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Pushing the boundaries of fluid dynamics

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Theoretical Foundations of Relativistic Hydrodynamics

1) What new phenomena appear when viscous fluids are coupled to strong gravitational fields?

What new phenomena appear when viscous fluids are coupled to strong gravitational fields?

Neutron Star Mergers

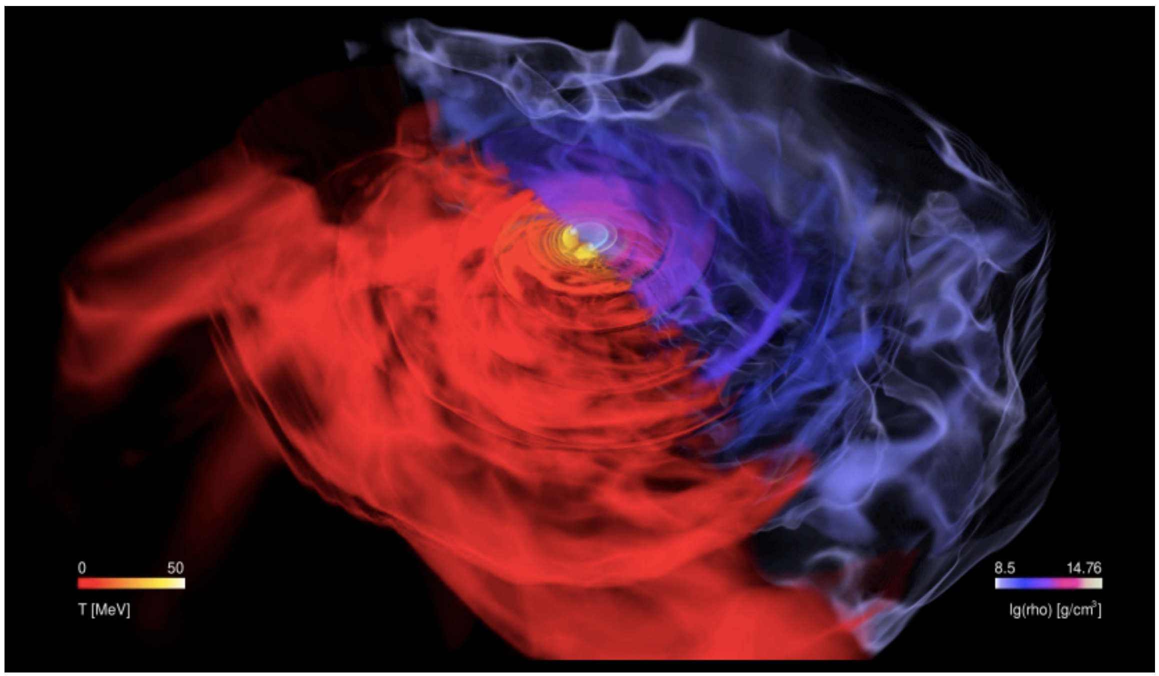


Fig. from L. Rezzolla's talk at QM2019

Reference: Most, Papenfort, Dexheimer, Hanauske, Schramm, Stöcker, Rezzolla, PRL (2019)

How does a lump of baryon rich QCD matter flow under strong gravitational fields?

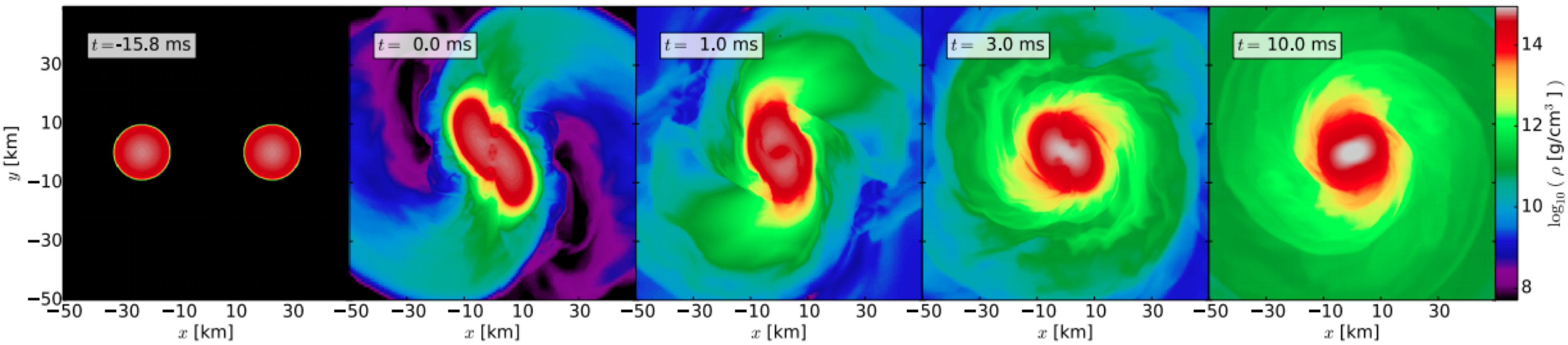


Fig. by L. Rezzolla

New signatures for deconfinement/phase transitions?

e.g. Most et al., PRL (2019)

Viscous fluid dynamics + strong gravitational fields?

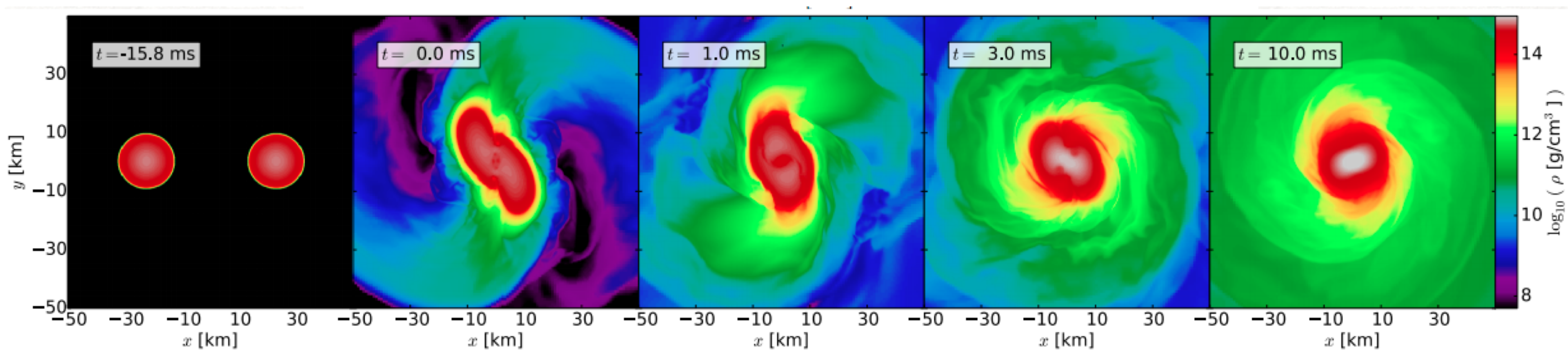
Viscous effects in neutron star mergers?

Duez et al PRD (2004), Shibata et al. PRD (2017), Alford et al. PRL (2018)

Viscous effects in binary neutron-star mergers?

Previous assumption (since 1992): viscous effects do not matter

Bildsten and Cutler, Astrophys. J. (1992)



Why? Based on the simulations/knowledge at that time:

- Transport time scales estimated to be far from ~ microseconds
- Temperatures not so large, system very smooth, gradients too small

Viscous effects in binary neutron-star mergers

Previous exceptions: Duez et al PRD (2004), Shibata et al. PRD (2017)

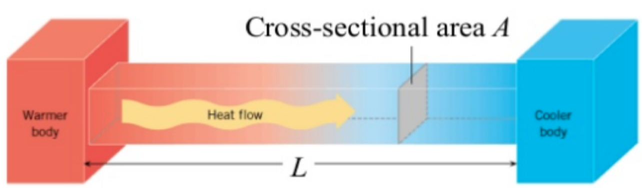
Alford, Bovard, Hanauske, Rezzolla, Schwenzer, PRL (2018):

Post-merger phase

Shear dissipation:

Relevant for trapped neutrinos if T > 10 MeV and gradients at small scales ~ 0.01 km (e.g, turbulence).

Thermal transport:



"heat conductivity"

Relevant for trapped electron neutrinos if T > 10 MeV and gradients ~ 0.1 km

Viscous effects in binary neutron-star mergers

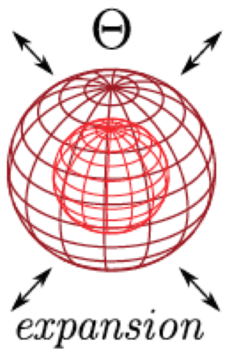
Alford et al. PRL (2018)

If suppressed

$B_1 \to B_2 + \ell + \bar{\nu}_\ell$;

$B_2 + \ell \to B_1 + \nu_\ell$,

Bulk viscosity:



Should affect density oscillations after merger!!!

See also Alford, Harris, PRC (2019)

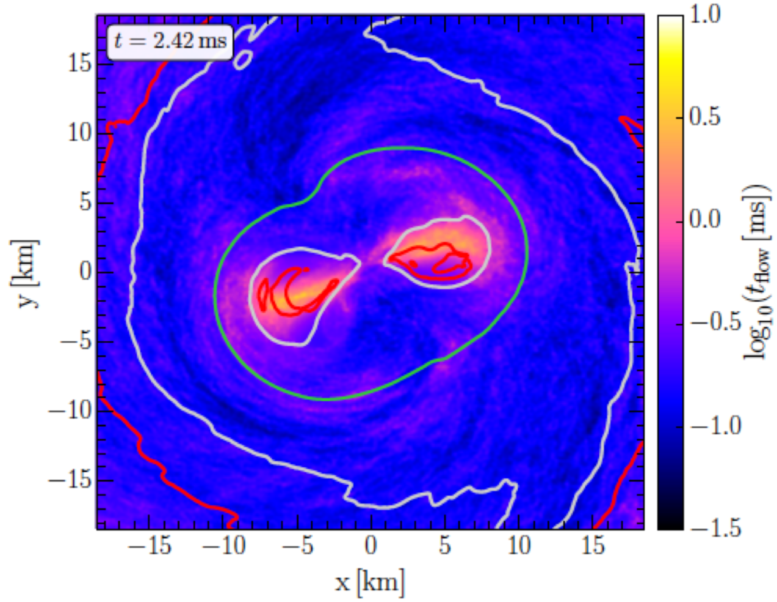
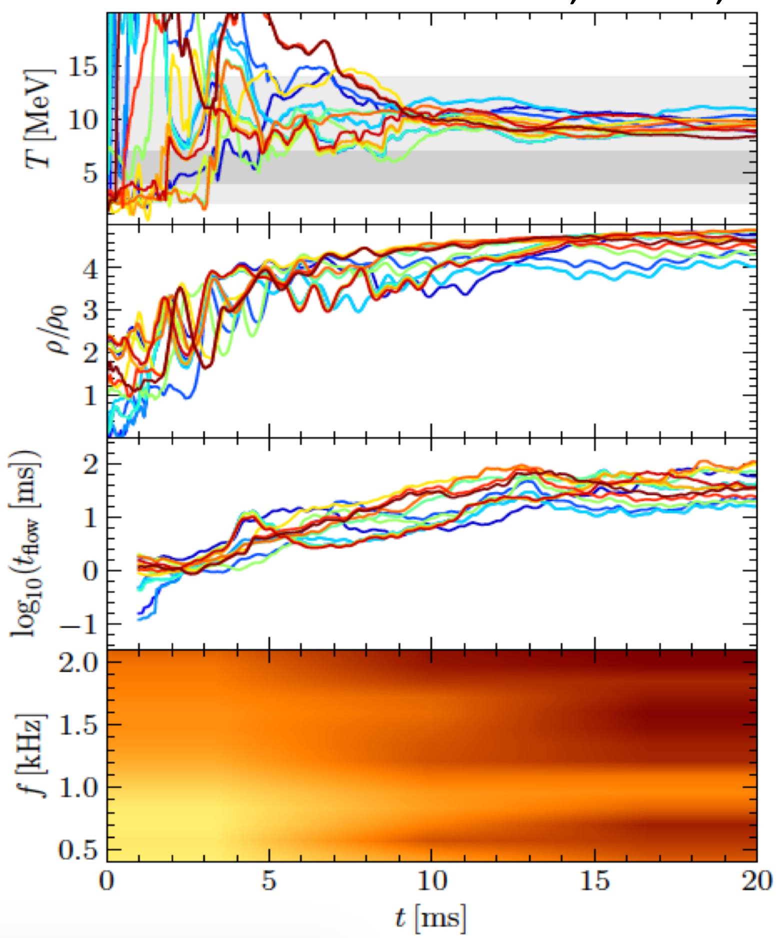


Figure 3: The flow timescale $t_{\rm flow}$ obtained from a numerical-relativity simulation of two 1.35 $M_\odot$ neutron stars [40]. The red (4 MeV) and gray (7 MeV) contours show the boundaries of the temperature range in which the bulk viscosity roughly takes its maximum value, while the green contour shows the inner region where the rest-mass density exceeds nuclear saturation density.



Viscous effects in binary neutron-star mergers

Alford et al. PRL (2018)

"The effects of bulk viscosity should be consistently included in future merger simulations. This has not been attempted before and requires a formulation of the relativistic-hydrodynamic equations that is hyperbolic and stable".

Challenge: Prove that the solutions $\{g_{\mu\nu}, T_{\mu\nu}, J^{\mu}_{B}\}$ are **well posed** (existence, uniqueness) and **causal in the full nonlinear regime.**

Einstein's equations

$$R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

Conservation laws

$$\nabla_{\mu} T^{\mu\nu} = 0 \qquad \nabla_{\mu} J^{\mu}_{B} = 0$$

+ Bulk Viscosity

Eckart
1940's

Relativistic Navier-Stokes equations

Energy-momentum tensor

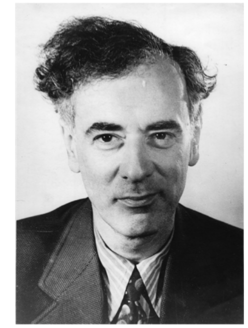
$$T^{\mu\nu} = \varepsilon u^\mu u^\nu + (P + \Pi)\Delta^{\mu\nu}$$

Bulk scalar

$$\Pi = -\zeta \nabla_\lambda u^\lambda$$

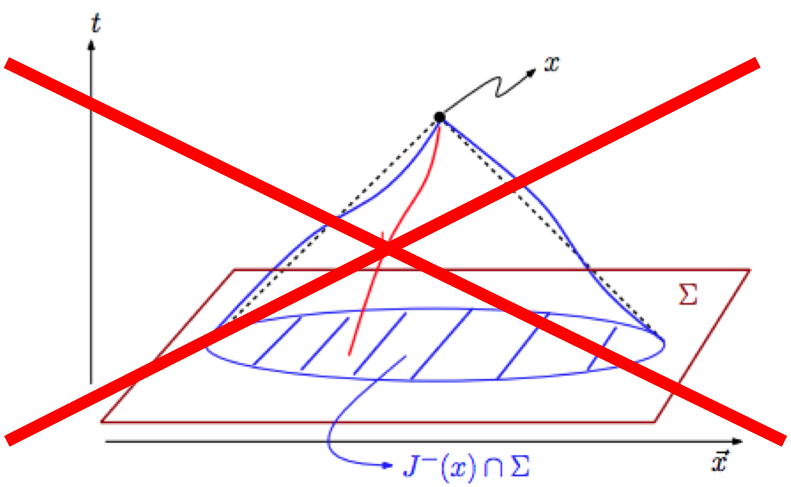
Conservation law

$$\nabla_\mu T^{\mu\nu} = 0$$



Landau
1950's

This theory is **acausal**
(mathematical proof by Pichon, 1965)



Nonlinear diffusion equations

$$\partial_t u = \frac{\zeta}{sT} \nabla^2 u + \dots$$

Also **unstable!!!**
(Hiscock, Lindblom, 1984)

How can one solve this problem?

- Heavy ion collisions require state-of-the-art developments in relativistic viscous fluid dynamics.

- Let us check how this is handled there …

"Hydro" in HIC is not simple textbook hydro

Israel-Stewart theory

Israel, Stewart, Ann. Phys. 118, 341 (1979)

Energy-momentum tensor

$$T_{\mu\nu} \Rightarrow \varepsilon, u_\mu, \pi_{\mu\nu}, \Pi$$ as dynamical variables

A theory for **hydrodynamic fields** and **non-hydrodynamic** fields

Dynamics: $\nabla_\mu T^{\mu\nu} = 0$ (energy-momentum conservation)

$$u^\lambda \nabla_\lambda \Pi + F(\varepsilon, \nabla_\alpha u_\beta, \pi^{\alpha\beta}, \Pi) = 0 \quad \text{(bulk)}$$

Many terms!
Many coefficients!

$$u^\lambda \nabla_\lambda \pi^{\mu\nu} + F^{\mu\nu}(\varepsilon, \nabla_\alpha u_\beta, \pi^{\alpha\beta}, \Pi) = 0 \quad \text{(shear)}$$

Despite the impressive progress in recent years, outstanding questions remain in the nonlinear regime:

Open problems in physics and mathematics (since 1940)

- **Can one formulate a theory of relativistic viscous fluids where causality holds in the full nonlinear regime?**
- **Is the mathematical problem well-posed?**

Results in IS are known only in the linearized regime around equilibrium

See, e.g, Hiscock, Lindblom, Ann. Phys. (1983)
Pu, Koide, Rischke, PRD (2010)

*Exception (nonlinear symmetric flow) in IS: Floerchinger, Grossi, JHEP (2018)

Nonlinearity is essential for mergers!!

Why is this so hard to do?

- 16 coupled nonlinear PDE's (Einstein-Israel-Stewart)

energy-momentum tensor

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu + (P + \Pi)\Delta^{\mu\nu}$$

conserved baryon current

$$J^\mu = n u^\mu$$

Israel-Stewart equation

$$\tau_\Pi u^\alpha \nabla_\alpha \Pi + \Pi + \zeta \nabla_\alpha u^\alpha + f(\varepsilon, n, \Pi) = 0$$

- Nonlinearity in hydrodynamics is notoriously hard to handle.

- Standard lore: Causality in the nonlinear Israel-Stewart eqs. seem to be beyond current mathematical techniques …

Bemfica, Disconzi, JN, PRL 122, 221602 (2019)

Einstein-Israel-Stewart equations can be written as (FOSH)

$$A_0(\Phi)\nabla_t\Phi + A^i(\Phi)\nabla_i\Phi + B(\Phi)\Phi = 0$$

where $\Phi = (\varepsilon, n, u_\mu, \Pi, g_{\mu\nu}, \partial g_{\mu\nu})$

Only bulk viscosity is included.

First mathematical proof of existence, uniqueness, hyperbolicity and causality in IS in the full nonlinear regime (fully embedded in GR)

Theorem 1. *Let $(\varepsilon, u^\alpha, \Pi, n, g_{\alpha\beta})$ be a solution to the generalized EIS equations defined on a globally hyperbolic spacetime M, and let Σ be a Cauchy surface. Suppose $\frac{\zeta}{\tau_\Pi(\varepsilon+P+\Pi)} + \alpha_1 + \frac{\alpha_2 n}{\varepsilon+P+\Pi} \geq 0$ and that (1) holds. Then, for any $p \in M$ in the future of Σ, $(\varepsilon(p), u^\alpha(p), \Pi(p), n(p), g_{\alpha\beta}(p))$ depends only on $(\varepsilon, u^\alpha, \Pi, n, g_{\alpha\beta}, \kappa_{\alpha\beta})|_{\Sigma\cap J^-(p)}$, where $J^-(p)$ is the causal past of p and κ is the extrinsic curvature of Σ in M.*

Theorem 2. *Let $\mathcal{I} = (\Sigma, \mathring{\varepsilon}, \mathring{u}^i, \mathring{\Pi}, \mathring{n}, \mathring{g}_{ij}, \mathring{\kappa}_{ij})$ be an initial data set for the generalized EIS system, with an equation of state $P = P(\varepsilon, n)$, a bulk viscosity $\zeta = \zeta(\varepsilon, n)$, and a relaxation time $\tau_\Pi = \tau_\Pi(\varepsilon, n)$. Assume that $\mathring{\varepsilon} + P(\mathring{\varepsilon}, \mathring{n}) + \mathring{\Pi}$, $\tau_\Pi(\mathring{\varepsilon}, \mathring{n})$, $\zeta(\mathring{\varepsilon}, \mathring{n}) > 0$, $\frac{\partial P}{\partial \varepsilon}(\mathring{\varepsilon}, \mathring{n}) + \frac{\partial P}{\partial n}(\mathring{\varepsilon}, \mathring{n})\mathring{n}/(\mathring{\varepsilon} + P(\mathring{\varepsilon}, \mathring{n}) + \mathring{\Pi}) \geq 0$, and that $\mathring{n}$, $\frac{\partial P}{\partial \varepsilon}(\mathring{\varepsilon}, \mathring{n})$, and $\frac{\partial P}{\partial n}(\mathring{\varepsilon}, \mathring{n})$ are nonzero. Suppose that $\mathring{g}_{ij} \in H^{s+1}_{ul}(\Sigma)$, $\mathring{\varepsilon}, \mathring{u}^i, \mathring{\Pi}, \mathring{n}, \mathring{\kappa}_{ij} \in H^s_{ul}(\Sigma)$, and that $P, \zeta, \lambda, \tau_\Pi \in C^s(\mathbb{R}^2)$, where $s \geq 3$. Suppose that (1) holds for $\mathcal{I}$. Then, there exists a globally hyperbolic development of $\mathcal{I}$.*

New nonlinear constraint

$$\left[\frac{\zeta}{\tau_\Pi} + n\left(\frac{\partial P}{\partial n}\right)_\varepsilon\right]\frac{1}{\varepsilon + P + \Pi} < 1 - \left(\frac{\partial P}{\partial \varepsilon}\right)_n$$

- The initial value problem of the full nonlinear set of Einstein-Israel-Stewart equations can now be solved.

- Valid for arbitrary EOS. No symmetry assumptions.

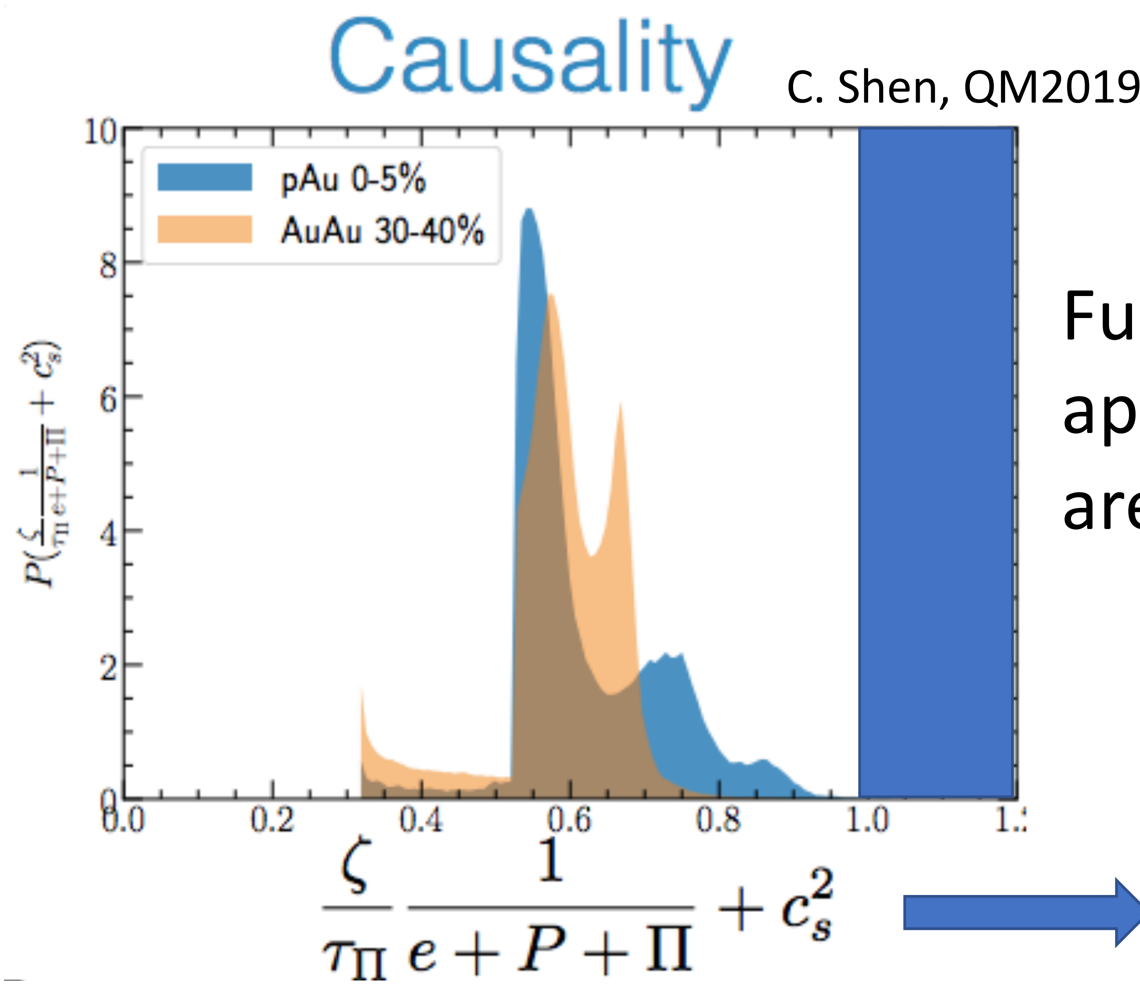
- Valid in the nonlinear regime (but no shear or diffusion).

- No fundamental issues appear if $P + \Pi < 0$ as long as $\varepsilon + P + \Pi > 0$

Causality constraint and small systems in heavy ions

$$\frac{\zeta}{\tau_\Pi(\varepsilon + P + \Pi)} < 1 - c_s^2$$

Zero baryon density



Further constraints will appear when shear effects are included!

Dependence on transport coeff. !!!

2) Is the description of relativistic viscous fluids unique?

Is the description of relativistic fluid dynamics unique?

- Israel-Stewart theory not unique (eg. rBRSSS vs. DNMR).

- **Transient** behavior of strongly coupled (holographic) liquids has different properties than Israel-Stewart theory.

Denicol, Niemi, JN, Rischke, PRD (2011)
Heller, Janik, Spalinski, Witaszczyk, PRL (2014)
See also Grozdanov, Lucas, Poovuttikul, PRD (2019)

Is there another way to describe the motion of viscous fluids that is compatible with (general) relativity?

Revisiting the gradient expansion

Hydrodynamics: Simplest effective theory for $\{T, u^\mu\}$

$\mu_B = 0$

$$T^{\mu\nu} = T^{\mu\nu}_{eq}(T, u^\lambda) + T^{\mu\nu}_{viscous}$$

Viscous correction? Gradient expansion

Chapman-Enskog, 1930's
BRSSS, JHEP (2008)
BMHR, JHEP (2008)

$$T^{\mu\nu}_{viscous} = \underbrace{T^{\mu\nu}_{viscous}(\nabla T, \nabla u)}_{\text{All possible 1st order terms}} + \underbrace{\mathcal{O}(\nabla^2)}_{\text{higher orders}}$$

All possible 1st order terms
with no time derivatives in the local rest frame

higher orders

Revisiting the gradient expansion

IMPORTANT: **If** Landau definition is assumed to be valid throughout $\Rightarrow$ $u_\mu T^{\mu\nu} = -\varepsilon u^\nu$

- First order truncation leads to relativistic Navier-Stokes theory, which is **acausal and unstable**.

- This is not fixed by going to 2^{nd} order in gradients (note this is **NOT** Israel-Stewart theory).

- Can one make the gradient expansion causal and stable?

A new approach to relativistic viscous fluid dynamics

Based on Bemfica, Disconzi, JN, PRD (2017) and PRD (2019)
See also P. Kovtun, JHEP (2019) and his talk later in the week!

Effective theory: Space-time derivative expansion

$$T^{\mu\nu}_{viscous}(\nabla T, \nabla u) \Rightarrow$$ Most general **derivative** expansion compatible with symmetries

- Definition of T and u^{μ} not unique out of equilibrium.
- No reason to expect a priori that Landau's definition is a reasonable frame choice.

Tsumura, Kunihiro, PLB (2008)
Van, Biro, EPJ ST (2008)

A new approach to relativistic viscous fluid dynamics

Based on Bemfica, Disconzi, JN, PRD (2017) and PRD (2019)
See also P. Kovtun, JHEP (2019) and his talk later in the week!

Most general derivative expansion compatible with symmetries

$$T^{\mu\nu} = (\varepsilon + \mathcal{A}_1) u^\mu u^\nu + (P(\varepsilon) + \mathcal{A}_2)\,\Delta^{\mu\nu} - 2\eta\sigma^{\mu\nu} + u^\mu \mathcal{Q}^\nu + u^\nu \mathcal{Q}^\mu$$

where to 1st order in derivatives

$$\mathcal{A}_1 = \chi_1 \frac{u^\alpha \nabla_\alpha \varepsilon}{\varepsilon + P} + \chi_2 \nabla_\alpha u^\alpha, \qquad \mathcal{A}_2 = \chi_3 \frac{u^\alpha \nabla_\alpha \varepsilon}{\varepsilon + P} + \chi_4 \nabla_\alpha u^\alpha, \qquad \mathcal{Q}_\mu = \lambda \left(\frac{c_s^2 \Delta_\mu^\nu \nabla_\nu \varepsilon}{\varepsilon + P} + u^\alpha \nabla_\alpha u_\mu \right)$$

Energy density correction — Pressure correction — Heat flow

Equations of motion: $\nabla_\mu T^{\mu\nu} = 0$

Causality and well-posedness are valid in the full nonlinear regime, also including Einstein's equations, when

Rigorous theorems in Bemfica, Disconzi, JN, PRD (2019)

$$\lambda, \chi_1 > 0, \; \eta \geq 0$$

$$\lambda \geq \eta,$$

$$3\chi_4 \geq 4\eta,$$

$$\lambda\chi_1 + c_s^2\lambda\left(\chi_4 - \frac{4\eta}{3}\right) \geq c_s^2\lambda\chi_2 + \lambda\chi_3 + \chi_2\chi_3 - \chi_1\left(\chi_4 - \frac{4}{3}\eta\right) > 0.$$

Arbitrary EOS

Linear stability also holds:

P. Kovtun, JHEP (2019).
Bemfica, Disconzi, JN, PRD (2019).

- Heat flow coefficient lower bound from causality.
- Only 6 coefficients (Israel-Stewart > 10).
- No additional fields besides $\{T, u^\mu\}$.
- Very different δf at freeze-out (scalar, vector, tensor).

Causal and stable 1st order theory: Benefits

- Universal behavior using only hydro variables (in contrast to IS): Causality and stability define a class of “good” frames!!

- Initial value problem and causality understood in the nonlinear regime + Einstein’s equations (in contrast to IS).

- Can be readily applied in HIC, neutron star mergers, and cosmology (numerical simulations!!).

- Derivation from kinetic theory is understood. AdS/CFT?

BDN, PRD (2017) and (2019)

- Natural starting point to solve relativistic hydro with fluctuations (connection to action principle approaches?).

New causal 1^{st} order theory
Many remaining questions

- Does not describe all info contained in the energy-mom. tensor – Does this matter for HIC/Cosmo/Mergers?

- Although attractors appear here, the domain of the 1^{st} order theory has not yet been understood.

- Causality and stability beyond first order must be done.

- 1^{st} order causality and stability at $\mu \neq 0$

  P. Kovtun's talk for $\mu \neq 0$

- Large order behavior of this derivative series not known.

Conclusions

- Neutron star mergers can also be used to obtain essential info about viscous properties of ultradense matter.

- New results allow the investigation of dissipative phenomena in general relativistic fluids for the first time.

- Nonlinear causality may impose severe new constraints on the IS description of small heavy-ion collision systems.

- New 1st order approach to relativistic viscous hydro offers a simple and natural solution to known problems and creates new questions/opportunities.

Next steps

- Extend proof in Israel-Stewart theory to include shear viscous effects (to appear soon).

- Implement new viscous hydro equations in a code and investigate phenomenological consequences in heavy ions.

- Novel viscous effects in the evolution of the universe.

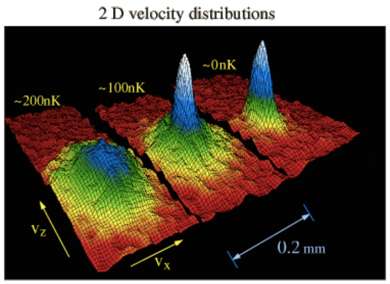
  Bemfica, Disconzi, Noronha, Scherrer, to appear

- Stochastic formulation of new viscous hydro equations (applications in heavy ions and cosmology).

- Inclusion of critical phenomena (BES II, FAIR).

ADDITIONAL SLIDES

Emergence of fluid dynamics: A bird's-eye view

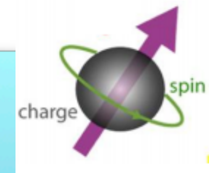


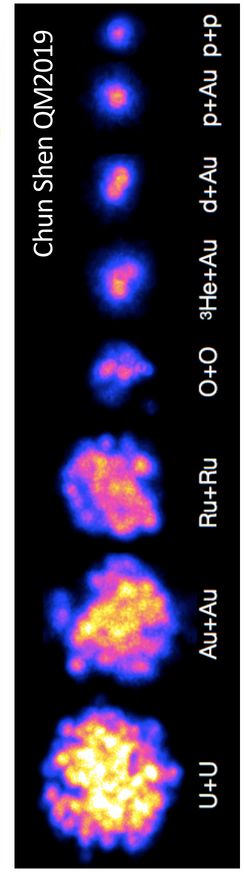
Bose-Einstein
condensate

$\hbar$

Quantumness

"Fluid"

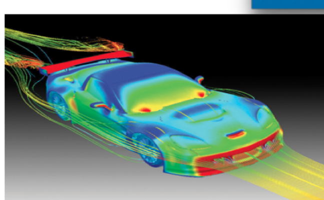




small
systems

HIC

Large
systems

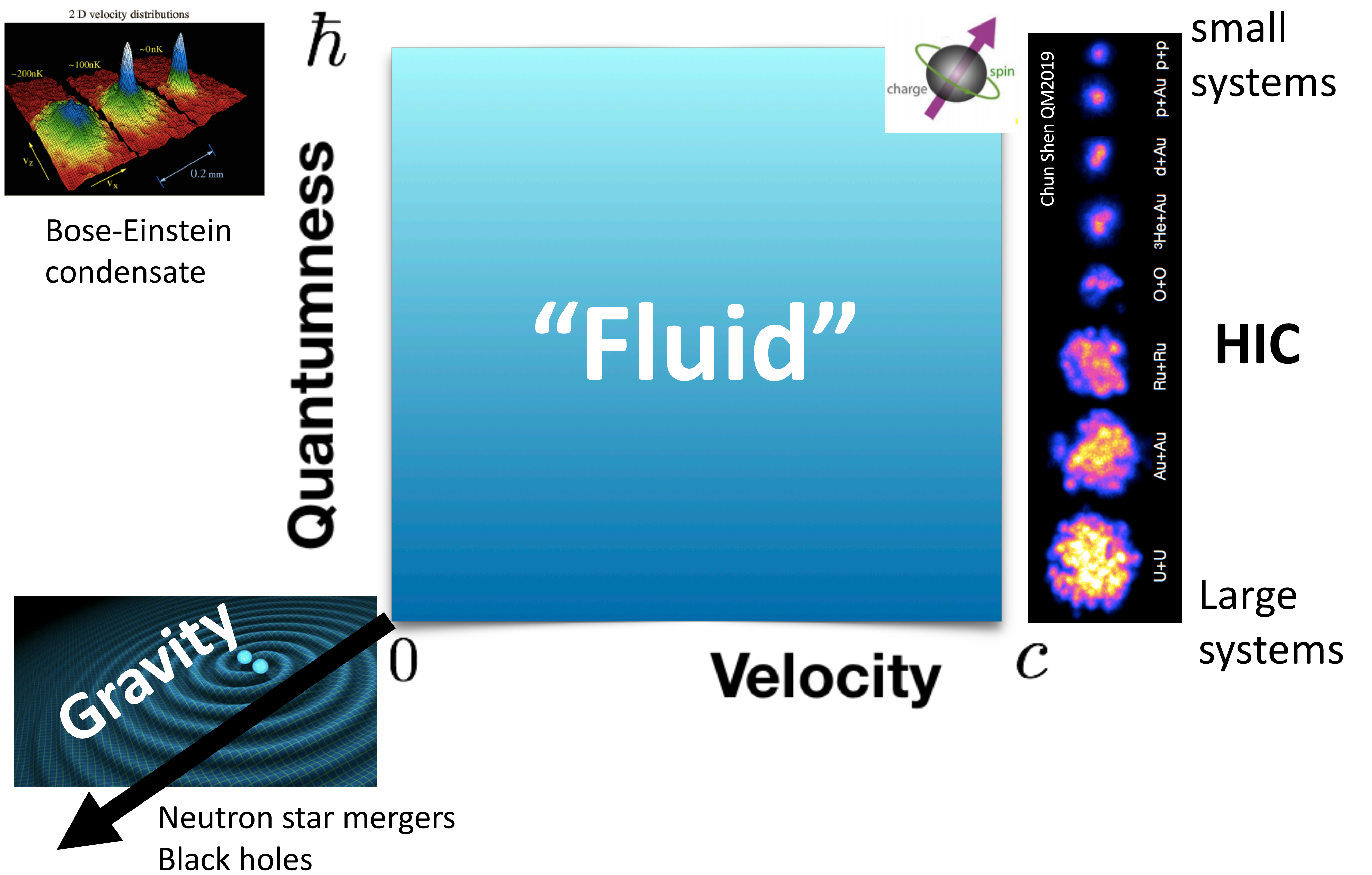


Daily life

Velocity

c

Emergence of fluid dynamics: A bird's-eye view



Bemfica, Disconzi, JN, PRD (2019)

Theorem I. Consider the energy-momentum tensor (1) and assume that λ, χ_a, $a = 1, \ldots, 4$, η, and P are given real valued functions with domain $(0, \infty)$, where we recall that in (1) these quantities are functions of ε, i.e., $\lambda = \lambda(\varepsilon)$, $\chi_a = \chi_a(\varepsilon)$, $\eta = \eta(\varepsilon)$, and $P = P(\varepsilon)$. Suppose that λ, χ_a, η, and P are $G^{(s)}$ regular. Let $\mathcal{I} = (\Sigma, \mathring{\varepsilon}, \mathring{\mathcal{E}}, \mathring{u}, \mathring{U})$ be an initial data set for Einstein's equations coupled to (1). Assume that the initial data belongs to $G^{(s)}(\Sigma)$. Suppose that Σ is compact and that $\mathring{\varepsilon} > 0$. Suppose that $P' \geq 0$, that $\lambda > 0$, $\chi_1 > 0$, $\eta \geq 0$, and that conditions (6) and (7) hold. Finally, assume that $1 < s < 20/19$. Then, there exist a four-dimensional Lorentzian manifold (M, g), a vector field u and a real valued function ε, both defined on M, such that:

(1) Einstein's equations coupled to (1) hold in M.

(2) There exists an isometric embedding $i : (\Sigma, \mathring{g}) \to (M, g)$ with second fundamental form κ.

(3) Identifying Σ with its image $i(\Sigma)$ in M, we have $\varepsilon|_\Sigma = \mathring{\varepsilon}$ and $\Pi_\Sigma(u) = \mathring{u}$, where Π_Σ :

that $\{x^i\}_{i=1}^3$ are coordinates on Σ, then $\partial_0 \varepsilon|_\Sigma = \mathring{\mathcal{E}}$ and $\partial_0 u^i|_\Sigma = \mathring{U}$.

(4) (M, g) is globally hyperbolic with Cauchy surface $i(\Sigma)$.

(5) (M, g) is causal, in the following sense: for any x in the future[5] of $i(M)$, $(g(x), u(x), \varepsilon(x))$ depends only on $\mathcal{I}|_{i(\Sigma) \cap J^-(x)}$, where $J^-(x)$ is the causal past of x (with respect to the metric g).

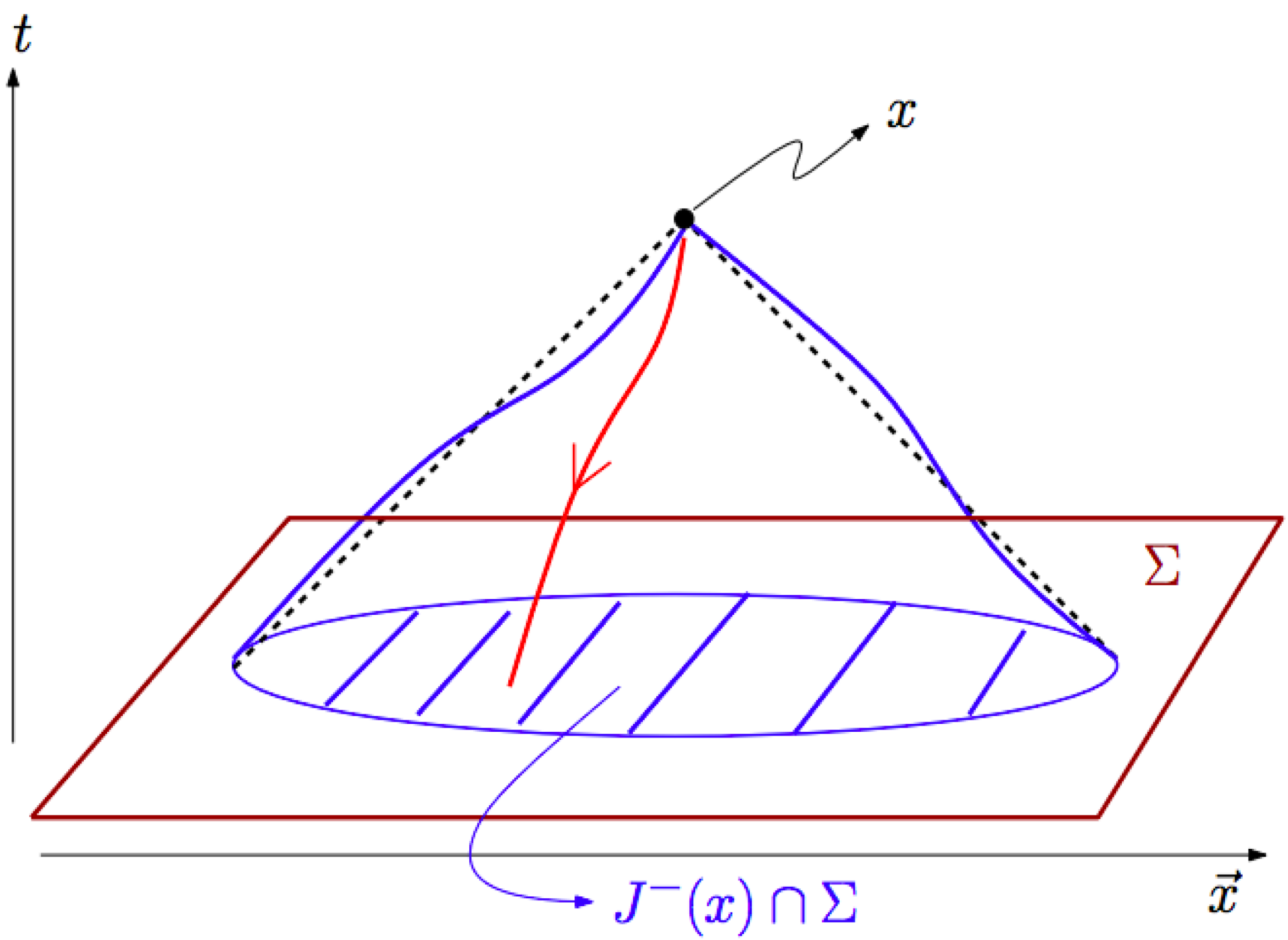
(6) (M, g) is unique up to actions of diffeomorphisms of M.

Theorem II. The same conclusions of Theorem I hold if one assumes that the initial data belongs to the Sobolev space H^s for sufficiently large s.

See also Bemfica, Disconzi, Rodriguez, Shao, arXiv:1911.02504 [math.AP].

Why is this so hard to do?

Causality in the nonlinear regime in curved spacetime



How can a pp system behave like a fluid?

Proton at high energies: average shape but with strong color fluctuations

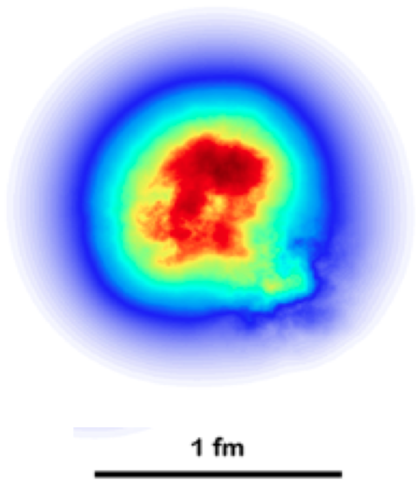


Fig. from Mantysaari, Schenke, PRL (2016)

A simple uncertainty principle estimate

$$\Delta x \sim 0.1\,\mathrm{fm}$$

$$\Delta p \sim \frac{1}{\Delta x} \sim 2\,\mathrm{GeV} > \langle p_T \rangle$$

Quantum correlations should be important !!

$$\langle \hat{T}^{\mu\nu} \rangle, \langle \hat{T}^{\mu\nu} \hat{T}^{\alpha\beta} \rangle, \ldots$$

- **Opportunity:** Investigate quantum entanglement in a non-Abelian theory.
- Here we should really go beyond the “everything is hydro approach” …

Ideal (Euler) relativistic fluid dynamics

Energy-momentum tensor

$$T^{\mu\nu} = \epsilon u^\mu u^\nu + P(\epsilon, n)\Delta^{\mu\nu}$$

Conserved charge

$$J^\mu = n u^\mu$$

$$\Delta_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$$

Assuming that speed of sound and $\epsilon + P > 0$

$$u_\mu u^\mu = -1$$

- System is well-posed (Sobolev space H^s with s > 5/2)
- - Existence, uniqueness, continuous dependence on initial data

- Causal (domain of dependence property in the sense of relativity)

Remember that a system of PDE's is well-posed (Hadamard) when:

http://www.math.ucla.edu/~tao/Dispersive/

1) A solution exists.

2) The solution is unique.

3) The solution depends continuously on initial data (e.g., initial conditions, boundary conditions).

Ex.: $u_{xx} + u = 0$

a) $u(0) = 0, u(\frac{\pi}{2}) = 1 \Rightarrow$ unique solution $u(x) = \sin(x)$
b) $u(0) = 0, u(\pi) = 1 \Rightarrow$ no solution
c) $u(0) = 0, u(\pi) = 0 \Rightarrow$ infinitely many solutions: $u(x) = A\sin(x)$

Ex.:

$$\left.\begin{cases} u_t = u_{xx} & \text{heat equation} \\ u(0,t) = u(1,t) = 0 & \text{boundary conditions} \\ u(x,0) = u_0(x) & \text{initial conditions} \end{cases}\right\} \text{well-posed}$$

$$\left.\begin{cases} u_t = -u_{xx} & \text{backwards heat equation} \\ u(0,t) = u(1,t) & \text{boundary conditions} \\ u(x,0) = u_0(x) & \text{initial conditions} \end{cases}\right\} \begin{array}{l}\text{no continuous dependence} \\ \text{on initial data}\end{array}$$

Mathematical definition of causality (relativity)

See, e.g., Choquet-Bruhat, Wald

Consider a system of (linear or nonlinear) PDE's

N unknowns

$$P^I_K \varphi^K = 0 \qquad \{\phi^K\}_{K=1}^N$$

The system is causal if for any point x in the future of Σ

$\varphi^K(x)$ depends only on $J^-(x) \cap \Sigma$

causal past

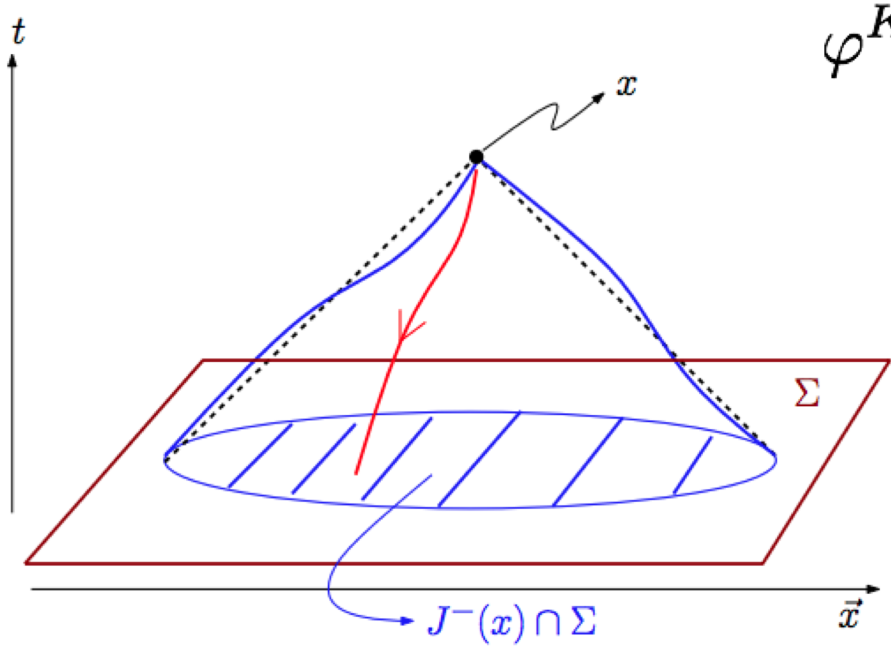


FIG. 1: (color online) Illustration of causality. In curved spacetime $J^-(x)$ looks like a distorted light-cone opening to the past (blue region); in flat spacetime the cone would be straight (dotted line). Points inside $J^-(x)$ can be joined to a point x in spacetime by a causal past directed curve (e.g. the red line). The Cauchy surface Σ supports the initial data and the value of the field $\varphi(x)$ depends only on the initial data on $J^-(x) \cap \Sigma$.

Fluid dynamics in the relativistic regime

- Einstein + Euler equations: Locally well-posed and causal

Choquet-Bruhat 1958, 1966
Lichnerowicz, 1967

- Einstein + Euler equations: Not globally well-posed (schocks occur)

Christodoulou, 2007

What about dissipative fluids? $T^{\mu\nu} = T^{\mu\nu}_{ideal} + \pi^{\mu\nu}$

(a) Einstein+viscous fluid admit existence + uniqueness of solutions?
(b) Causality?
(c) Stability (at least in the linear regime)?
(d) Does the solution really describe the physics of the system?

Causality does not imply (linear) stability

$$T^{\alpha\beta} = T^{\alpha\beta}_{ideal} + \left(\lambda + \frac{\nu}{4}\right) g^{\alpha\beta}(\nabla_\lambda C^\lambda) - \frac{\nu}{2}\left(\nabla^\alpha C^\beta + \nabla^\beta C^\alpha\right)$$

Dynamic velocity $C^\alpha = (\epsilon + P)u^\alpha$

Choquet-Bruhat, 2006

$\nabla_\mu T^{\mu\nu} = 0$ → Principal part: $\frac{\nu}{2}\{\nabla_\alpha\nabla^\alpha C^\beta + \nabla_\alpha\nabla^\beta C^\alpha\} + \left(\lambda - \frac{1}{3}\nu\right)\nabla^\beta\nabla_\alpha C^\alpha.$

Characteristic matrix: $\frac{\nu}{2}X_\alpha X^\alpha C^\beta + aX_\alpha X^\beta C^\alpha \quad a := \frac{\nu}{6} + \lambda.$

Characteristic determinant: $\frac{\nu}{2}\left(\frac{\nu}{2} + a\right)(X^\alpha X_\alpha)^4$ → **CAUSAL**

However, this system is linearly unstable around global equilibrium

Bemfica, Disconzi, JN, 2017

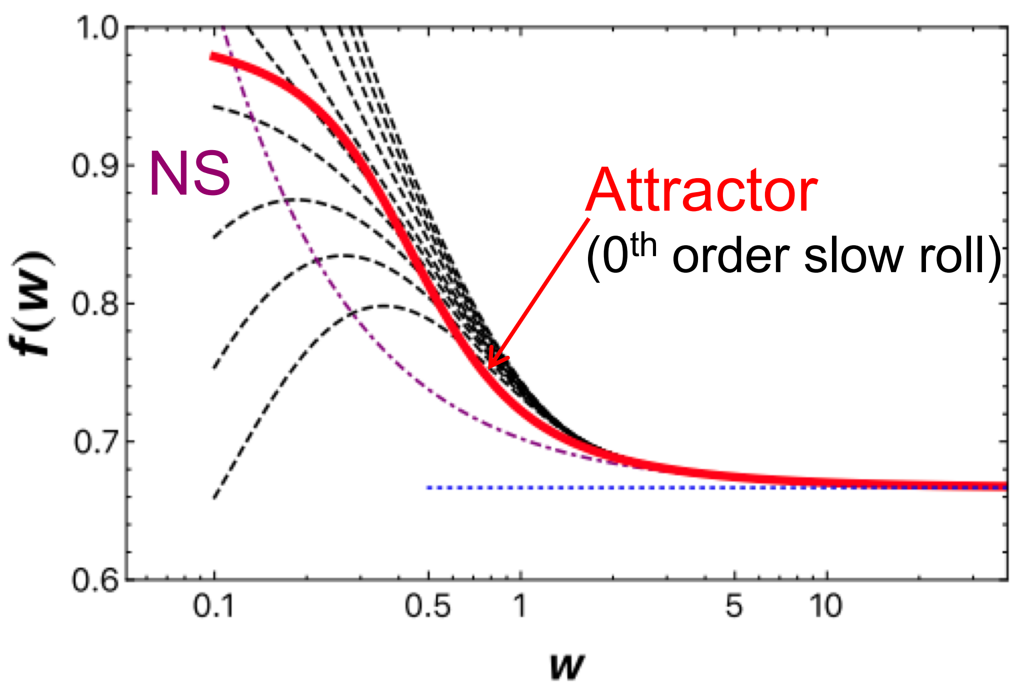
Applications: Bjorken flow

Milne coordinates $u_\mu = (-1,0,0,0)$ $T \to T(\tau)$

Equation of motion: $w = \tau T$ and $f = 1 + \tau \partial_\tau T / T$

Heller-Spalinski form

$$\bar{\chi} w f(w) \frac{df(w)}{dw} + 3\bar{\chi} f(w)^2 + f(w)\left(w - \frac{14}{3}\bar{\chi}\right) + \frac{16\bar{\chi}}{9} - \frac{4\bar{\eta}}{9} - \frac{2w}{3} = 0$$



$\eta/s = 0.08$

$\chi = 4\eta$

Applications: Gubser flow

$dS_3 \otimes \mathbb{R}$ spacetime $\quad u_\mu = (-1, 0, 0, 0) \qquad T \to \hat{T}(\rho)$

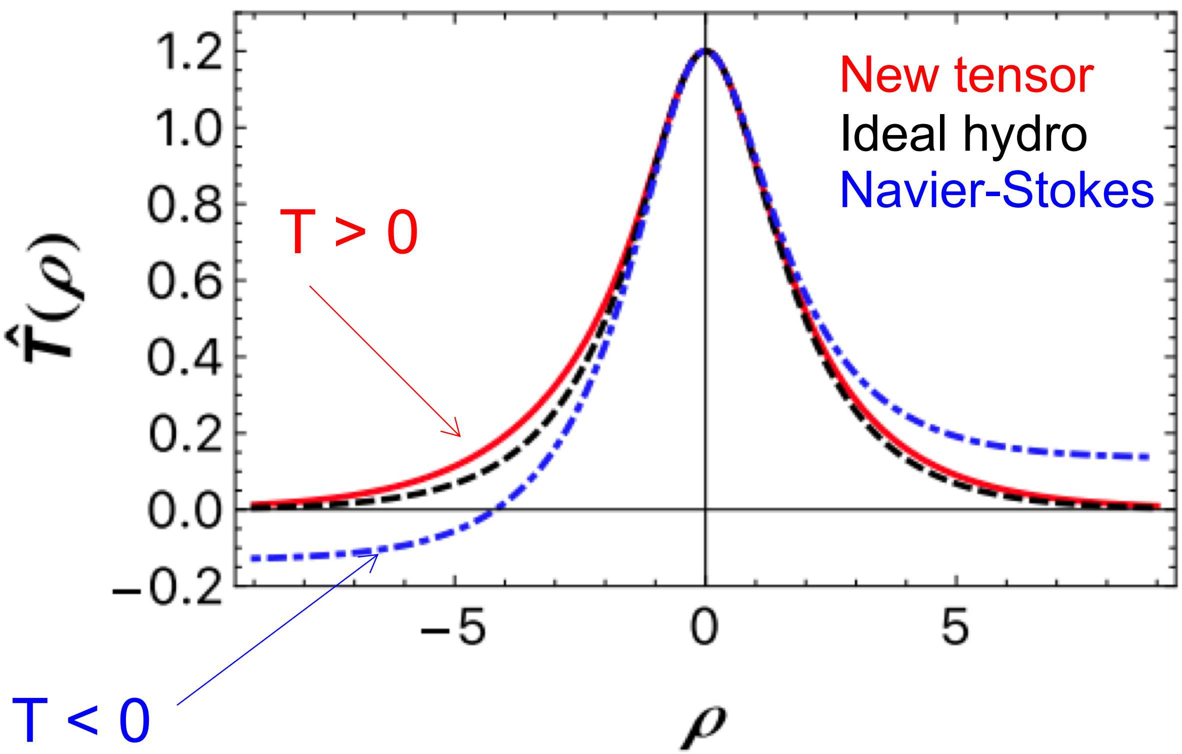
Equation of motion (written in 1st order form):

$$\frac{1}{\hat{T}}\frac{d\hat{T}}{d\rho} + \frac{2}{3}\tanh\rho = \hat{\mathcal{F}}(\rho) \qquad \bar{\chi}\frac{d\hat{\mathcal{F}}}{d\rho} + 3\bar{\chi}\hat{\mathcal{F}}^2 + \frac{2}{3}\bar{\chi}\hat{\mathcal{F}}\tanh\rho + \hat{T}\hat{\mathcal{F}} - \frac{4}{9}\bar{\eta}(\tanh\rho)^2 = 0$$

$\eta/s = 0.2$

$\chi = 4\eta$

$\hat{T}_0 = 1.2$



Definition 8.6 (Strongly causal spacetime)
A spacetime $\mathscr{M}$ is strongly causal *if given an arbitrarily chosen event $\mathbf{p} \in \mathscr{M}$ for each $U \subset \mathscr{M}$ open neighborhood of $\mathbf{p}$ there exist another open neighborhood of $\mathbf{p}$, $V \subset U$, such that no casual curve intersects it more than once.*

Definition 8.7 (Inextendible causal curve)
A causal curve γ_C is called future *(resp.* past*)* inextendible *if it is impossible to find an event $\mathbf{p} \in \mathscr{M}$ such that for all $U \subset \mathscr{M}$, U neighborhood of $\mathbf{p}$, there exist a t' such that $\gamma_C(t) \in U$ for all $t > t'$ (resp $t < t'$)*

In more concrete words, this means that γ_C has no *future* (resp. *past*) *endpoint*.

Definition 8.10 (Domains of dependence)
Let $\mathscr{A}$ be a closed achronal set. The set $D^+(\mathscr{A})$ (resp. $D^-(\mathscr{A})$) of all spacetime events $\mathbf{p}$ such that every past (resp. future) inextendible causal curve passing through $\mathbf{p}$ intersects $\mathscr{A}$ is called the future *(resp.* past*)* domain of dependence of $\mathscr{A}$. *The set $D(\mathscr{A}) = D^+(\mathscr{A}) \cup D^-(\mathscr{A})$, union of the past and of the future domains of dependence is the* domain of dependence *of $\mathscr{A}$.*

Definition 8.11 (Cauchy surface and global hyperbolicity)
Let $\mathscr{A} \subset \mathscr{M}$ be an achronal set such that $D(\mathscr{A}) = \mathscr{M}$. Then $\mathscr{A}$ is called a Cauchy surface *(we instead use the denomination* partial Cauchy surface *for a closed achronal set without edge). A spacetime $\mathscr{M}$ which admits a Cauchy surface is called* globally hyperbolic.

Analytic functions obey (α = multi-index):

$$|\partial^\alpha f| \leq C^{|\alpha|+1}\,\alpha!$$

The Gevrey class $\gamma^{(\sigma)}$, $\sigma > 1$, consists of C^∞ functions that obey the weaker inequality:

$$|\partial^\alpha f| \leq C^{|\alpha|+1}(\alpha!)^{\sigma}.$$

Advantage: large class of functions, including compactly supported (not determined by values on an open set).

The larger the σ, the larger the space. Larger σ: more general results.

$\gamma^{(\infty)}$ = Sobolev space.

$\gamma^{(\sigma)}$: used in the study of non-relativistic viscous fluids; also have had applications in General Relativity (magneto-hydrodynamics).

Sobolev H_s

M. Disconzi

$$||f||_s^2 = \sum_{|j|\leq s} ||\partial^j u||_{L^2}^2$$

Characteristics.

Consider the linear differential operator:

$$Lu = a^{\mu\nu}(x)\frac{\partial^2 u}{\partial x^\mu \partial x^\nu} + b(x, \partial u)$$

or, more generally (α=multi-index),

$$Lu = \sum_{|\alpha|=m} a^\alpha(x)\partial_\alpha^{|\alpha|} u + b(x, \partial^{m-1}u, \dots, \partial u, u).$$

We define the characteristic cone V_x of L at $T_x^* M$ by

$$h(x, \xi) \equiv \sum_{|\alpha|=m} a^\alpha(x)\xi_\alpha = 0.$$

$h(x, \xi)$ (=characteristic polynomial) is a homogeneous polynomial of degree m.

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Hyperbolic polynomials (Leray).

$h(x,\xi)$ is called a hyperbolic polynomial (at x) if there exists $\zeta \in T^*_x M$ such that every line through ζ that does not pass through the origin intersects V_x at m real distinct points ($m =$degree of $h=$ order of L).

In this case, the set of $\zeta \in T^*_x M$ with this property forms the interior of two opposite convex half-cones $\Gamma^\pm_x$.

The differential operator L is called hyperbolic (at x) if $h(x,\xi)$ is hyperbolic.

Dualizing, one obtains $C^\pm_x \subset T_x M$. For example

$$C^+_x = \{ v \in T_x M \,|\, \zeta(v) \geq 0 \text{ for all } \zeta \in \Gamma^+_x \}.$$

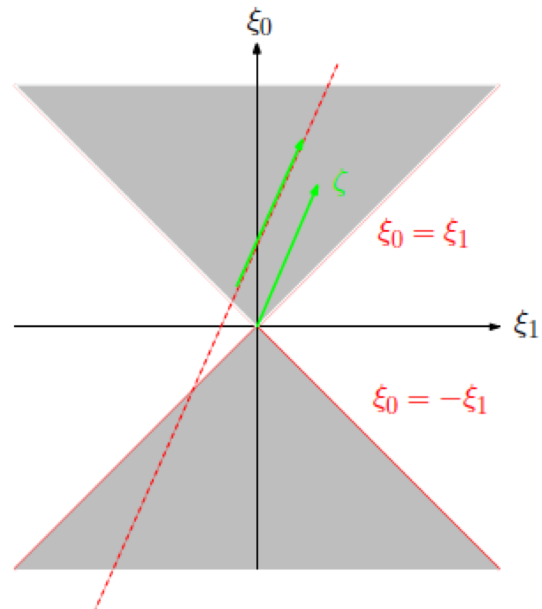
$\Sigma^n = \{\varphi(x) = 0\} \subset M^{n+1}$ is characteristic for L if

$$\sum_{|\alpha|=m} a^\alpha(x)\partial_\alpha \varphi = 0.$$

Wave equation: characteristics.

Consider $Lu = u_{tt} - u_{xx}$, $\xi = (\xi_0, \xi_1)$. Then:

$$\xi_0^2 - \xi_1^2 = 0 \Rightarrow \xi_0 = \pm\xi_1.$$



Γ_x and C_x are both given by the "light-cone".

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Hyperbolic and weakly hyperbolic operators.

Hyperbolic operators (sometimes called strictly hyperbolic) have a Cauchy problem that is well-posed in Sobolev spaces.

When the definition of a hyperbolic polynomial is weakened to:

there exists $\zeta \in T_x^* M$ such that every line through ζ that does not pass through the origin intersects V_x at m, not necessarily distinct, real points, we obtain weakly hyperbolic polynomials and operators (m =degree of h=order of L).

Weakly hyperbolic operators are well-posed in Gevrey spaces, but there are counter-examples to well-posed in Sobolev spaces.

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