From spin chains to real-time thermal field theory using tensor networks

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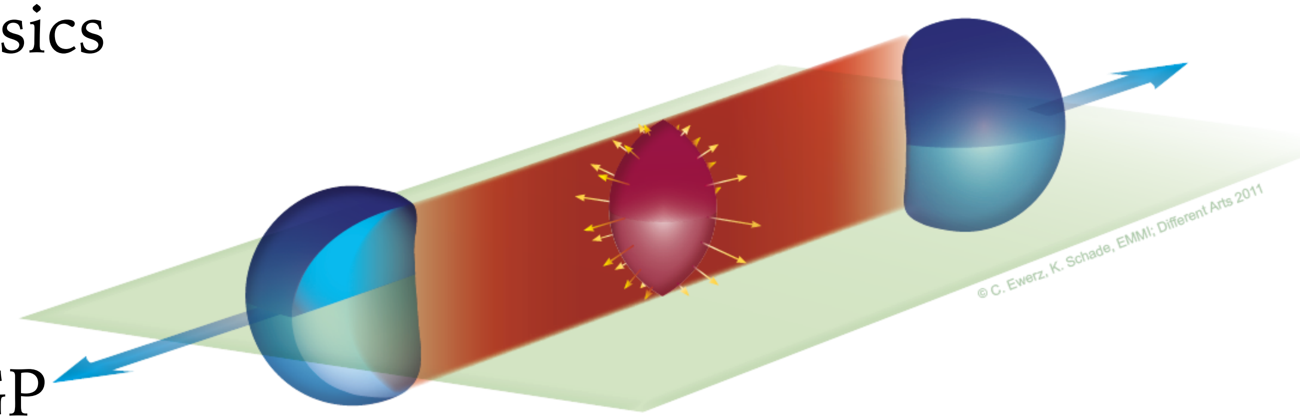
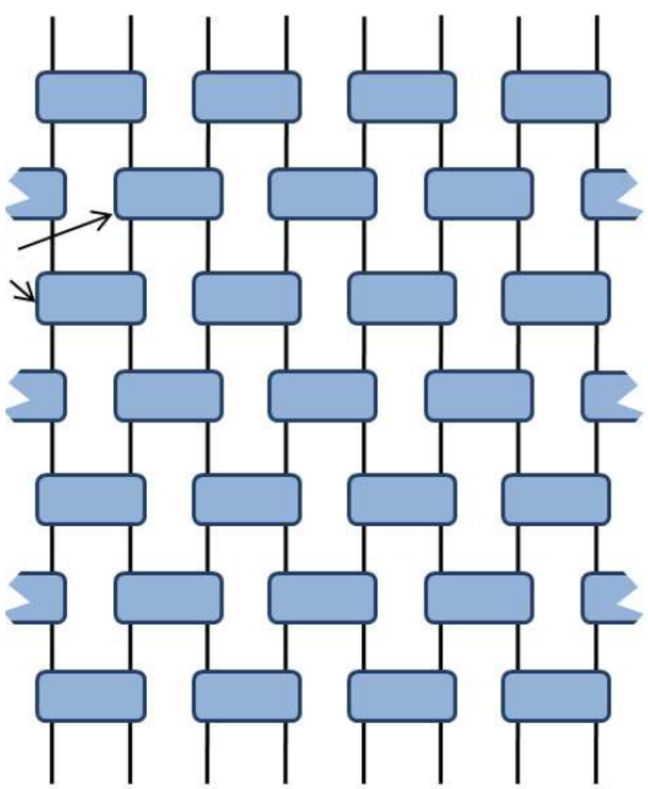
Introduction and Motivation

- the understanding of quantum many-body systems is of central interest in condensed matter and high-energy physics
- collective phases of QCD matter are probed in heavy-ion collisions:

  relaxation from non-equilibrium to QGP





- Tensor Networks (TNs) are representations of quantum many-body states in a tensor product basis
- They capture relevant entanglement properties and allow efficient time simulation

$\Rightarrow$ explore thermal quenches of 1D Ising spin chain to extract real-time QFT dynamics

The quantity we are interested in...

- dynamics in linear response theory:

$$\delta\langle\mathcal{O}(t,x)\rangle = \int d\tilde{t}\, d\tilde{x}\, G_R(t-\tilde{t}, x-\tilde{x})\, \delta J(\tilde{t},\tilde{x})$$

local operator

source of Hamiltonian H

retarded 2-point function at non-zero temperature $T = 1/\beta$

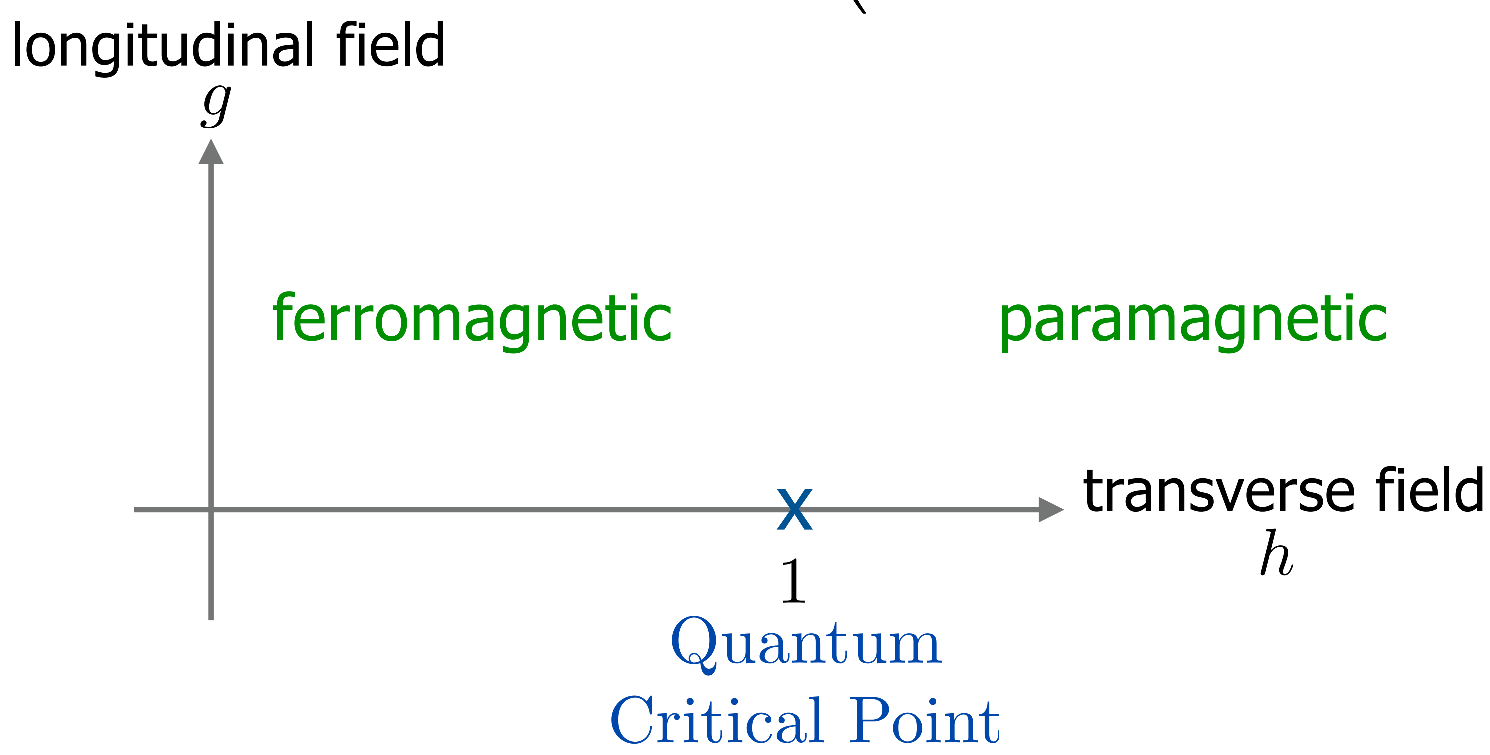
$$G_R(t-\tilde{t}, x-\tilde{x}) = i\,\theta(t-\tilde{t})\ \mathrm{Tr}\left(Z_\beta^{-1}\, e^{-\beta H}\, [\mathcal{O}(t,x), \mathcal{O}(\tilde{t},\tilde{x})]\right)$$

- in Fourier space, time response is governed by structure of $G_R(\omega, p)$ in complex ω plane:

$$\delta\langle\mathcal{O}(t,p)\rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega\, G_R(\omega,p)\, \delta J(-\omega,-p)\, e^{-i\,\omega\, t}$$

The quantum Ising model

$$H = -J\left(\sum_{j=1}^{N-1}\sigma_z^j\sigma_z^{j+1} + h\sum_{j=1}^{N}\sigma_x^j + g\sum_{j=1}^{N}\sigma_z^j\right)$$



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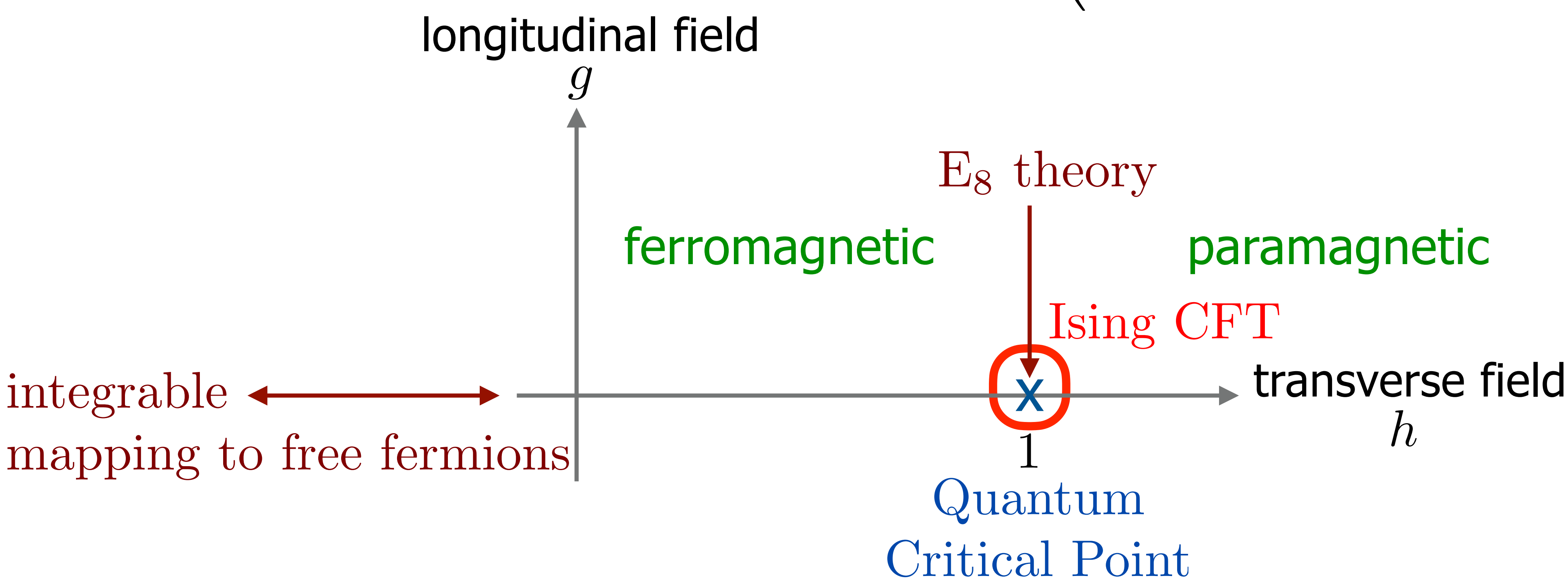

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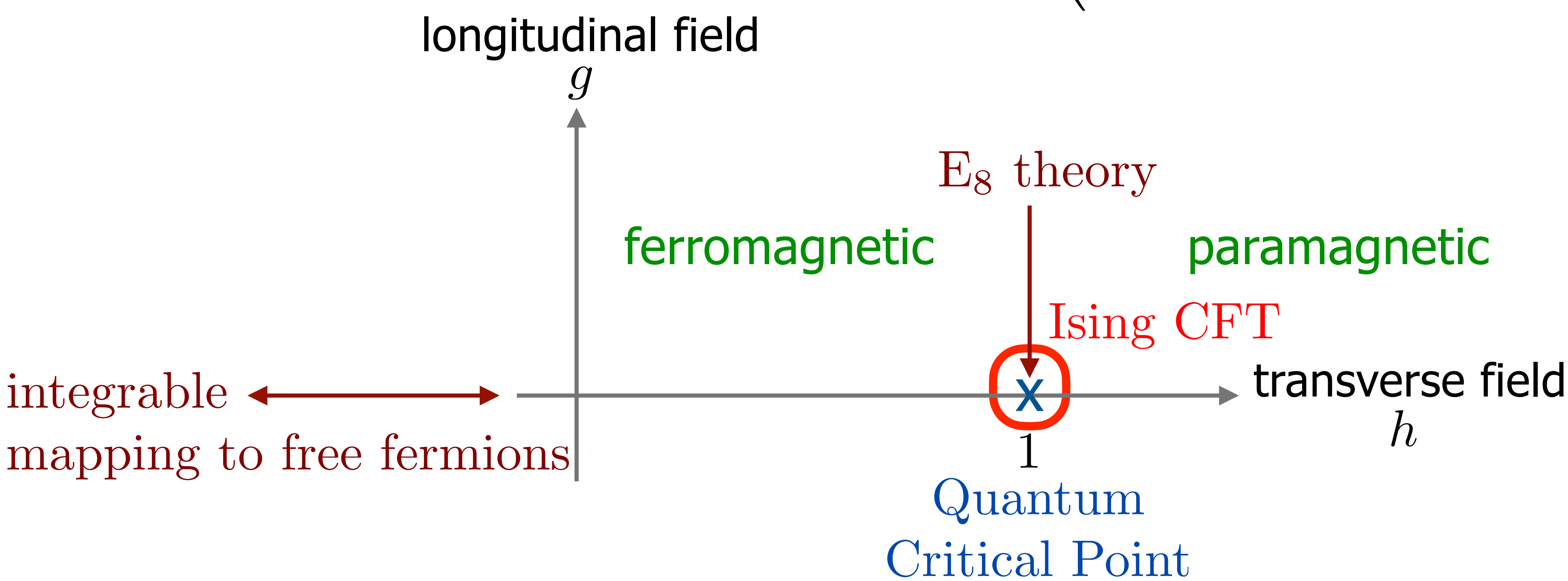


- the full scaling Ising field theory Hamiltonian in presence of transverse and longitudinal perturbations has the form [Rakovszky et al. 2016]:

$$H = \int_{-\infty}^{\infty} dx \left\{ \frac{1}{2\pi}\left[\frac{i}{2}\left(\psi(x)\partial_x\psi(x) - \bar{\psi}(x)\partial_x\bar{\psi}(x)\right) - iM_h\bar{\psi}(x)\psi(x)\right] + \bar{g}\sigma(x)\right\}$$

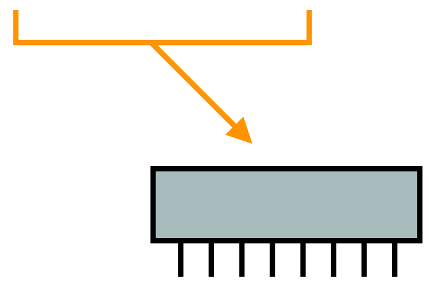
continuum limit: $M_h = 2J|1-h|$ $\beta\omega = F\left[\beta J,\ \beta M_h,\ \beta M_{\bar{g}}\right]$

$N \to \infty,\ \beta J \gg 1$ $\bar{g} = \frac{2}{\bar{s}}J^{15/8}g,\ M_{\bar{g}} = \bar{\eta}|\bar{g}|^{8/15}$

Tensor Networks *… a classical simulation for a quantum problem*

- the Hilbert space of a generic quantum state is huge:

$$|\Psi\rangle = \sum_{i_1, i_2, \dots, i_N} \psi_{i_1, i_2, \dots, i_N} |i_1\rangle |i_2\rangle \cdots |i_N\rangle \, , \quad i_n = 1 \dots d$$



N-legged tensor:
exponentially many coefficients
in N-body Hilbert space: d^N

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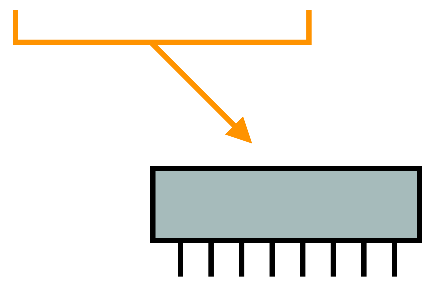

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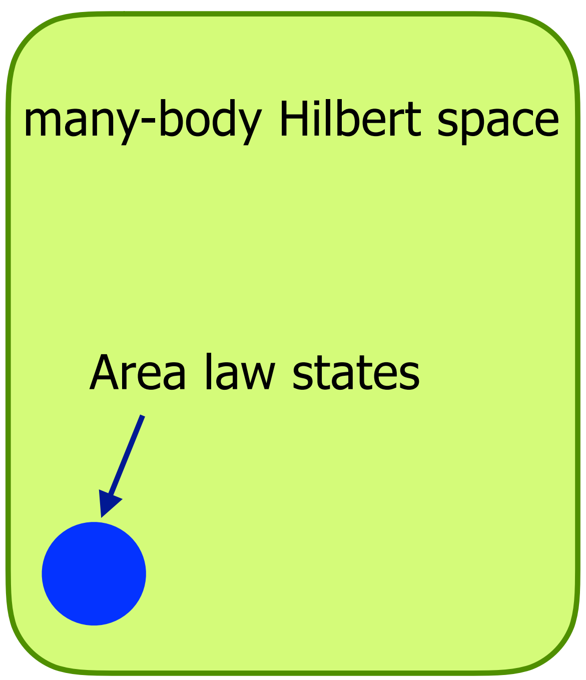


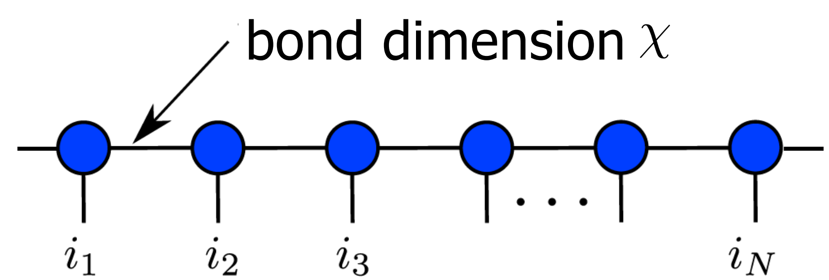
- ground states of local gapped Hamiltonians satisfy Area law for entanglement entropy [Hastings 2007]:

$$S(L) \sim L^{D-1}$$

- Matrix Product States (MPS) as ansätze satisfy this by construction [Schollwöck 2011]:

$$|\Psi\rangle = \sum_{i_1,i_2,\ldots,i_N} A^1_{i_1} A^2_{i_2} \cdots A^N_{i_N} |i_1\rangle |i_2\rangle \cdots |i_N\rangle$$

$\psi_{i_1,i_2,\ldots,i_N} \sim$



$\mathcal{O}(Nd\chi^2)$ parameters

efficient

- advantages and properties of MPS as TN states:

  + efficient description of wave function for large (!) quantum systems
  + no sign problem $\Rightarrow$ application to gauge theories
  + non-perturbative for Hamiltonian systems
  + quantum phases, connection to holography and RG, topological order, higher dimensions...

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- time evolution:

  We use the TEBD algorithm (time-evolving block decimation [Vidal 2004]) to construct thermal states and perform real-time evolution.

  Trotter decomposition: $H = \sum_{i=1}^{N-1} h_{i,i+1} \qquad e^{-\tau H} = e^{-\tau H_{odd}} e^{-\tau H_{even}} + \mathcal{O}(\tau^2)$

  Expectation values are calculated as:

$$\langle O_2^{[n_2]}(t) O_1^{[n_1]}(0) \rangle_\beta = \mathrm{Tr}\left[U^\dagger(t) O_2^{[n_2]} U(t) O_1^{[n_1]} \rho(\beta) \right]$$

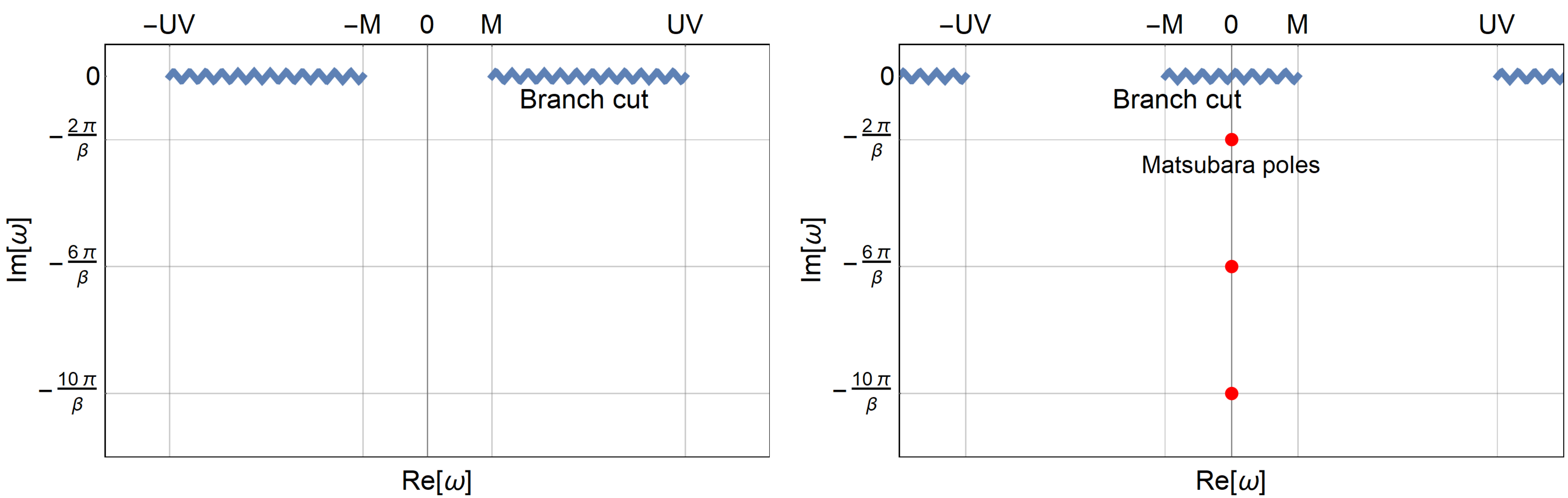
  for Pauli matrices $O_j^{[n_j]} = \sigma_{x,z}^{n_j}\,, \qquad \rho(\beta) = e^{-\beta H}\,, \quad U(t) = e^{-itH}$

CFT results for correlators

- (1+1)D CFT: $\omega = \pm p - i\,2\pi\,T\,(\Delta + 2\,n)$ for $n \in \{0, 1, 2, \ldots\}$

$$\Delta_{\bar{\psi}\psi} = 1 \qquad \Delta_\sigma = \frac{1}{8}$$

- massive free fermions (transverse Ising model at zero momentum):
  $\Rightarrow$ two equivalent representations



+ branch points at physical mass $M = 2M_h$, lattice UV scale at $8J - M$
+ complex structures govern relaxation behavior
+ lowest decaying pole sets thermalization scale

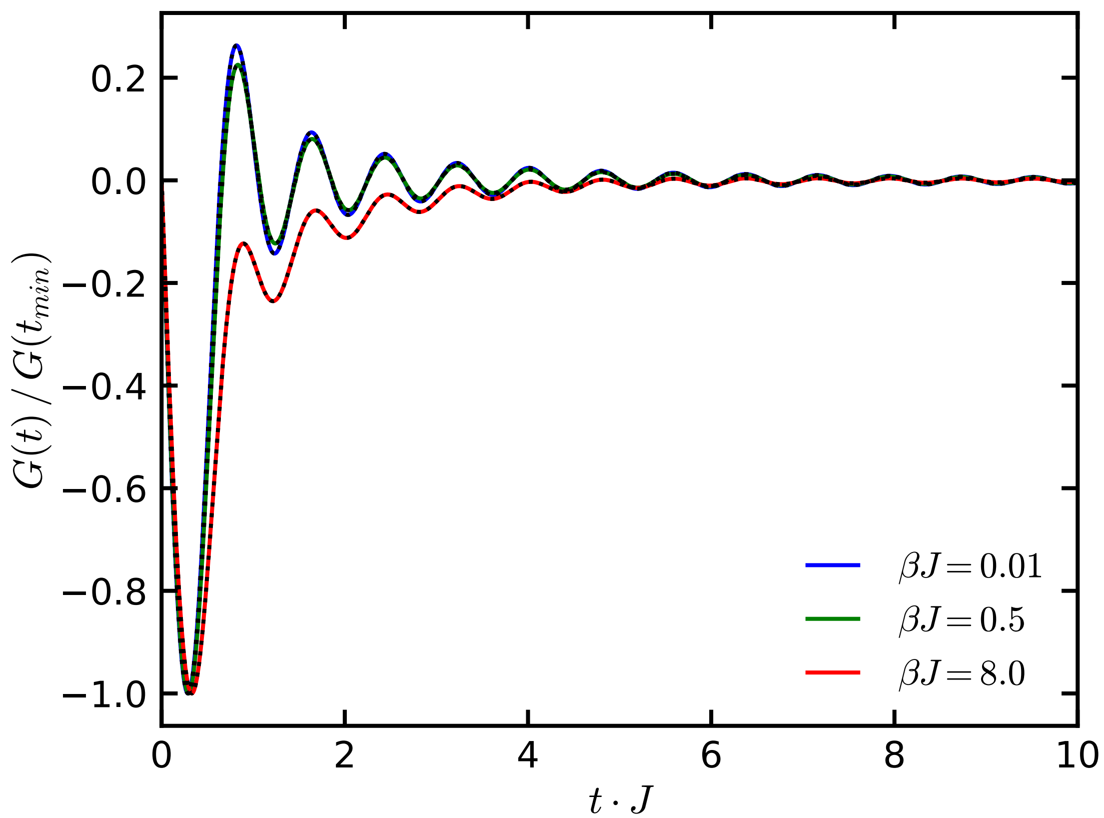
$\Rightarrow$ holographic interpretation as BH quasi normal modes [Sachs et al. 2002]

Numerics with MPS

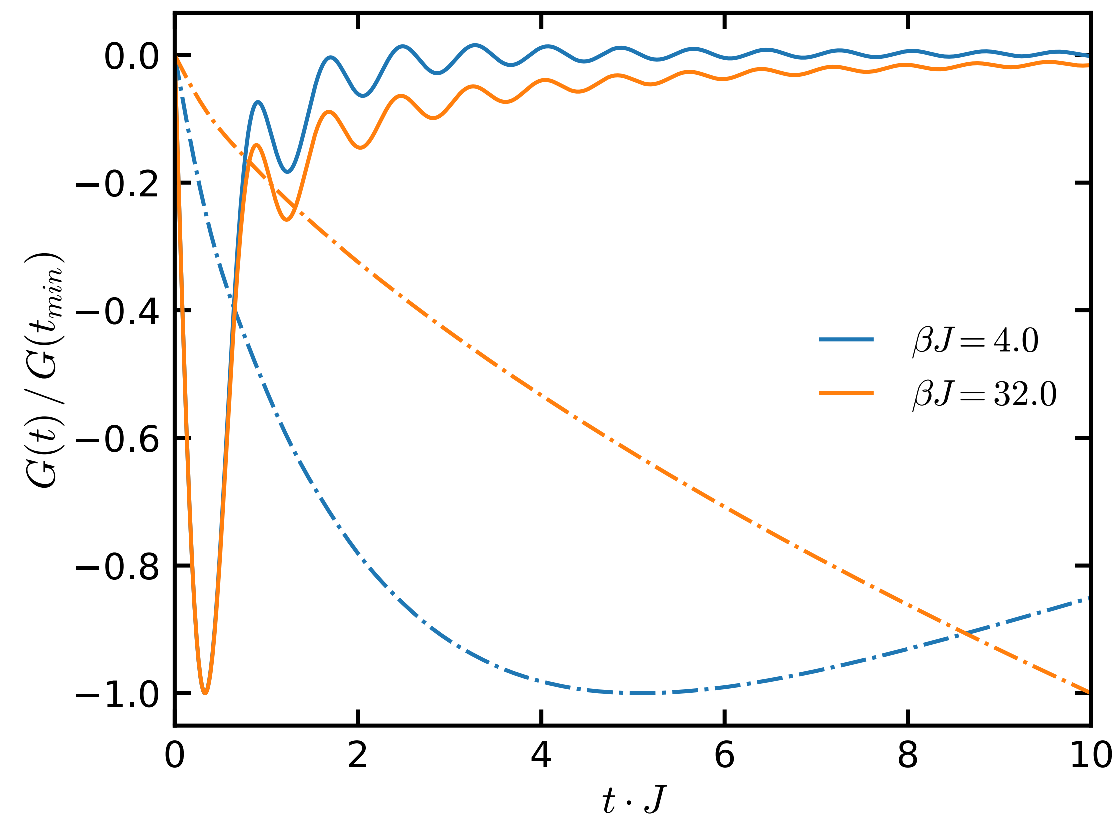
- retarded thermal 2-point function:

$$G_R(t-\tilde{t}, x-\tilde{x}) = i\,\theta(t-\tilde{t})\ \mathrm{Tr}\left(Z_\beta^{-1}\, e^{-\beta H}\, [\mathcal{O}(t,x), \mathcal{O}(\tilde{t},\tilde{x})]\right)$$

for operators $\sigma_{x,z}^{N/2}(t)$ and $\sigma_x(0)$ (global transverse perturbation at zero momentum)



agreement with free fermion mapping at criticality



transverse vs. longitudinal response function

- transverse magnetization follows from convolution:

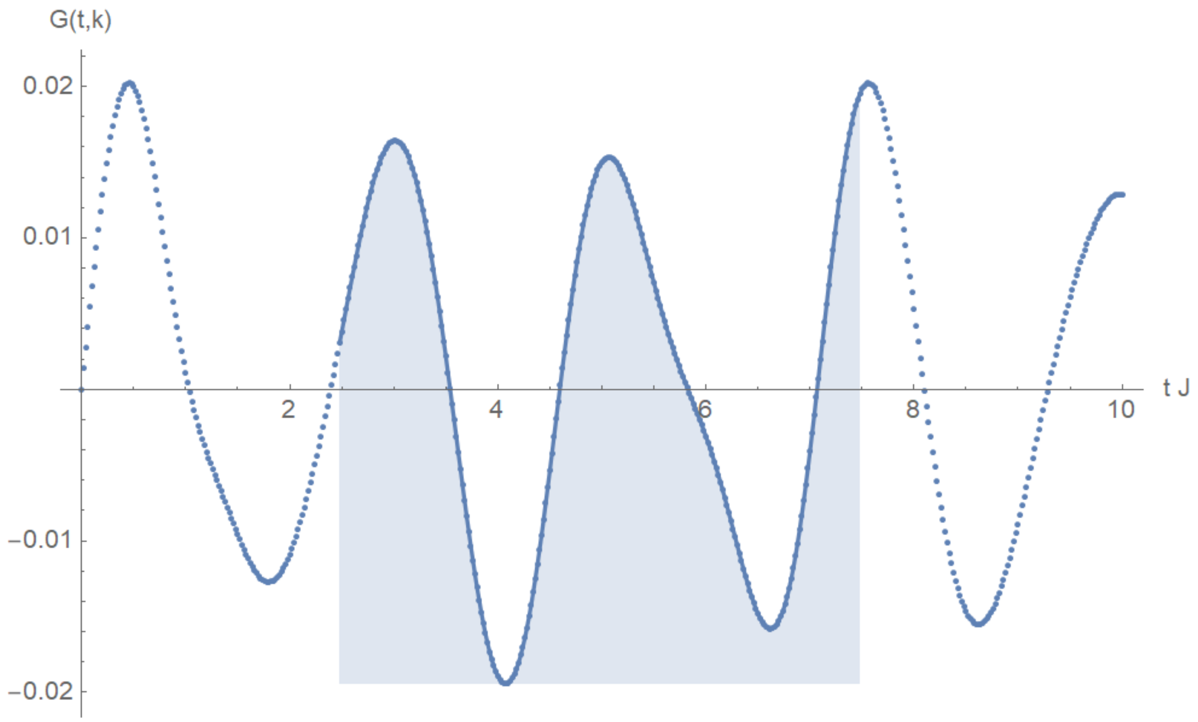
$$\langle \sigma_x^{N/2} \rangle(t) = \int_0^t dt' G_R(t') h(t')$$

Signal analysis with Prony

- represent function as sum of complex exponentials:

$$G(t) = \sum_{k=1}^{M} c_k e^{\omega_k t} \qquad c_k,\ \omega_k \in \mathbb{C}$$

1. Determine ω_k independent of c_k (ESPRIT)
2. Fit c_k by least squares

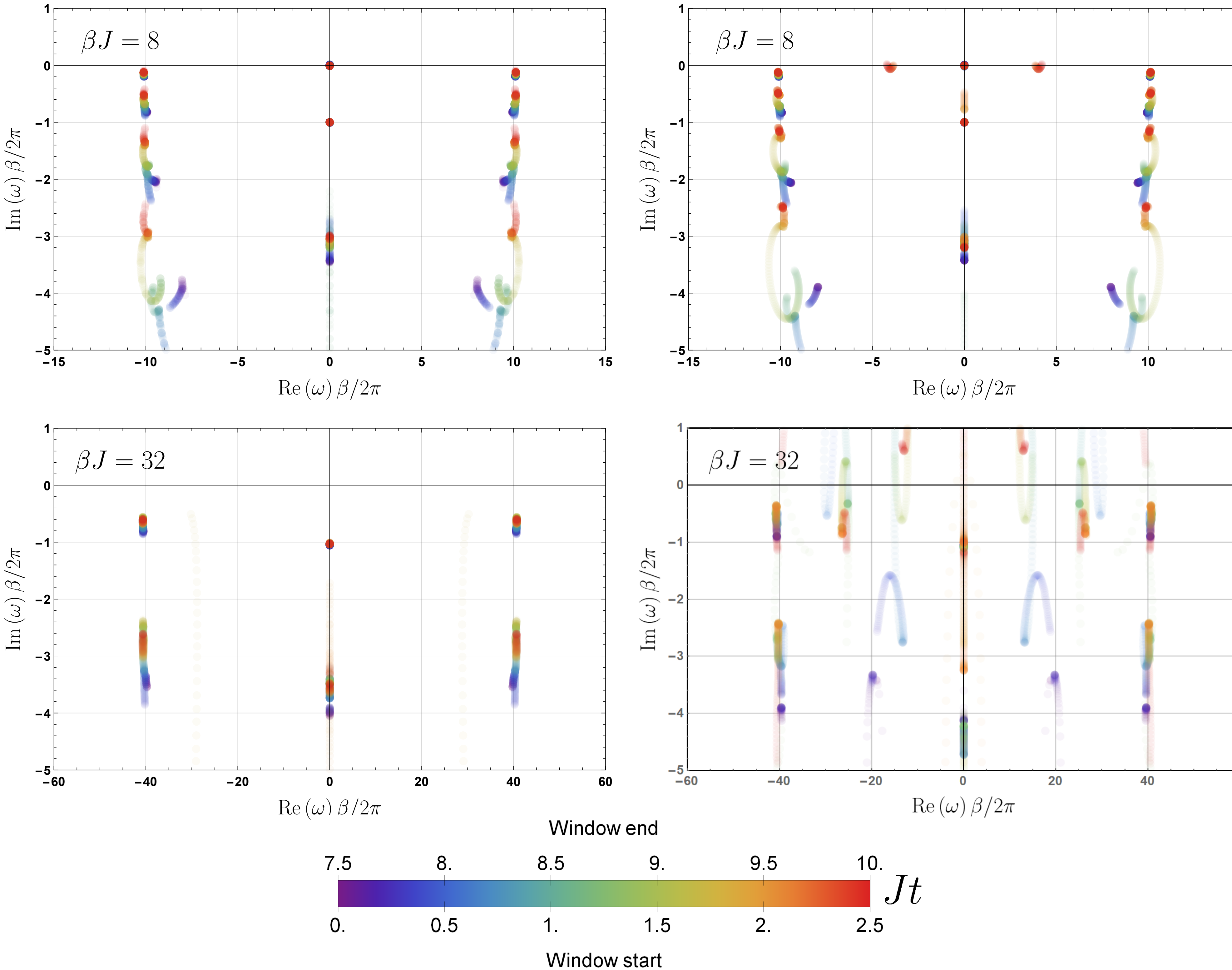


$\Rightarrow$ estimation of stability and uncertainty of poles from parameter variation in Prony and time-shifted analysis window

The integrable QFT limit: MPS results

$\beta M_h = 0.2$ (ferromagnetic), $\beta M_{\bar{g}} = 0$, $\beta J = \{2, 4, 8, 12, 16, 32\}$
$N = 100$, $t = 0 \ldots 10$

free fermion calculation | MPS simulation



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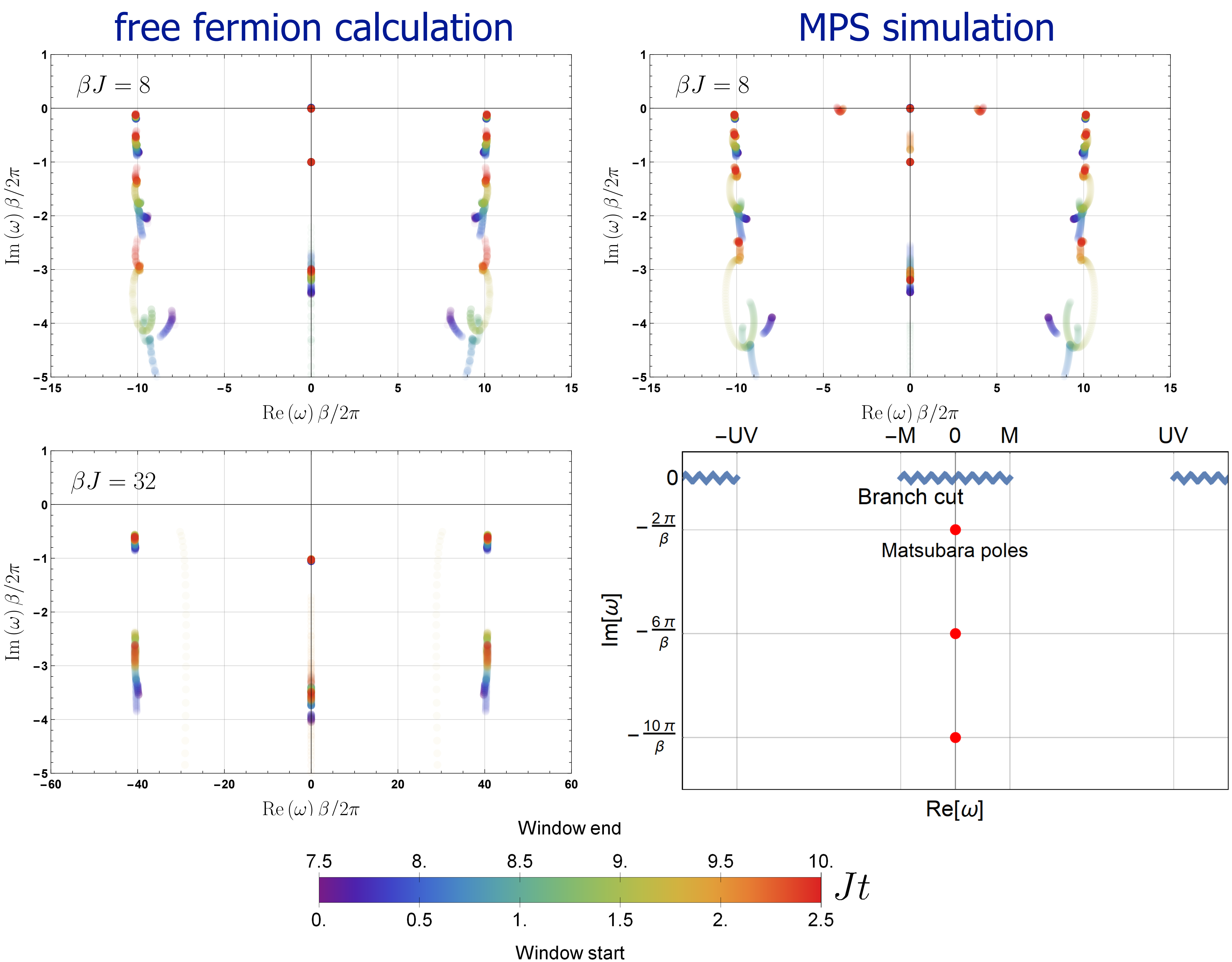
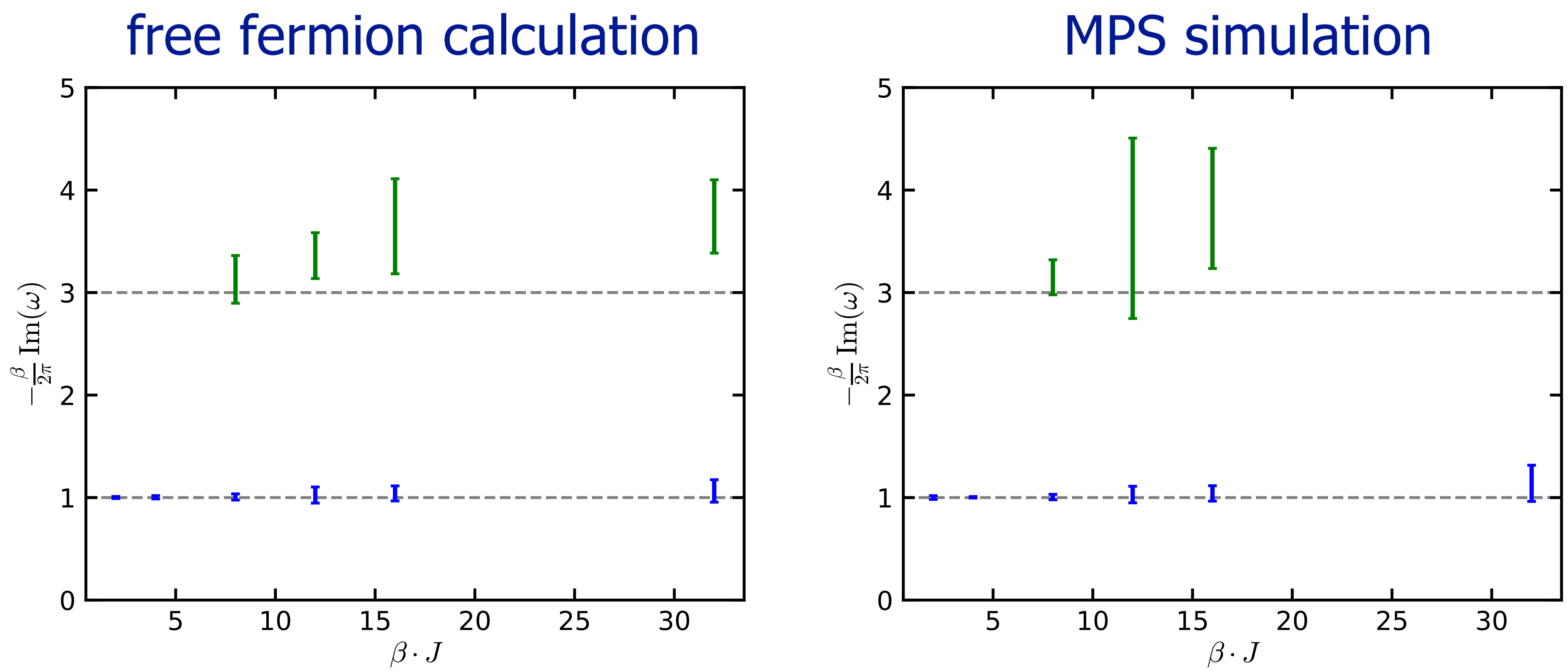


- extracted decaying thermodynamic poles:



+ good agreement with analytical result for first pole

+ second pole partially identifiable

- extracted decaying thermodynamic poles:

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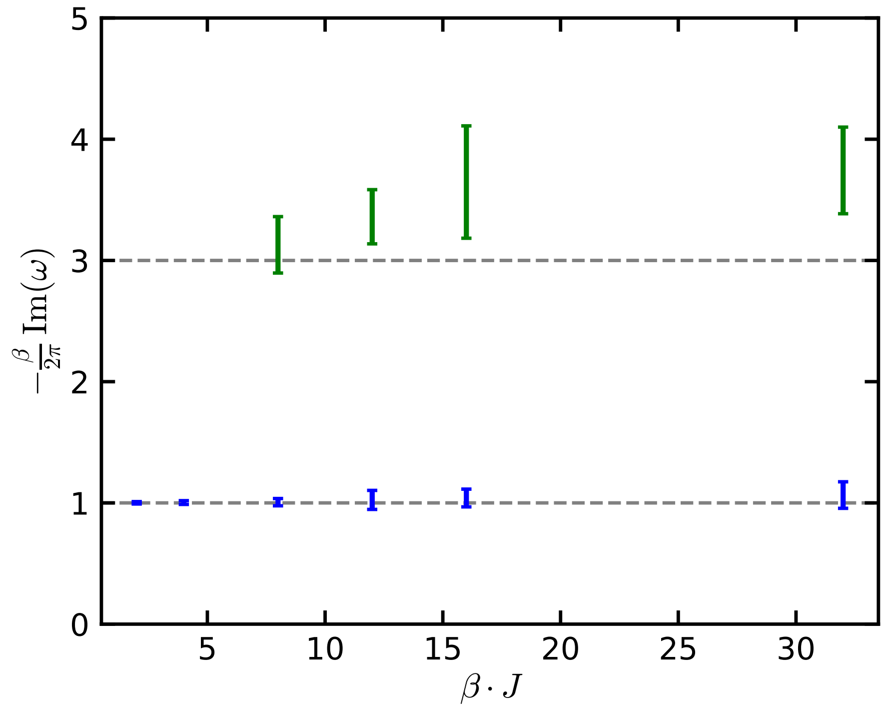
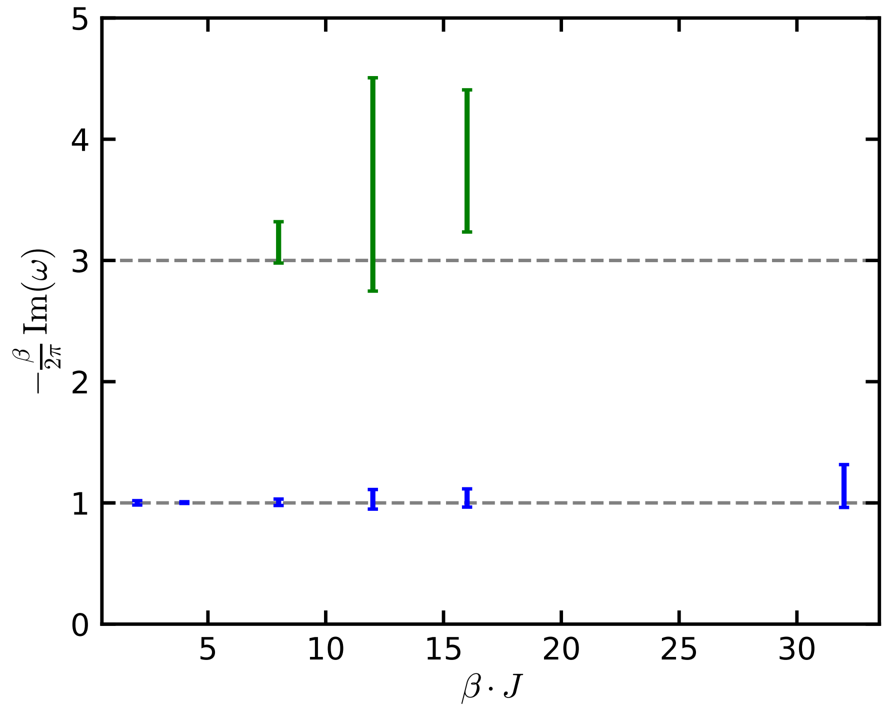


MPS simulation

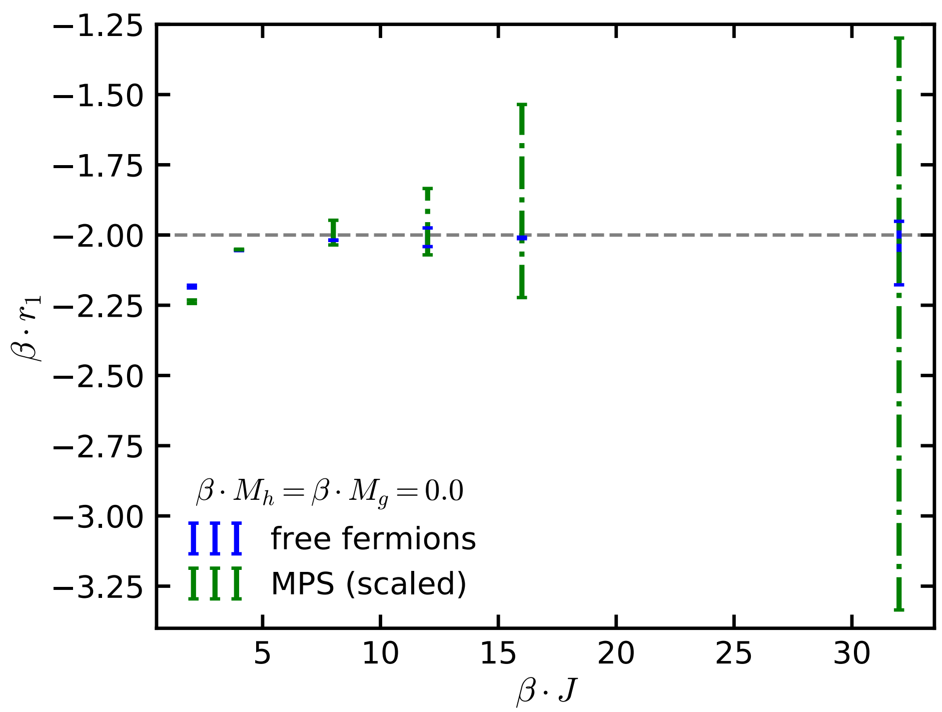


+ good agreement with analytical result for first pole

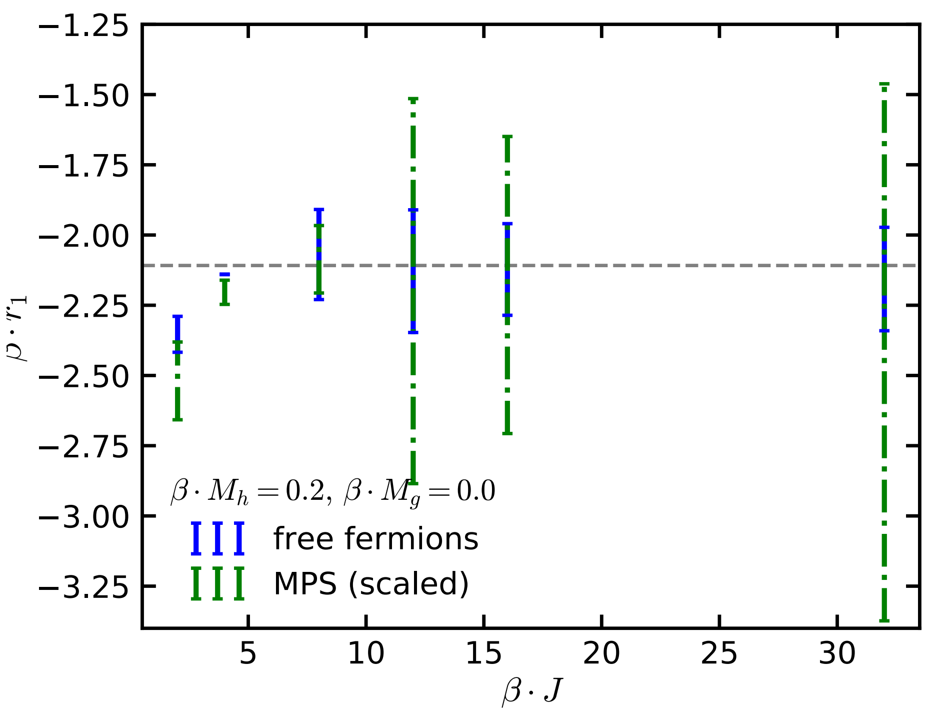
+ second pole partially identifiable

- residues consistent with analytical result in continuum limit:

at criticality



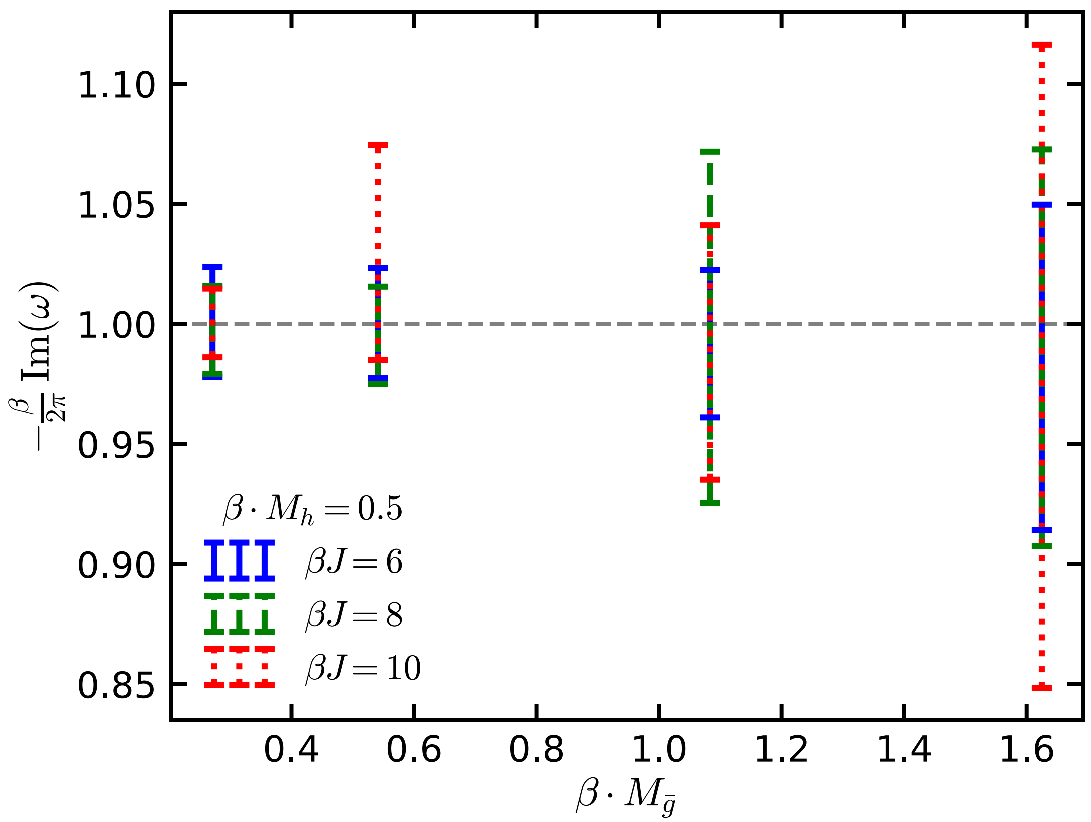
integrable ferromagnetic



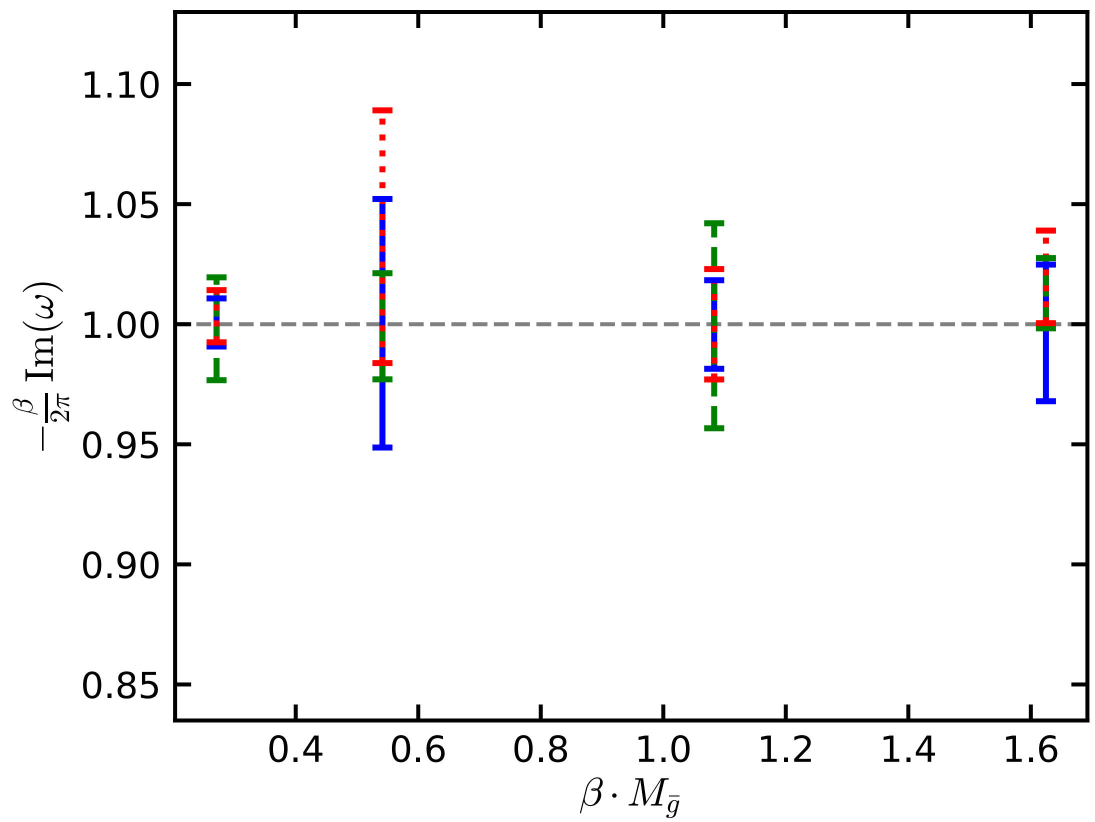
The non-integrable QFT limit: MPS predictions

$$\beta M_h = 0.5, \quad \beta J = \{6, 8, 10\}, \qquad N = 100, \quad t = 0 \dots 10$$

ferromagnetic



paramagnetic

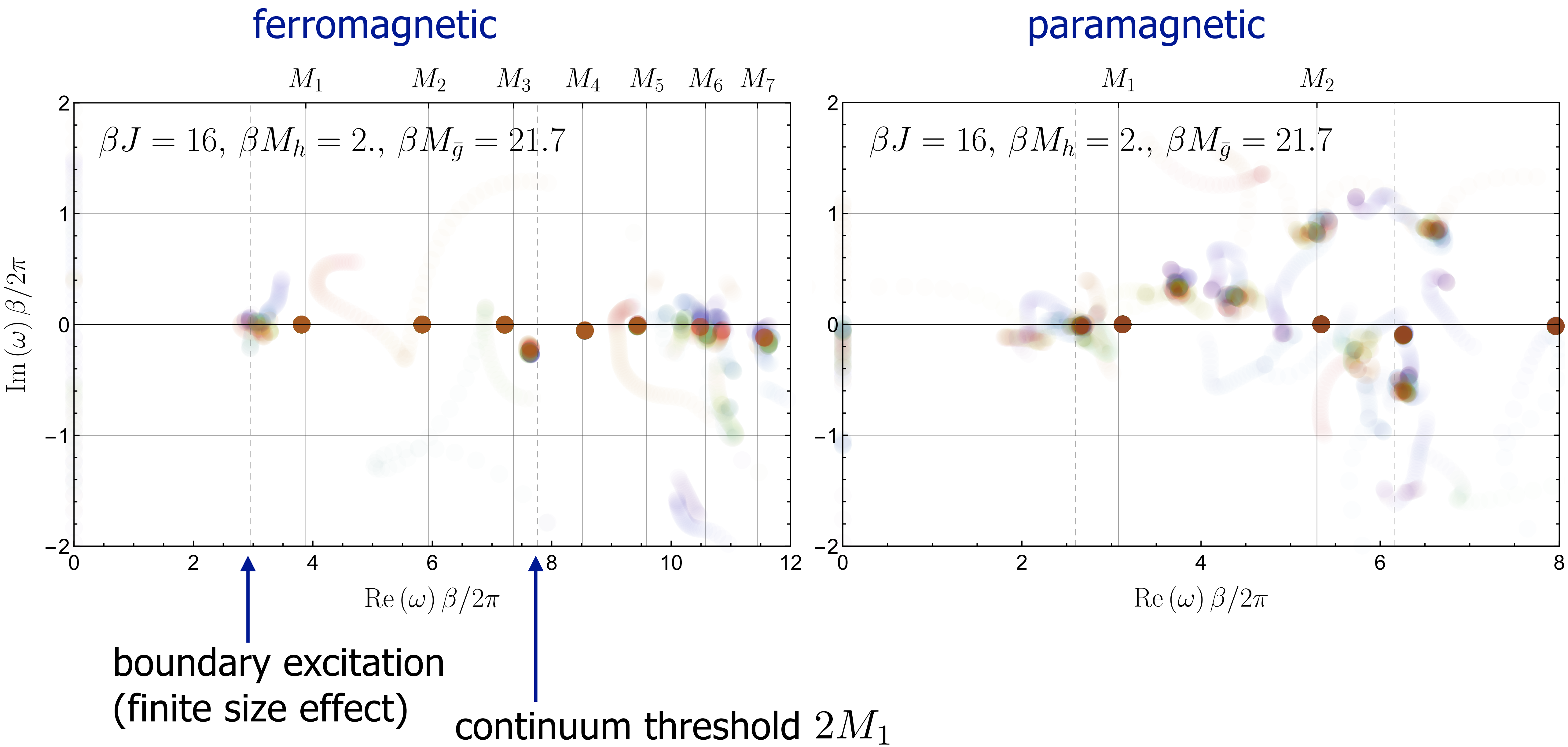


⇒ no movement of poles visible within uncertainties

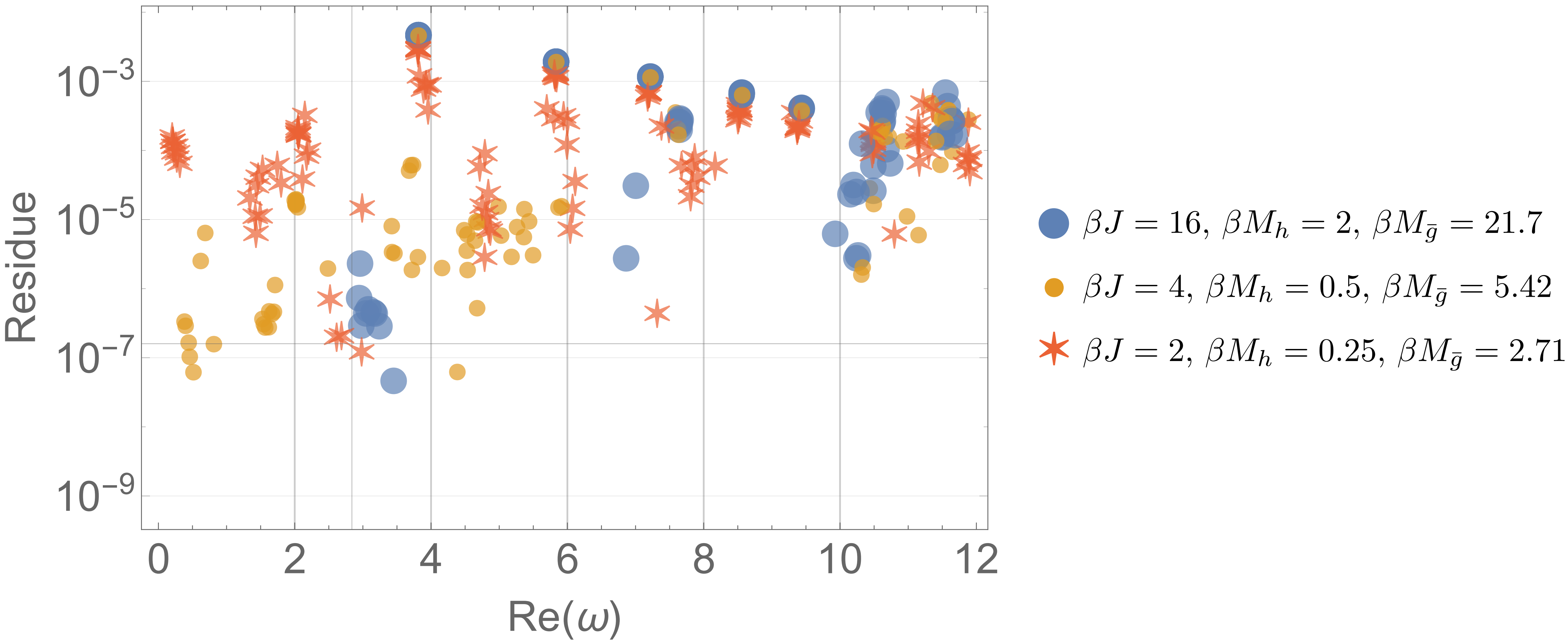
(zero momentum)

- cross-checks in QFT regime:

identification of nontrivial meson / particle masses and their decay rates of perturbed Ising CFT in different vacuum phases
[Zamolodchikov 2006, 2013; Delfino et al. 2006]



- temperature dependence of residues of meson states:



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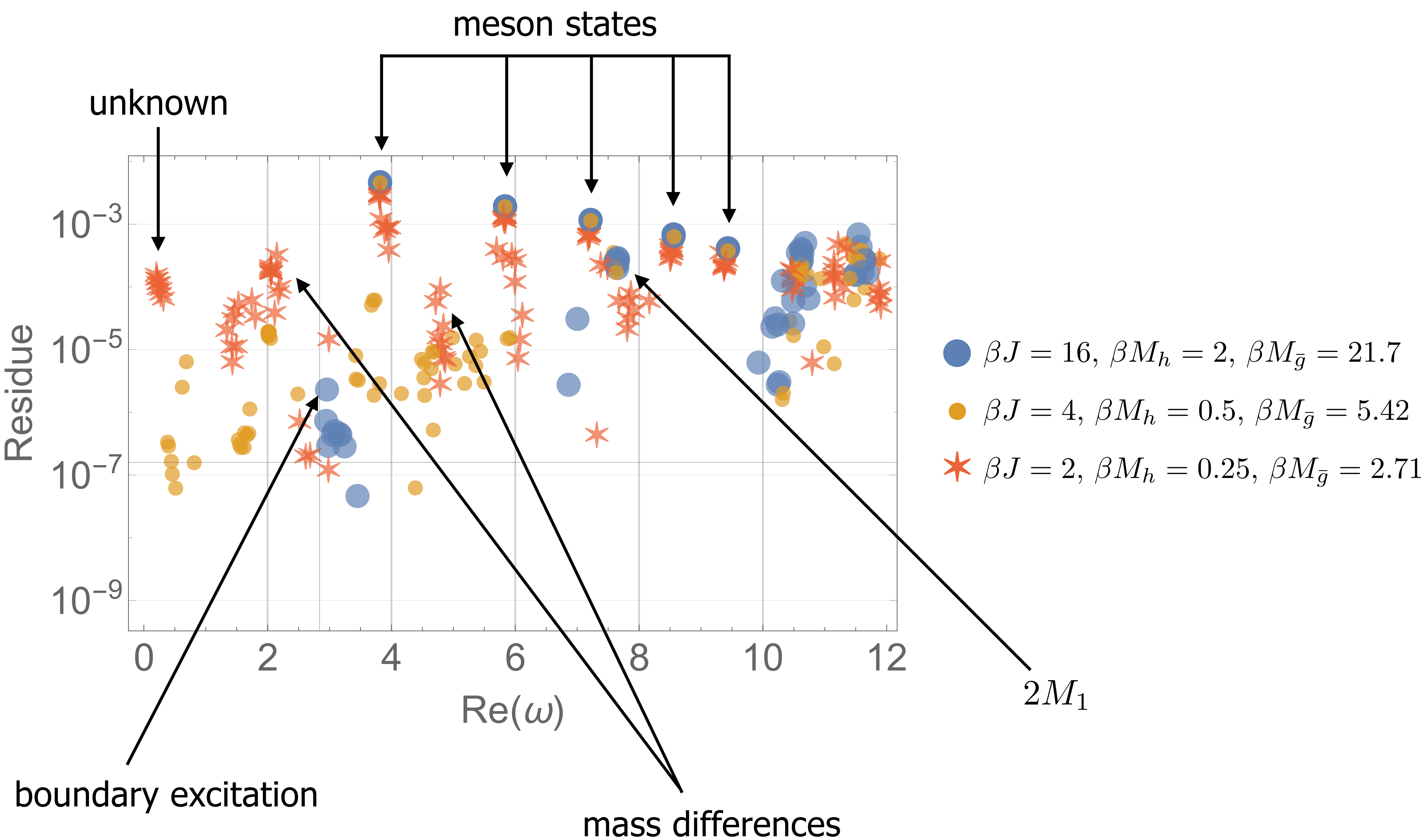


SUMMARY

- Tensor network techniques can be used to extract nontrivial real-time thermal field theory dynamics.
- Prony method can be used to numerically evaluate structure of retarded 2-point function in frequency plane

$\Rightarrow$ agreement with CFT result / free fermions in integrable regime

$\Rightarrow$ no movement of first decaying thermodynamic pole for non-integrable perturbations
$\Rightarrow$ meson / particle masses and decay rates match predictions from Ising QFT

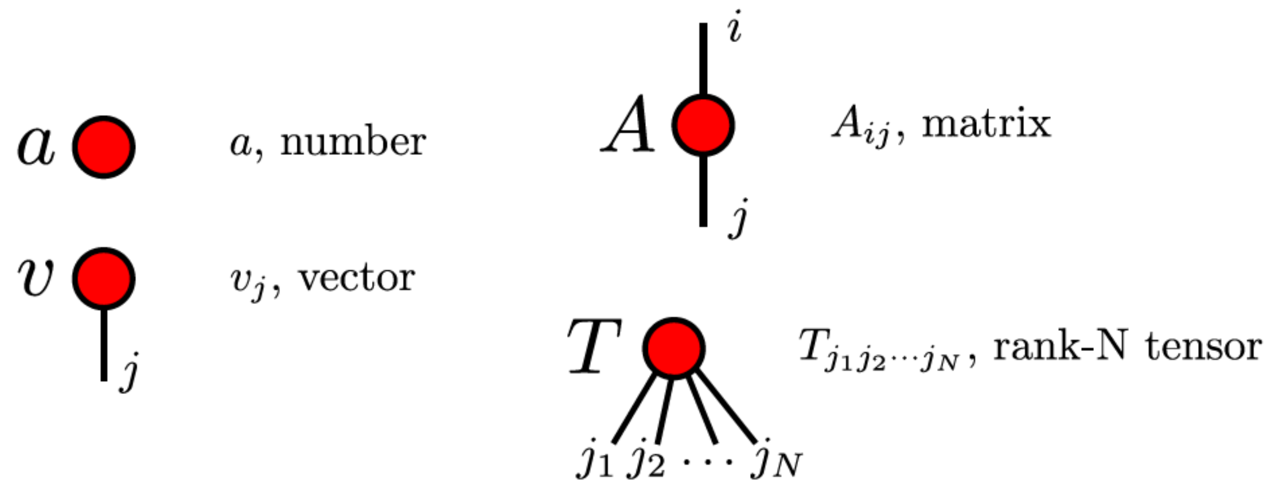
Outlook:

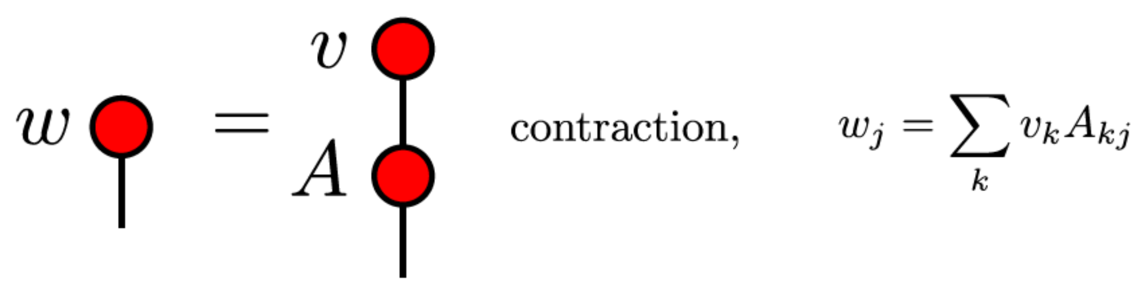
- residues of transients in non-integrable regime
- finite momentum calculations easily possible

$\Rightarrow$ possible change of frequency structure

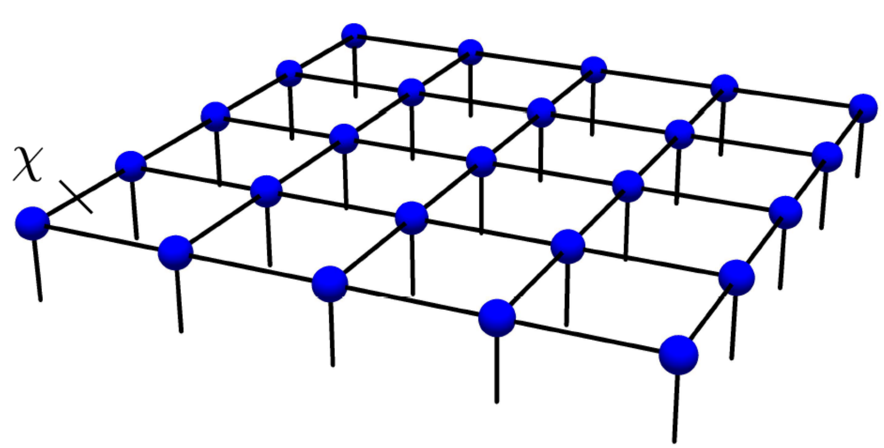
BACKUP

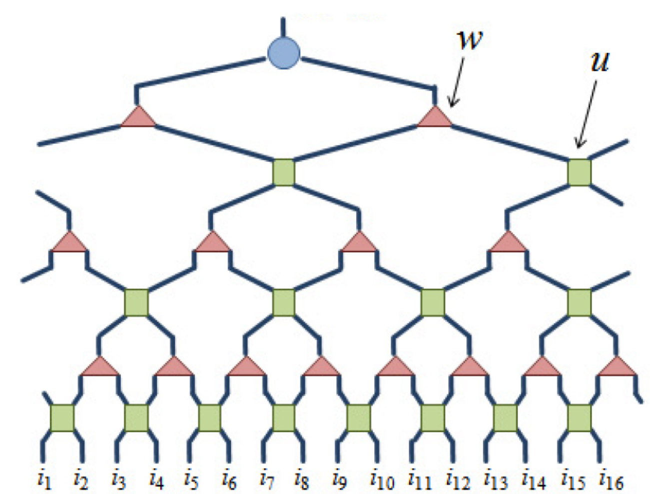
- TNs conventions:





- other types of TN states:

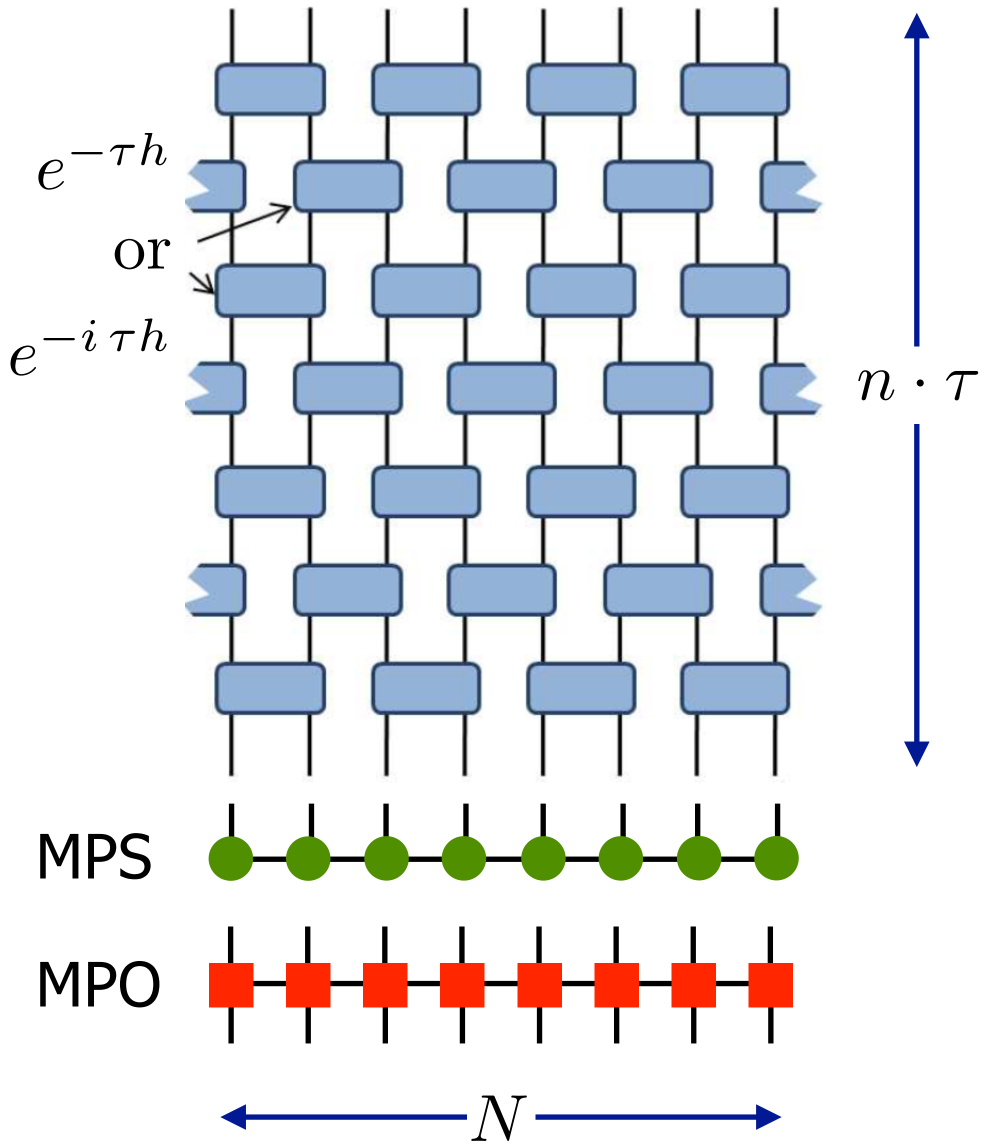




PEPS [Verstraete, Cirac 2004]   MERA [Vidal 2006]

upper slide and PEPS from talk „Tensor networks" by Lukasz Cincio; MERA from [arXiv:0707.1454]

- time evolution:



$$H = \sum_{i=1}^{N-1} h_{i,i+1}$$

Trotter decomposition:

$$e^{-\tau H} = e^{-\tau h_{1,2}} e^{-\tau h_{2,3}} \cdots e^{-\tau h_{N-1,N}} + \mathcal{O}(\tau^2)$$

$$e^{-\tau H} = e^{-\tau H_{odd}} e^{-\tau H_{even}} + \mathcal{O}(\tau^2)$$

MPO: Matrix Product Operator

$\Rightarrow$ operator representation in tensor product basis

- TEBD algorithm to construct thermal states and real-time evolution: [Vidal 2004]

$$\rho_0 \equiv e^{-\beta H} = e^{-\beta/2H} \mathbb{1} \cdot \mathbb{1} e^{-\beta/2H}$$

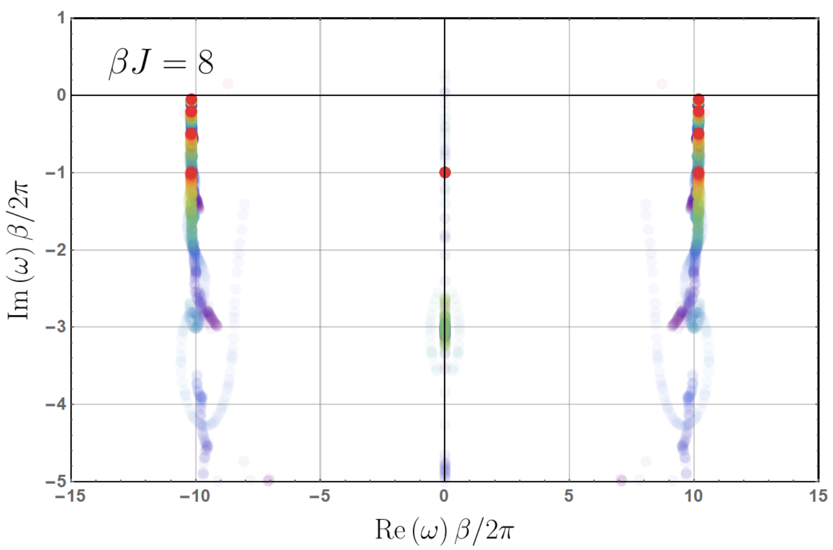
$$\rho(t) = e^{-itH(t)} \rho_0 e^{itH(t)}$$

$$\langle O_2^{[n_2]}(t) O_1^{[n_1]}(0) \rangle_\beta = \mathrm{Tr}\left[U^\dagger(t) O_2^{[n_2]} U(t) O_1^{[n_1]} \rho(\beta) \right]$$

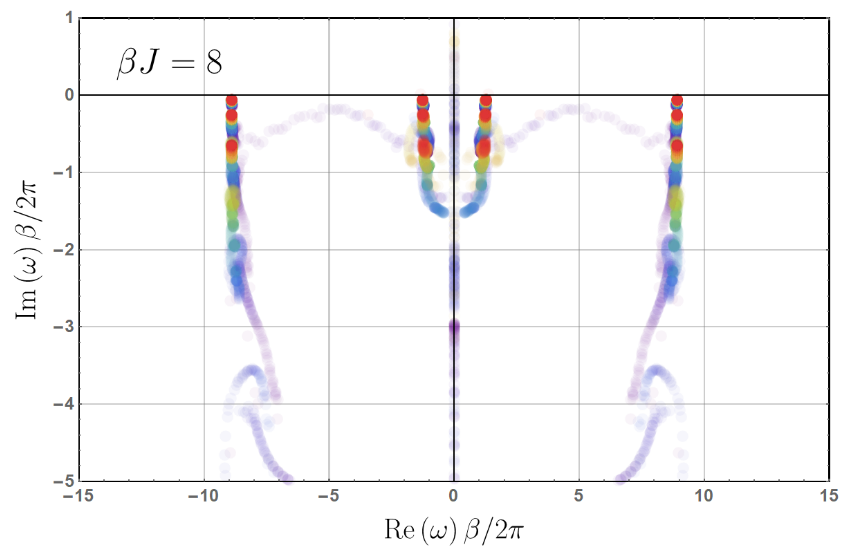
Signal analysis with Prony

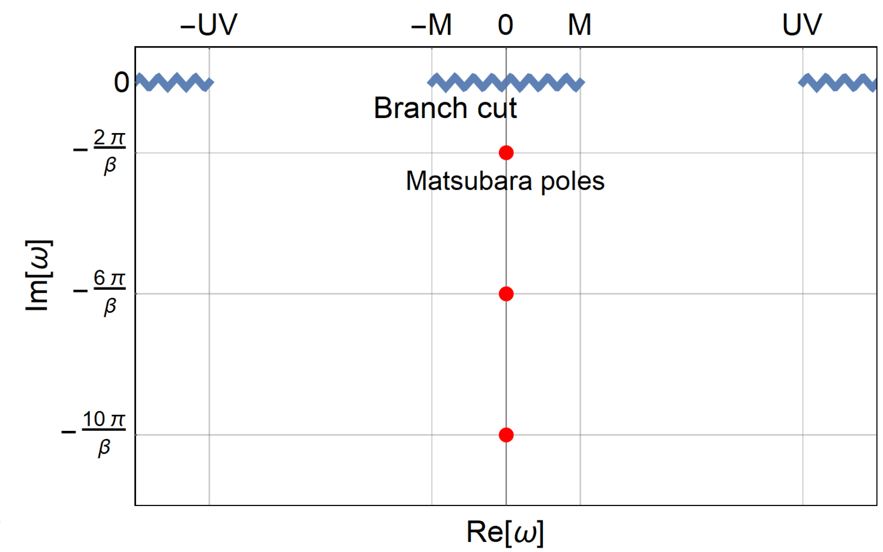
- examples of Prony analyses of the retarded transverse 2-point correlator (from numerical evaluation of integral):

zero (transverse) mass

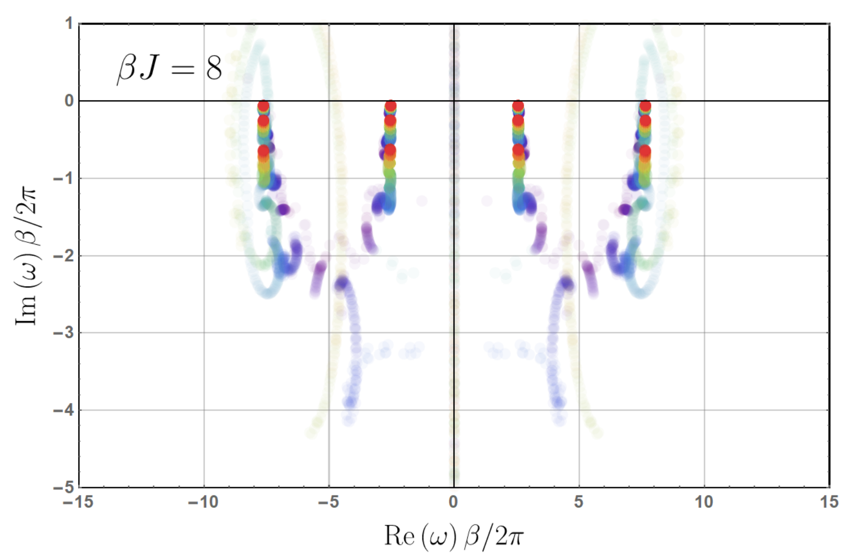


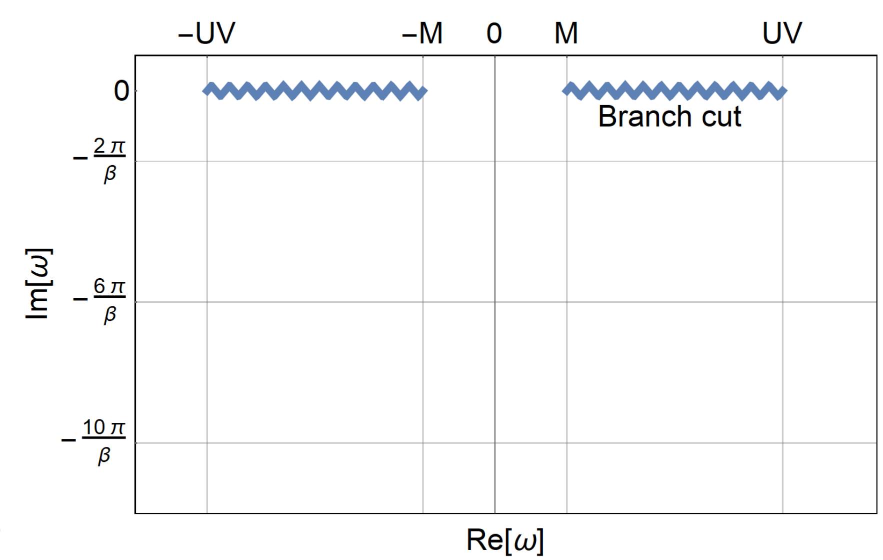
intermediate mass





large mass





- mapping of the integrable transverse field Ising model to massive free fermions:

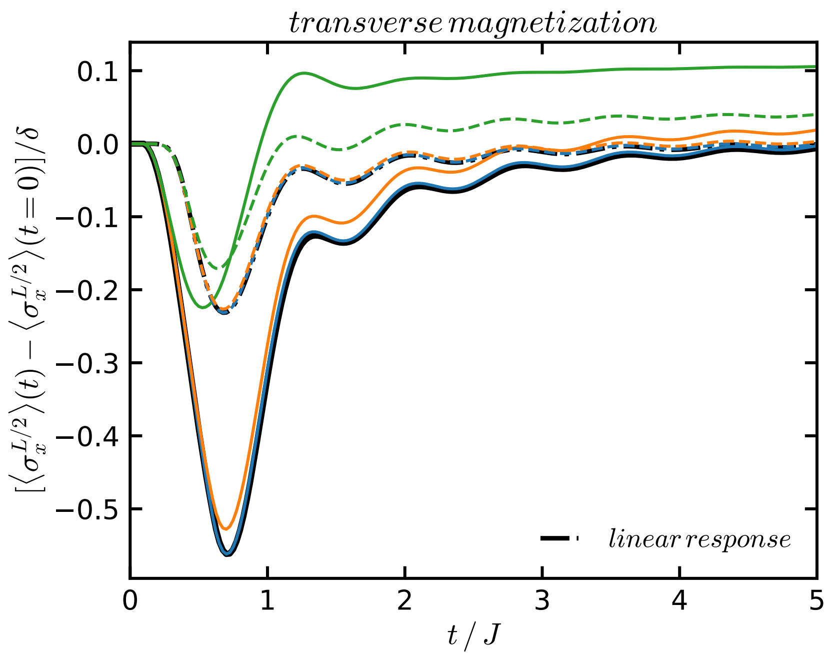
$$H = \sum_k \varepsilon_k (\gamma_k^\dagger \gamma_k - 1/2)$$

$$\varepsilon_k = 2\sqrt{J^2(1 + h^2 - 2h\cos k)}$$

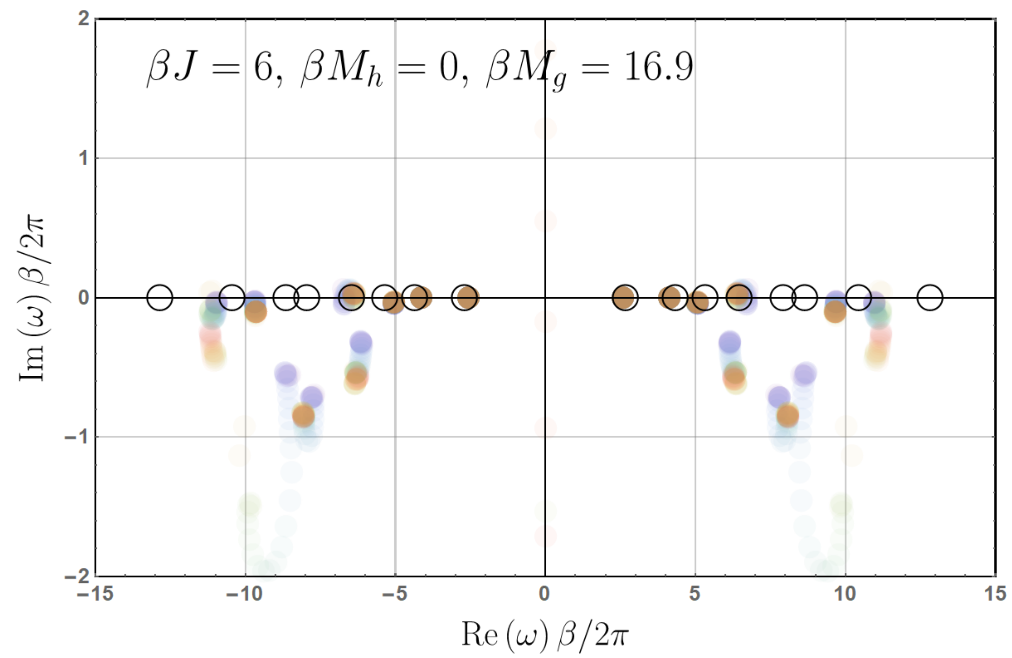
- retarded correlator:

$$G(t, p = 0) = \int_{-\pi}^{\pi} dk \frac{\sin(2\varepsilon_k t)}{e^{\beta\varepsilon_k} + 1} f(k)$$

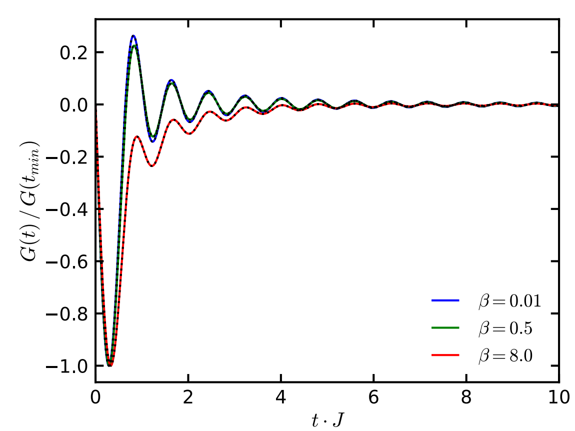
(other) Things to do with TNs…



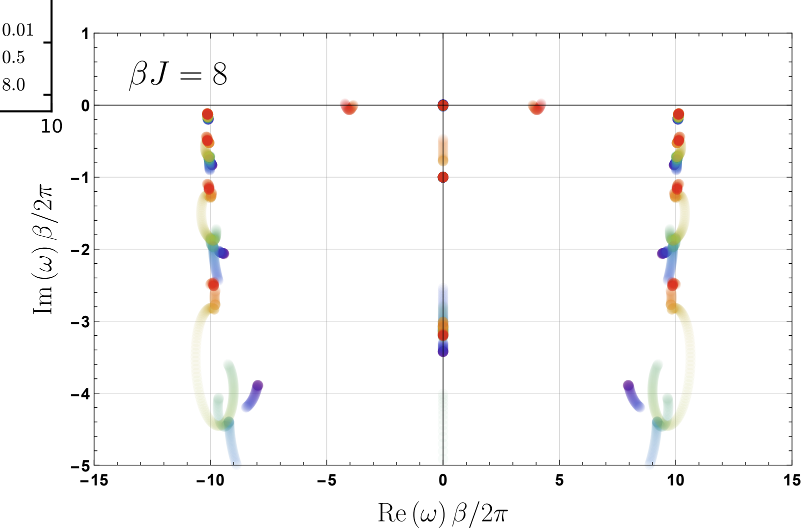
nonlinear behavior in thermal quantum quenches, relaxation, thermalization…

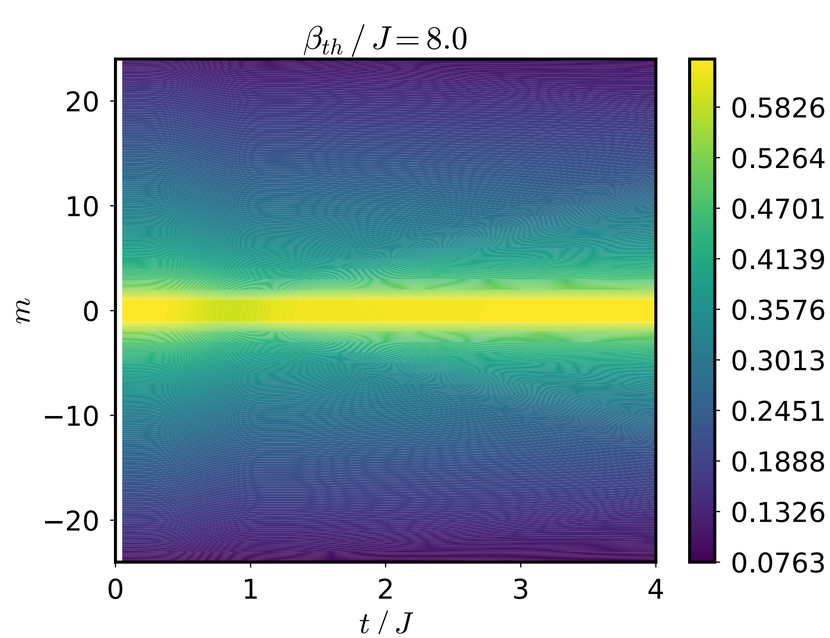


meson studies



real-time dynamics





spatial correlations

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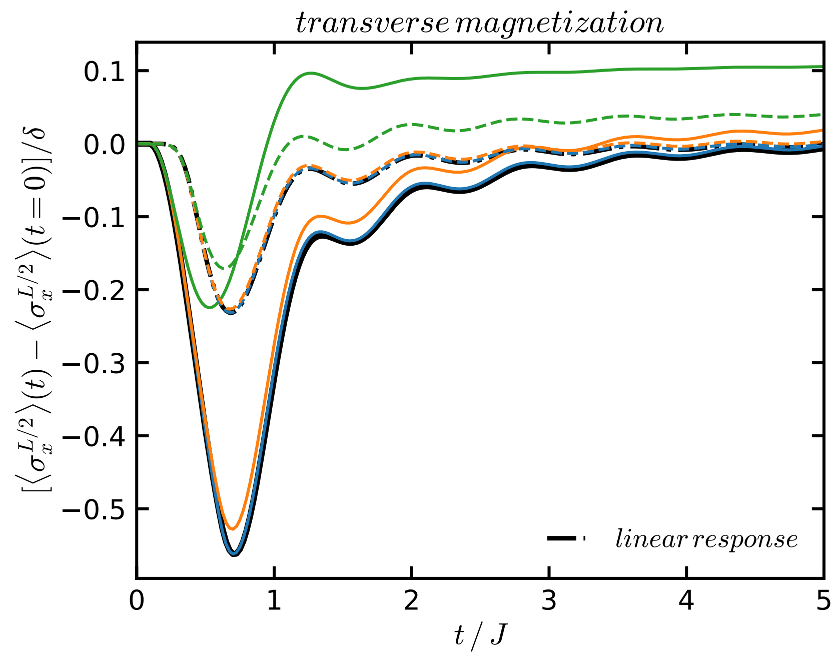

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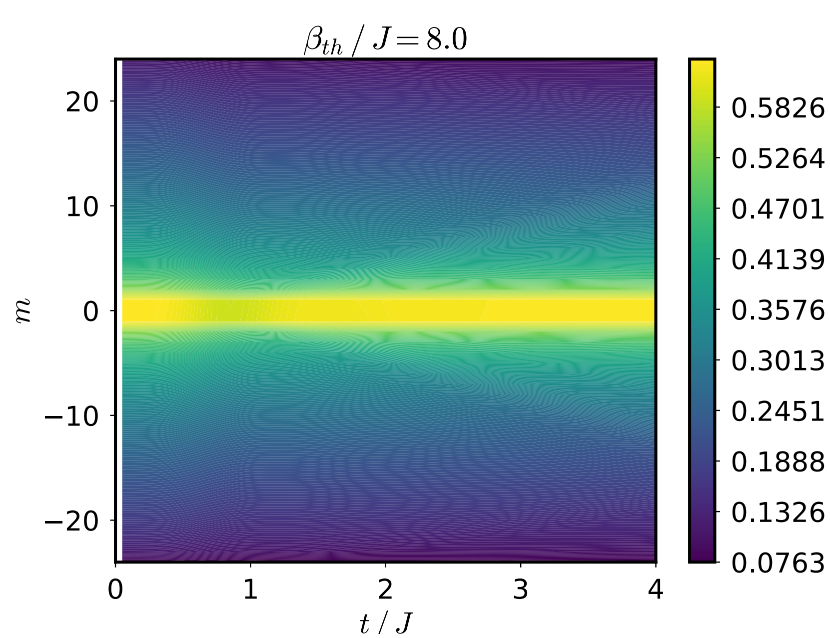

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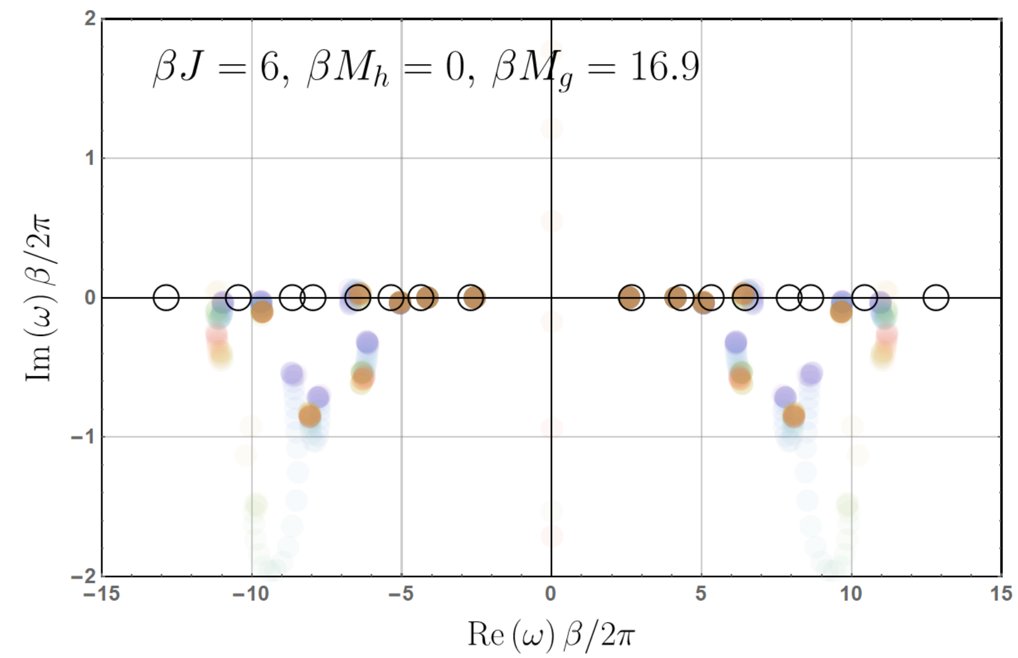
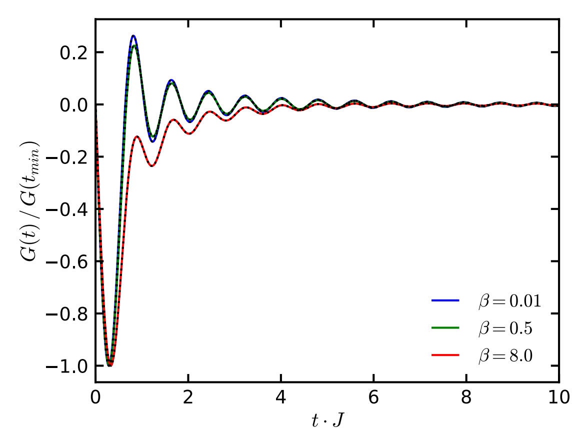


meson studies

QFT regime



real-time dynamics