the knowledge of the evolution of $H$ (can be met in many cases), then the Anomaly flow is of the form

$$\partial_t(\|\Omega\|_\omega \cdot \omega^2) = i\partial\overline{\partial}\omega - \frac{\alpha'}{4}\left(\mathrm{Tr}(R \wedge R) - \Phi(t)\right)$$

with given $\Phi$, which can be further written as a flow of metrics (PPZ'16)

$$\partial_t\omega = \frac{1}{2\|\Omega\|_\omega}\left(-\widetilde{\mathrm{Ric}} + \frac{\alpha'}{4}Rm * Rm + \mathrm{l.o.t.}\right),$$

where $\widetilde{\mathrm{Ric}}$ is the second Chern-Ricci curvature.

In this way, we may think of the Anomaly flow as a modification of Ricci flow with $\alpha'$-correction term involving quadratic form of curvature tensor.

# What's proved

For general $\alpha'$

- Short-time existence (PPZ'15): provided $|\alpha' Rm| \ll 1$ initially.
- PPZ'16: The Fu-Yau ansatz is preserved. Get convergence under large initial data.
- PPZ'17: Complex Lie group cases.

For $\alpha' \neq 0$ and general set-up, it is very hard to study due to high nonlinearity.

# What's proved cont'd

Toy model $\alpha' = 0$: the flow can be generalized to incorporate any $n(\geq 3$ dimensional Calabi-Yau manifolds (after Fu-Yau')

$$\partial_t(\|\Omega\|_\omega \cdot \omega^{n-1}) = i\partial\bar{\partial}(\omega^{n-2})$$

- Shi-type estimate (PPZ'16): In finite time, if $\|\Omega\|_\omega$ bounded below, $|Rm|^2 + |T|^4 + |\nabla T|^2$ bounded, then the flow can be continued.
- PPZ'18: preserves the conformally Kähler condition, gives a new parabolic proof (after Cao'85) to Calabi conjecture.
  In fact F.-Picard'19 showed this flow is related to the inverse MA flow by Cao-Keller'13, Collins-Hisamoto-Takahashi'17 via T-duality.
- F.-Picard'19: Dilaton functionals are monotone, derive uniform estimate $\|\Omega\|_\omega \leq C$.

# Anomaly flow on Riemann surfaces

Based on F.-Huang-Picard's construction of solutions to the Hull-Strominger system on generalized Calabi-Gray manifolds, we may want to test the Anomaly flow on these manifolds.

- Our ansatz on generalized Calabi-Gray manifolds is preserved by the Anomaly flow, therefore the anomaly flow reduces to a parabolic PDE on Riemann surfaces,
- Short-time existence for free,
- Conserved quantities and monotonicity formula.

To be specific, the flow is

$$\partial_t(e^f) = \Delta(e^f + \alpha'\kappa e^{-f}) - 2\kappa(e^f + \alpha'\kappa e^{-f}),$$

where $\Sigma$ is equipped with a fixed metric with Gauss curvature $\kappa \leq 0$, $\Delta$ is the Laplace operator with respect to this canonical metric.

# Anomaly flow on Riemann surfaces cont'd

If we forget about the Calabi-Gray construction, we may interpret this flow as an intrinsic flow of metrics on a Riemann surface $\Sigma$ associated to a fixed map

$$\Sigma \xrightarrow{\varphi} (\mathbb{CP}^1, \omega_{FS}).$$

Naively, if we look at the flow

$$\partial_t(e^f) = \Delta(e^f + \alpha'\kappa e^{-f}) - 2\kappa(e^f + \alpha'\kappa e^{-f}).$$

If $\alpha'\kappa e^{-f}$ is negligible, then the dominant term $e^f$ satisfies a linear equation and we definitely expect long-time existence. On the other hand, if the term $\alpha'\kappa e^{-f}$ dominates, then bad things may occur. The key is to understand whether either dominance is preserved under the flow.

Indeed, F.-Huang-Picard '17 proved the following results:

- The flow can be continued as long as $e^f$ is bounded below by a positive number,
- If $|e^f|_{L^1}$ is sufficiently small, then the flow develops finite-time singularity and we have an estimate of the maximal existence time. This can be interpreted as that if the initial area of the Riemann surface is small, then its area keeps decreasing. In general, it does not shrink uniformly like the Ricci flow, singularities appear at certain points when the Riemann surface still has positive area.

# Anomaly flow on Riemann surfaces cont'd

- If $e^f \geq C$ initially for some constant $C$, then we have long-time existence. In this case $e^f$ grows exponentially and the growth rate is given by the first eigenvalue of the Jacobi operator $\Delta - 2\kappa$. After suitable normalization, $e^f$ converges to the first eigenfunction of $\Delta - 2\kappa$.
  Geometrically this can be interpreted as that generalized Calabi-Gray manifolds collapse their hyperkähler fibers under the Anomaly flow.
- Combining above results, one may construct two sets of initial data of the same de Rham conformally balanced cohomology class, such that for one choice of initial data we have finite time singularity and for the other choice of initial data we have long-time existence. It shows that the behavior of Anomaly flow is far more complicated than that of the Kähler-Ricci flow.

# Remaining mysteries

The above-mentioned results leave the region of medium initial data, where we have the stationary points of the flow. This case is very interesting since one would expect to detect the "hemisphere condition", which is an obstruction to the existence of solutions to the elliptic problem.

Naively one should expect singularities corresponding to the failure of hemisphere condition. For these singularities, the flow can be continued after reparametrization the map to $\mathbb{CP}^1$ (surgery).

# A modest motivation

In order to study the Anomaly flow, we need to first understand the simplest case $\alpha' = 0$.

PPZ'18 tells us that this flow is useful in complex geometry: it provides a "non-Kähler" proof of Calabi-Yau theorem in Kähler geometry.

Natural questions:

- Extension to complex manifolds not necessarily Calabi-Yau?
- Does the flow preserve any curvature condition?
- Perelman's theory?

# A modest motivation cont'd

To answer these questions, need to calculate evolution equation for various tensors.

E.g. $n = 3$:

$$\partial_t g_{\bar{k}j} = \frac{1}{2\|\Omega\|_\omega}\left(-\widetilde{Ric}_{\bar{k}j} + g^{s\bar{r}} g^{m\bar{l}} T_{\bar{r}jm} \bar{T}_{s\bar{l}\bar{k}}\right)$$

E.g. general $n$:

$$\begin{aligned}\partial_t g_{\bar{k}j} =& \frac{1}{(n-1)\|\Omega\|_\omega}\left(-\widetilde{Ric}_{\bar{k}j} + \frac{1}{2(n-2)}(\|T\|^2 - 2\|\tau\|^2) g_{\bar{k}j}\right.\\ & \left. -\frac{1}{2} g^{q\bar{p}} g^{s\bar{r}} T_{\bar{k}qs} \bar{T}_{j\bar{p}\bar{r}} + g^{s\bar{r}}(T_{\bar{k}js}\bar{\tau}_{\bar{r}} + \tau_s \bar{T}_{j\bar{k}\bar{r}}) + \tau_j \bar{\tau}_{\bar{k}}\right)\end{aligned}$$

The two expressions are the same when $n = 3$ due to the torsion identity discovered in F.-Picard'19.

# A modest motivation cont'd

Observations:

- Comparing to Ricci flow, we have the extra factor $1/\|\Omega\|_\omega$
- The evolution equation of metric is too complicated (for general $n$) to carry out exact calculations.

Dream: get rid of $1/\|\Omega\|_\omega$ and simplify the evolution equation of metric.

Consider the conformally scaled metric $\eta = \|\Omega\|_\omega \omega$, its evolution equation is extremely simple

$$\partial_t \eta_{\bar{k}j} = \frac{1}{n-1}\left(-\widetilde{Ric}_{\bar{k}j} - \frac{1}{2} g^{s\bar{r}} g^{m\bar{l}} T_{\bar{k}sm} \bar{T}_{j\bar{r}\bar{l}}\right).$$

Here every tensor is with respect to the new metric $\eta$.

# Remarks and Applications

- This formulation allows us to generalize the Anomaly flow to arbitrary complex manifolds.
- The evolution equation

$$\partial_t \eta_{\bar{k}j} = -\widetilde{Ric}_{\bar{k}j} - \frac{1}{2} g^{s\bar{r}} g^{m\bar{l}} T_{\bar{k}sm} \bar{T}_{j\bar{r}\bar{l}}$$

  was identified by Ustinovskiy'18, who proved this flow preserves many positivity notions such as Griffiths positivity and dual-Nakano positivity.
- This new formulation allows us to establish sharper Shi-type estimates.

# Applications cont'd

**Theorem (F.-Phong'19)**

Consider the Anomaly flow

$$\partial_t(\|\Omega\|_\omega \omega^{n-1}) = i\partial\bar{\partial}(\omega^{n-2}),$$

if in finite time we know that

$$\frac{1}{\|\Omega\|_\omega^2}\left(|Rm|_\omega^2 + |T|_\omega^4 + |\nabla_\omega T|_\omega^2\right) \leq C,$$

then the flow can be continued.

# Applications cont'd

Ustinovskiy'18 asked about how to characterize periodic solutions to the flow

$$\partial_t \eta_{\bar{k}j} = -\widetilde{Ric}_{\bar{k}j} - \frac{1}{2} g^{s\bar{r}} g^{m\bar{l}} T_{\bar{k}sm} \bar{T}_{j\bar{r}\bar{l}}.$$

By making use of the connection with the Anomaly flow, we proved

**Proposition (F.-Phong'19)**

All periodic solutions to the above flow are stationary points, which are exactly Kähler-Ricci-flat metrics.

# Sketch of proof

- Ustinovskiy showed that if one has a periodic solution to the flow, then it lands in the realm of Anomaly flow (up to a covering map). So we can apply tools from Anomaly flow.
- Consider the monotone dilaton functionals in Anomaly flow, for a periodic solution it implies that the monotone quantities are constants and we get an extra equation.
- By straightforward calculation, one easily shows that the only solutions to this equation are Kähler-Ricci-flat metrics.

# Recent Progress in Anomaly Flow

Teng Fei

Rutgers University - Newark

Bridging the Gap between Kähler and non-Kähler Complex Geometry

BIRS, Oct. 29, 2019