Deformation theory of non-Kähler holomorphically symplectic manifolds

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Bogomolov-Tian-Todorov

DEFINITION: A **holomorphic symplectic form** on a complex manifold is a holomorphic, non-degenerate 2-form Ω.

REMARK: The top power of Ω is a holomorphic volume form. Therefore, **any holomorphically symplectic manifold has trivial canonical bundle.**

DEFINITION: A manifold with trivial canonical bundle is called **Calabi-Yau**.

THEOREM: (Bogomolov-Tian-Todorov) The Kuranishi deformation space of complex structures on a compact, Kähler Calabi-Yau manifold M **is smooth,** and **its tangent space is $H^1(TM)$,** that is, the deformations of complex structures on M are **unobstructed**

Bogomolov-Tian-Todorov on holomorphically symplectic manifolds

REMARK: Non-Kähler Calabi-Yau manifolds **can have obstructed deformations.** In *É. Ghys,* Déformations des structures complexes sur les espaces homogènes de SL(2,C), *J. Reine Angew. Math.* **468** *(1995), 113–138,* it was shown that **the deformation space of a locally homogeneous manifold $SL(2,\mathbb{C})/\Gamma$ can be obstructed,** for a cocompact and discrete subgroup $\Gamma \subset SL(2,\mathbb{C})$.

QUESTION: Is there any compact, simply connected holomorphically symplectic non-Kähler manifold with obstructed deformations?

The main result of today's talk: **Bogomolov-Tian-Todorov holds for a class of holomorphically symplectic non-Kähler manifolds, called Bogomolov-Guan manifolds.**

Hilbert schemes

DEFINITION: Hilbert scheme of points on a variety M is the Hilbert space of ideal sheaves $I \subset \mathcal{O}_M$ with $F := \mathcal{O}_M/I$ supported in a finite subset of M; dimension of $H^0(F)$ is called **length**. Hilbert scheme of points for length n is denoted by $M^{[n]}$

REMARK: When M is a complex surface, $M^{[n]}$ **is a smooth resolution of the n-th symmetric power of M,** denoted $M^{(n)}$.

REMARK: If the surface M is holomorphically symplectic, $M^{[n]}$ is also holomorphically symplectic (follows easily from Serre's duality).

REMARK: Hilbert scheme of a K3 surface is a simply connected, holomorphically symplectic manifold. Hilbert scheme of a torus T is not simly connected, but the fiber of its Albanese map $T^{[n]} \longrightarrow T$ has finite fundamental group. The universal cover of this fiber is called **generalized Kummer variety**.

This way one obtains **two main examples of simply connected holomorphically symplectic manifolds.**

Kodaira-Thurston surface

REMARK: A. Todorov conjectured that any compact, simply connected holomorphically symplectic manifold is Kähler. **This is false.** D. Guan has constructed examples of manifolds which are compact, simply connected, holomorphically symplectic but non-Kähler. His example was explicated by Bogomolov.

DEFINITION: Let L be a line bundle on an elliptic curve E with the first Chern class $c_1(L) \neq 0$. Denote by $\tilde{S}$ the corresponding $\mathbb{C}^*$-bundle on E obtained by removing the zero section, $\tilde{S} = \operatorname{Tot}(L)\backslash 0$. Fix a complex number λ with $|\lambda| > 1$, and let $h_\lambda : \tilde{S} \longrightarrow \tilde{S}$ be the corresponding homothety of $\tilde{S}$. The quotient $\tilde{S}/\langle h_\lambda \rangle$ is called **a primary Kodaira-Thurston surface**, or simply **Kodaira-Thurston surface**.

REMARK: Kodaira-Thurston surface is an isotrivial elliptic fibration over the elliptic curve E, with the fiber identified with the elliptic curve $E_L := \mathbb{C}^*/\langle \lambda \rangle$. Therefore, **it is holomorphically symplectic.**

Bogomolov-Guan manifolds

DEFINITION: Let S be a Kodaira-Thurston surface, $S^{[n]}$ its Hilbert scheme, $S^{(n)}$ its symmetric power, and $\pi_S : S \longrightarrow E$ the elliptic fibration constructed above. π_S to each component of $S^{(n)}$ and summing up, we obtain a holomorphic projection from $S^{(n)}$ to E; taking the composition with the resolution $r : S^{[n]} \longrightarrow S^{(n)}$, we obtain an isotrivial fibration $\pi : S^{[n]} \longrightarrow E$. Denote its fiber by $F^{[n]}$. Then $F^{[n]}$ is a smooth divisor in a holomorphically symplectic manifold $(S^{[n]}, \Omega)$. The restriction of Ω to $F^{[n]}$ has rank $2n-2$, because $F^{[n]} \subset S^{[n]}$ is a divisor. Denote by $K \subset TF^{[n]}$ the kernel of $\Omega\Big|_{F^{[n]}}$, that is, the set of all $x \in TF^{[n]}$ such that $\Omega\Big|_{F^{[n]}}(x, \cdot) = 0$. The corresponding foliation is called **the characteristic foliation**.

REMARK: The leaf space W of K is a holomorphically symplectic orbifold, but it is never smooth. When the degree of the line bundle L over E is divisible by n, the space W has a smooth finite covering, of order n^2, ramified in the singular points of W. This covering is called **the Bogomolov-Guan manifold**. By construction, **it is compact, simply connected, holomorphically symplectic.**

REMARK: Since the Bogomolov-Guan manifold contains a blown-up Kodaira-Thurston surface, **it is non-Kähler.**

Bogomolov-Tian-Todorov for holomorphically symplectic manifolds

This is the version of Bogomolov-Tian-Todorov which can be applied to Bogomolov-Guan manifolds.

THEOREM: Let (M, I, Ω) be a compact holomorphically symplectic manifold (not necessarily Kähler). Assume that the Dolbeault cohomology group $H^{0,2}_{\bar\partial}(M) = H^2(\mathcal{O}_M)$ is generated by ∂-closed $(0,2)$-forms, and all ∂-exact holomorphic 3-forms on M vanish. **Then the holomorphic symplectic deformations of (M, I, Ω) are unobstructed.** If, in addition, all classes in the Dolbeault cohomology group $H^{1,1}_{\bar\partial}(M)$ are represented by closed $(1,1)$-forms, **the complex deformations of M are also unobstructed,** and all sufficiently small complex deformations remain holomorphically symplectic.

I would try to explain how such a result can be obtained.

Schouten brackets

DEFINITION: Let M be a complex manifold, and $\Lambda^{0,p}(M) \otimes T^{1,0}M$ the sheaf of $T^{1,0}M$-valued $(0,p)$-forms. Consider the commutator bracket $[\cdot,\cdot]$ on $T^{1,0}M$, and let $\overline{\mathcal{O}}_M$ denote the sheaf of antiholomorphic functions. Since $[\cdot,\cdot]$ is $\overline{\mathcal{O}}_M$-linear, it is naturally extended to $\Lambda^{0,p}(M) \otimes_{C^\infty M} T^{1,0}M = \overline{\Omega^p M} \otimes_{\overline{\mathcal{O}}_M} T^{1,0}M$, giving a bracket

$$[\cdot,\cdot]: \Lambda^{0,p}(M) \otimes T^{1,0}M \times \Lambda^{0,q}(M) \otimes T^{1,0}M \longrightarrow \Lambda^{0,p+q}(M) \otimes T^{1,0}M.$$

This bracket is called **Schouten bracket**.

REMARK: Since $[\cdot,\cdot]$ is $\overline{\mathcal{O}}_M$-linear, the Schouten bracket satisfies the Leibnitz identity:

$$\overline{\partial}([\alpha,\beta]) = [\overline{\partial}\alpha,\beta] + [\alpha,\overline{\partial}\beta].$$

This allows one to extend the Schouten bracket to the $\overline{\partial}$-cohomology of the complex $(\Lambda^{0,*}(M) \otimes T^{1,0}M, \overline{\partial})$, which coincide with the cohomology of the sheaf of holomorphic vector fields: $[\cdot,\cdot]: H^p(TM) \times H^q(TM) \longrightarrow H^{p+q}(TM)$.

Tian-Todorov lemma

DEFINITION: Assume that M is a complex n-manifold with trivial canonical bundle K_M, and Φ a non-degenerate section of K_M. We call a pair (M, Φ) **a Calabi-Yau manifold**. Substitution of a vector field into Φ gives an isomorphism $TM \cong \Omega^{n-1}(M)$. Similarly, one obtains an isomorphism

$$\Lambda^{0,q}M \otimes \Lambda^p TM \longrightarrow \Lambda^{0,q}M \otimes \Lambda^{n-p,0}M = \Lambda^{n-q,p}M. \qquad (*)$$

Yukawa product $\bullet: \Lambda^{p,q}M \otimes \Lambda^{p_1,q_1}M \longrightarrow \Lambda^{p+p_1-n,q+q_1}M$ is obtained from the usual product

$$\Lambda^{0,q}M \otimes \Lambda^p TM \times \Lambda^{0,q_1}M \otimes \Lambda^{p_1}TM \longrightarrow \Lambda^{0,q+q_1}M \otimes \Lambda^{p+p_1}TM$$

using the isomorphism (*).

TIAN-TODOROV LEMMA: Let (M, Φ) be a Calabi-Yau manifold, and

$$[\cdot,\cdot]: \Lambda^{0,p}(M) \otimes T^{1,0}M \times \Lambda^{0,q}(M) \otimes T^{1,0}M \longrightarrow \Lambda^{0,p+q}(M) \otimes T^{1,0}M.$$

its Schouten bracket. Using the isomorphism (*), we can interpret Schouten bracket as a map

$$[\cdot,\cdot]: \Lambda^{n-1,p}(M) \times \Lambda^{n-1,q}(M) \longrightarrow \Lambda^{n-1,p+q}(M).$$

Then, for any $\alpha \in \Lambda^{n-1,p}(M)$, $\beta \in \Lambda^{n-1,p_1}(M)$, one has

$$[\alpha, \beta] = \partial(\alpha \bullet \beta) - (\partial\alpha) \bullet \beta - (-1)^{n-1+p} \alpha \bullet (\partial\beta),$$

where $\bullet$ denotes the Yukawa product.

Maurer-Cartan equation and deformations

CLAIM: Let (M, I) be an almost complex manifold, and B an abstract vector bundle over $\mathbb{C}$ isomorphic to $\Lambda^{0,1}(M)$. Consider a differential operator $\overline{\partial}: C^\infty M \longrightarrow B = \Lambda^{0,1}(M)$ satisfying the Leibnitz rule. Its symbol is a linear map $u: \Lambda^1(M,\mathbb{C}) \longrightarrow B$. Then $B = \frac{\Lambda^1(M,\mathbb{C})}{\ker u} = \Lambda^{0,1}(M)$. Extend $\overline{\partial}: C^\infty M \longrightarrow B$ to the corresponding exterior algebra using the Leibnitz rule:

$$C^\infty M \xrightarrow{\overline{\partial}} B \xrightarrow{\overline{\partial}} \Lambda^2 B \xrightarrow{\overline{\partial}} \Lambda^3 B \xrightarrow{\overline{\partial}} ...$$

Then integrability of I is equivalent to $\overline{\partial}^2 = 0$.
Proof: This is essentially the Newlander-Nirenberg theorem. ■

REMARK: Almost complex deformations of I are given by the sections $\gamma \in T^{1,0}M \otimes \Lambda^{0,1}(M)$, with the integrability relation $(\overline{\partial} + \gamma)^2 = 0$ rewritten as **the Maurer-Cartan equation** $\overline{\partial}(\gamma) = -\{\gamma, \gamma\}$. Here $\overline{\partial}(\gamma)$ is identified with the anticommutator $\{\overline{\partial}, \gamma\}$, and $\{\gamma, \gamma\}$ is anticommutator of γ with itself, where γ is considered as a $\Lambda^{0,1}(M)$-valued differential operator. **This identifies $\{\gamma, \gamma\}$ with the Schouten bracket.**

REMARK: We shall write $[\gamma, \gamma]$ instead of $\{\gamma, \gamma\}$, because this usage is more common.

Maurer-Cartan equation and obstructions to the deformations

The Kuranishi deformation space of complex structures on M **is identified with the space of solutions of Maurer-Cartan equation $\overline{\partial}(\gamma) = -[\gamma, \gamma]$ modulo the diffeomorphism action.**

DEFINITION: Write γ as power series, $\gamma = \sum_{i=0}^{\infty} t^{i+1}\gamma_i$. **Then the Maurer-Cartan becomes**

$$\overline{\partial}\gamma_0 = 0, \quad \overline{\partial}\gamma_p = -\sum_{i+j=p-1} [\gamma_i, \gamma_j]. \quad (**)$$

We say that deformations of complex structures are **unobstructed** if the solutions $\gamma_1, ..., \gamma_n, ...$ of (**) can be found for γ_0 in any given cohomology class $[\gamma_0] \in H^1(M, TM)$.

REMARK: Unobstructedness means that **the Kuranishi deformation space K of (M, I) is smooth and the Kodaira-Spencer map $T_{(M,I)}K \longrightarrow H^1(M, TM)$ is an isomorphism.**

Tian-Todorov lemma and deformations

REMARK: Notice that the sum $\sum_{i+j=p-1}[\gamma_i,\gamma_j]$ is always $\overline{\partial}$-closed. Indeed, the Schouten bracket commutes with $\overline{\partial}$, hence

$$\overline{\partial}\left(\sum_{i+j=p-1}[\gamma_i,\gamma_j]\right) = -\sum_{i+j+k=p-1}[\gamma_i,[\gamma_j,\gamma_k]] + [[\gamma_i,\gamma_j],\gamma_k].$$

vanishes as a sum of triple supercommutators. **Obstructions to deformations** are given by cohomology classes of the sums $\sum_{i+j=p-1}[\gamma_i,\gamma_j]$, which are defined inductively. These classes are called **Massey powers** of γ_0.

REMARK: In the Kähler setting, **Tian-Todorov lemma immediately implies the unobstructedness of deformations for compact manifolds with trivial canonical bundle.** Indeed, we can always start from $\gamma_0 \in TM \otimes \Lambda^{0,1}(M) = \Lambda^{n-1,1}(M)$ which is harmonic. Then it is $\overline{\partial}$- and ∂-closed. Therefore, $[\gamma_0,\gamma_0] = \partial(\gamma_0 \bullet \gamma_0)$ is ∂-exact. It is also $\overline{\partial}$-closed, because the Yukawa product commutes with $\overline{\partial}$. Then $\partial\overline{\partial}$-lemma implies that $[\gamma_0,\gamma_0]$ is $\partial\overline{\partial}$-exact. Using induction, we may assume that the solutions of (**) for $p = 1, ..., n-1$ are all $\partial\overline{\partial}$-exact. To solve (**) for $p = n$, we use Tian-Todorov lemma again, obtaining $\gamma_n = -G_{\overline{\partial}}\left(\sum_{i+j=n-1}\partial(\gamma_i \bullet \gamma_j)\right)$, where $G_{\overline{\partial}}$ is the Green operator inverting $\overline{\partial}$.

Tian-Todorov lemma for holomorphically symplectic manifolds

Let now Ω be a holomorphically symplectic form on a complex manifold M, $\dim_{\mathbb{C}} M = 2n$. Then $TM \cong \Omega^1 M$, hence the Schouten bracket is defined as

$$\Lambda^{1,p}(M) \times \Lambda^{1,q}(M) \longrightarrow \Lambda^{1,p+q}(M).$$

LEMMA: Let M be a holomorphic symplectic manifold. Consider the operators $L_\Omega(\alpha) := \Omega \wedge \alpha$, H_Ω acting as multiplication by $n-p$ on $\Lambda^{p,q}(M)$, and $\Lambda_\Omega := \star\Lambda\star$. **Then $L_\Omega, H_\Omega, \Lambda_\Omega$ satisfy the $\mathfrak{sl}(2)$ relations, similar to the Lefschetz triple:** $[H_\Omega, L_\Omega] = 2L_\Omega, \quad [H_\Omega, \Lambda_\Omega] = -2\Lambda_\Omega, [L_\Omega, \Lambda_\Omega] = H_\Omega$. ■

LEMMA: (Tian-Todorov for holomorphically symplectic manifolds) Let (M, Ω) be a holomorphically symplectic manifold, and

$$[\cdot,\cdot]_\Omega : \Lambda^{1,p}(M) \times \Lambda^{1,q}(M) \longrightarrow \Lambda^{1,p+q}(M).$$

the Schouten bracket. **Then for any $a, b \in \Lambda^{1,*}(M)$, one has**

$$[a,b] = \delta(a \wedge b) - (\delta a) \wedge b - (-1)^{\tilde{a}} a \wedge \delta(b),$$

where $\tilde{a}$ is parity of a, and $\delta := [\Lambda_\Omega, \partial]$.

Proof: Same as for the usual Tian-Todorov. ■

Maurer-Cartan for Hamiltonian vector fields

REMARK: A solution of the Maurer-Cartan equation $\left(\overline{\partial}+\sum_{i=0}^{\infty} t^{i+1}\gamma_i\right)^2 = 0$ **gives a holomorphically symplectic deformation whenever all γ_i belong to $\Lambda^{0,1}(M)\otimes \mathrm{Ham}_M$.**

Using Ω to identify vector fields and 1-forms, the sheaf of Hamiltonian vector fields can be embedded to $\Lambda^{1,0}(M)$ as a sheaf of ∂-closed (1,0)-forms.

Similarly, if we use Ω to consider γ_i as sections of $\Lambda^{0,1}(M)\otimes T^{1,0}M = \Lambda^{1,1}(M)$, the condition $\gamma_i \in \Lambda^{0,1}(M)\otimes \mathrm{Ham}_M$ is interpreted as $\partial\gamma_i = 0$.

DEFINITION: Let (M, I, Ω) be a holomorphically symplectic manifold. We say that the **holomorphic symplectic deformations of (M, I, Ω) are unobstructed** if for any $\overline{\partial}$- and ∂-closed $\gamma_0 \in \Lambda^{1,1}(M)$ the Maurer-Cartan equation

$$\overline{\partial}\gamma_p = -\sum_{i+j=p-1}[\gamma_i,\gamma_j], \quad p = 1, 2, 3, ...$$

has a solution $(\gamma_1, \gamma_2, ...,)$, with $\gamma_i \in \Lambda^{1,1}(M)$ ∂-closed.

Bogomolov-Tian-Todorov for BG-manifolds

THEOREM 1: Let (M, I, Ω) be a compact holomorphically symplectic manifold (not necessarily Kähler). Assume that the Dolbeault cohomology group $H^{0,2}_{\overline{\partial}}(M) = H^2(\mathcal{O}_M)$ is generated by ∂-closed $(0,2)$-forms, and all ∂-exact holomorphic 3-forms on M vanish. **Then the holomorphic symplectic deformations of (M, I, Ω) are unobstructed.**

Proof: Let $\alpha, \beta \in \Lambda^{1,*}(M)$ be ∂-closed forms, and $[\alpha, \beta]$ the Schouten bracket. The holomorphic symplectic Tian-Todorov lemma gives $[\alpha, \beta] = \partial\Lambda_\Omega(\alpha \wedge \beta)$. Suppose we have solved the Maurer-Cartan equation for all $p < n$, and all $\gamma_p \in \Lambda^{1,1}(M)$ with $p < n$ are ∂-closed. The Maurer-Cartan for γ_n becomes

$$\overline{\partial}\gamma_n = \sum_{i+j=n-1} \partial\Lambda_\Omega(\gamma_i \wedge \gamma_j) \quad (MC)$$

The right hand side of (MC) is $\overline{\partial}$-closed by the standard argument with triple commutators. Then it is $\overline{\partial}$-exact by Lemma 1 below, applied to $\rho = \sum_{i+j=n-1} \Lambda_\Omega(\gamma_i \wedge \gamma_j)$. ■

LEMMA 1: In assumptions of Theorem 1, let $\rho \in \Lambda^{0,2}(M)$ be a form which satisfies $\overline{\partial}\partial\rho = 0$. **Then $\partial\rho$ is $\partial\overline{\partial}$-exact.**

Proof: Diagram chasing. ■

Bogomolov-Beauville-Fujiki form on hyperkähler manifolds

THEOREM: (Fujiki). Let $\eta \in H^2(M)$, and $\dim M = 2n$, where M is hyperkähler. Then $\int_M \eta^{2n} = cq(\eta,\eta)^n$, for some primitive integer quadratic form q on $H^2(M,\mathbb{Z})$, and $c > 0$ an integer number.

Definition: This form is called **Bogomolov-Beauville-Fujiki form**. **It is defined by the Fujiki's relation uniquely, up to a sign.** The sign is determined from the following formula (Bogomolov, Beauville)

$$\begin{aligned}\lambda q(\eta,\eta) = & \int_X \eta\wedge\eta\wedge\Omega^{n-1}\wedge\overline{\Omega}^{n-1} - \\ & - \frac{n-1}{n}\left(\int_X \eta\wedge\Omega^{n-1}\wedge\overline{\Omega}^{n}\right)\left(\int_X \eta\wedge\Omega^{n}\wedge\overline{\Omega}^{n-1}\right)\end{aligned}$$

where Ω is the holomorphic symplectic form, and $\lambda > 0$.

Bogomolov-Beauville-Fujiki form on Bogomolov-Guan manifolds

THEOREM 2: Let M be a Bogomolov-Guan manifold, $\dim_{\mathbb{C}} M = 2n$. Then the space $H^2(M)$ is equipped with a bilinear symmetric form q such that **for any $\eta \in H^2(M)$, one has $\int_M \eta^{2n} = \lambda q(\eta,\eta)^n$, where λ is a fixed constant.**

DEFINITION: Let M be a complex manifold. We say that **$H^2(M)$ admits the Hodge decomposition** if any cohomology class in $H^2(M)$ can be represented by a sum of closed (p,q)-forms.

CLAIM: BG-manifolds admit the Hodge decomposition in $H^2(M)$.

Proof: This is true for Kodaira surfaces, and then follows for BG-manifolds using the standard results about topology of Hilbert schemes. ■

Theorem 2 is proven as follows: we associate with each complex structure the standard Hodge-theoretic $U(1)$-action on $H^2(M,\mathbb{R})$, with

$$\rho_t(\eta^{p,q}) = e^{\sqrt{-1}\,(p-q)2\pi}\eta^{p,q},$$

notice that the polynomial $Q(\eta) = \int_M \eta^{2n}$ is invariant under this $U(1)$-action, and apply the following elementary result.

Bogomolov-Beauville-Fujiki form on Bogomolov-Guan manifolds (2)

PROPOSITION: Let V be a real vector space equipped with an action of a Lie group G, and Q a G-invariant polynomial function. Let $S\subset Gr(2,V)$ be an open subset in the Grassmannian of 2-planes. Assume that for any $P\in S$, there exists a subgroup $\rho_P\subset G$ isomorphic to S^1 acting by rotations on P and trivially on V/P. **Then Q is proportional to q^n, where q is a quadratic form on V.**

Proof: Let $P\in S$ be a 2-plane in V. Any rotation-invariant polynomial function on $\mathbb{R}^2$ is a power of quadratic form, hence $Q|_P=\lambda q^n|_P$. When n is odd, the n-th root of Q is well defined. When n is even, the restriction $Q|_P$ does not change sign, hence Q does not change sign on the set $U_S\subset V$ of all vectors passing through planes $P\in S$. The function $q:=\sqrt[n]{\pm Q}$ is well defined on the whole of V when n is odd, and on U_S when it is even. Also, this function is polynomial of second degree on all hyperplanes $P\in S$. Therefore, **the third derivative $\frac{d^3}{dx^3}q|_y$ vanishes on V or U_S when $\langle x,y\rangle\in S$.** This function is real algebraic, hence by the analytic continuation principle, $\frac{d^3}{dx^3}q=0$ everywhere, for all linear coordinate functions x, and this implies that q is a second degree polynomial. ■