On Exceptional Lie Geometries

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Dedicated to the memory of Ernie Shult

General context

- Buildings : geometric interpretation of semi-simple groups of algebraic origin (semi-simple algebraic groups, classical groups, groups of mixed type, (twisted) Chevalley groups).
- Projective spaces and polar spaces: excellent permutation representations for the classical groups. In fact, projective and polar spaces are *Grassmannians* of certain Tits-buildings.
- Parapolar spaces are point-line geometries introduced to approach in a geometrical way the spherical Tits-buildings and algebraic (Chevalley) groups mainly of exceptional types.

The Klein Quadric

Consider a line L in $\mathbb{P}^3(\mathbb{K})$ and consider two points $x = (x_0, \cdots, x_3)$ and $y = (y_0, \cdots, y_3)$ and map (x, y) to the point in $\mathbb{P}^5(\mathbb{K})$ with coordinates $p_{ij} = x_i y_j - y_i x_j, 0 \leq i < j \leq 3$.
This is a well-defined map from the lines of $\mathbb{P}^3(\mathbb{K})$ to $\mathbb{P}^5(\mathbb{K})$ and we obtain a set of points satisfying the Plücker relation

$$p_{01}p_{23} + p_{02}p_{31} + p_{03}p_{12} = 0$$

A more abstract definition of the Klein Quadric

$\mathcal{G}$: geometry defined from $\mathbb{P}^3(\mathbb{F})$ where $\mathbb{F}$ is a skew field as follows

- Points of $\mathcal{G}$: Lines of $\mathbb{P}^3(\mathbb{F})$
- Lines of $\mathcal{G}$: Incident point-plane pairs
- Planes of $\mathcal{G}$ are of two types: points and planes
- Point and line of $\mathcal{G}$ are incident if the line belongs to the plane pencil determined by the point and the plane.
- Line and Plane are incident if the point or plane is one of the elements of the incident pair.
- Point and Plane: natural incidence of $\mathbb{P}^3(\mathbb{F})$.

Classical polar spaces

- Describe the geometry of vector spaces carrying a form as projective spaces describe the geometry of vector spaces
- Alternating form: Symplectic polar space
- (Anti)-Hermitian form: unitary polar space
- quadratic form (which include symmetric bilinear forms $\mathrm{char} \neq 2$): orthogonal polar space

Axiomatic polar spaces (Buekenhout-Shult)

A point-line geometry $\Delta = (X, \mathcal{L})$ is called a (non-degenerate) *polar space* if the following axioms holds:

1. Every line contains at least three points.
2. No point is collinear to all other points.
3. Every nested sequence of singular subspaces is finite.
4. For any point x and any line L, either one or all points on L are collinear to x.

Remark All maximal chains of non-empty singular subspaces have the same length, called the *rank* of the polar space.

Parapolar spaces

A point-line geometry $\Omega = (X, \mathcal{L})$ is called a *parapolar space* if the following axioms hold:

- Ω is connected and, for each line L and each point $p \notin L$, p is collinear to either none, one or all of the points of L and there exists a pair $(p, L) \in X \times \mathcal{L}$ with $p \notin L$ such that p is collinear to no point of L.
- For every pair of non-collinear points p and q in $\mathcal{P}$, one of the following holds:
  - (a) the convex closure of $\{p, q\}$ is a polar space, called a *symplecton*;
  - (b) $p^\perp \cap q^\perp$ is a single point;
  - (c) $p^\perp \cap q^\perp = \emptyset$.
- Every line is contained in at least one symplecton.

Point-residual

Let $\Omega = (X, \mathcal{L})$ be a parapolar space and let p be one of its points. We define the *point-residual at p*, denoted $\Omega_p = (X_p, \mathcal{L}_p)$, as follows:

- X_p: set of lines through p;
- $\mathcal{L}_p$: set of planar line-pencils with vertex p contained in singular planes through p which are contained in a symplecton of Ω.

Fact: A parapolar space is strong and of symplectic rank at least 3 if and only if all point-residuals have diameter 2.

The haircut axiom (Shult)

Inspired by an axiom by Cooperstein and Cohen.

(H) For any point p and any symp ξ with $p \notin \xi$, the intersection $p^\perp \cap \xi$ is never a submaximal singular subspace of ξ.

The above is a residual property:

Lemma

Suppose Ω is a parapolar space of symplectic rank at least 3. Then Ω satisfies property (H) if and only if Ω_p also has the property (H) for each point p.

Shult's haircut theorem

Let Ω be a locally connected p.p.s. of symplectic rank at least 3, such that:

- each singular space possesses a finite projective dimension; moreover, there exists an upper bound on the rank of a symplecton.
- the Haircut Axiom (H) holds.

Then Ω has uniform symplectic rank $d \geq 3$ and Ω is

$d = 3$ Either $\mathrm{A}_{n,d}(\mathcal{L})$ or a homomorphic image $\mathrm{A}_{2n-1,n}(\mathcal{L})/\langle\sigma\rangle$ where σ is a polarity of $\mathrm{A}_{2n-1}(\mathcal{L})$ of Witt index at most $n-5$, $n \geq 5$

$d = 4$ A Y_1 geometry or a twisted version thereof, which includes $\mathrm{E}_{6,2}(\mathbb{K})$ and $\mathrm{D}_{n,n}(\mathbb{K}), n \geq 5)$;

$d = 5$ A homomorphic image of a building geometry $\mathrm{E}_{m+4,1}(*)$ with $m \geq 2$, which includes $\mathrm{E}_{6,1}(\mathbb{K})$, $\mathrm{E}_{7,1}(\mathbb{K})$, $\mathrm{E}_{8,1}(\mathbb{K})$;

$d = 6$ $\mathrm{E}_{7,7}(\mathbb{K})$;

$d = 7$ $\mathrm{E}_{8,8}(\mathbb{K})$.

Our Motivation

- Shult characterized many Lie incidence geometries satisfying the "Haircut Axiom" which expresses a gap in the spectrum of dimensions of singular subspaces of symplecta arising from intersecting the latter with the perp of a point.
- In many exceptional Lie geometries, gaps appear in the spectrum of the dimensions of the singular subspaces that occur as intersections of two symplecta.
- Intimate connections with our characterization work on the projective varieties associated to the Freudenthal-Tits magic square.

Lacunarity

A *k-lacunary* parapolar space is a parapolar space such that

- the intersection dimension of two symps is never exactly k,
- the symplectic rank is at least $k+1$.

If k is not specified, we just say that the parapolar space is *lacunary*.

A parapolar space has minimum symplectic rank d if it has symplectic rank at least d and there exists a symplecton of rank d.

Grassmannians of vector spaces|Projective spaces

Let $\dim_{\mathbb{L}} V = n+1 \in \mathbb{N} \cup \{\infty\}$ and let $\ell \leq (n+1)/2$.

- V_ℓ: ℓ-dimensional subspaces of V,
- $\mathcal{L}_\ell$ family of subsets $L(W,U)$ of V_ℓ consisting of all ℓ-spaces containing a given $(\ell-1)$-space W and being contained in a given $(\ell+1)$-space U, with $W \subseteq U$.
- point-line geometry $(V_\ell, \mathcal{L}_\ell)$ is denoted by $\mathsf{A}_{n,\ell}(\mathbb{L})$.
- $\ell = 1$, we obtain a projective space of dimension n.
- For $\ell > 1$, we can view $\mathsf{A}_{n,\ell}$ also as the Grassmannian of $(\ell-1)$-dimensional subspaces of a projective space over $\mathbb{L}$ of dimension n, defined similarly.
- If $n \geq 5$, it is a strong parapolar space of uniform symplectic rank 3; the symplecta are of hyperbolic type, namely, isomorphic to $\mathsf{A}_{3,2}(\mathbb{L})$.

Cartesian product spaces

Let $\Omega_i = (X_i, \mathcal{L}_i)$, $i = 1, 2$, be two parapolar spaces.
Define $\Omega := \Omega_1 \times \Omega_2$ as the point-line geometry with point set $X_1 \times X_2$ and line set

$$\{\{p_1\} \times L_2 : p_1 \in X_1, L_2 \in \mathcal{L}_2\} \cup \{L_1 \times \{p_2\} : L_1 \in \mathcal{L}_1, p_2 \in X_2\}.$$

This is again a parapolar space with symps

- $\{p_1\} \times \xi_2$, $p_1 \in X_1, \xi_2 \in \Xi_2$,
- $\xi_1 \times \{p_2\}$, $\xi_1 \in \Xi_1$, $p_2 \in X_2$
- $L_1 \times L_2$, $L_1 \in \mathcal{L}_1$, $L_2 \in \mathcal{L}_2$.

If both Ω_1 and Ω_2 are strong, then also Ω is strong. Its diameter equals $\mathrm{Diam}\,\Omega_1 + \mathrm{Diam}\,\Omega_2$, and it always has minimum symplectic rank 2.

Classification of minus one lacunary parapolar spaces

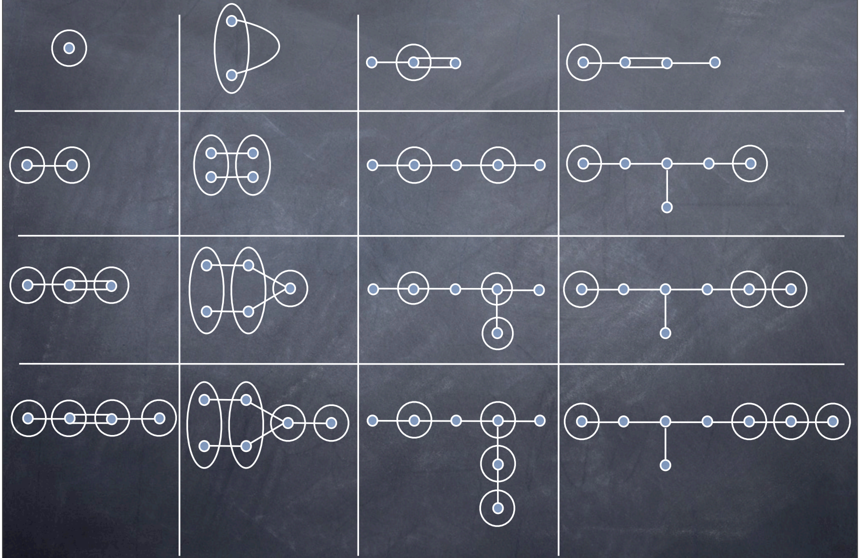
Let $\Omega = (X, \Xi)$ be a geometry with Ξ a family of subsets of X such that

- Each member of Ξ is a polar space whose point set is a subset of X;
- In $(X, \mathcal{L})$ each member of Ξ is a convex subspace;
- Every pair of points is contained in at least one member of Ξ;
- Every pair of members of Ξ intersects exactly at a nonempty singular subspace of both.

Then, Ω arises from one of:

- The Cartesian product of a thick line and an arbitrary projective plane
- The Cartesian product of two arbitrary projective planes
- The line-Grassmannian of any projective space of dimension 4 or 5;
- The Lie incidence geometry $\mathsf{E}_{6,1}(\mathbb{K})$, $\mathbb{K}$ any field.

Freudenthal-Tits Magic Square



Symplectic rank at least $k+3$

d	S	$k=-1$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$
$k+3$	$\{k+2, k+3\}$	$A_{1,1}(*) \times A_{2,1}(*)$	$A_{4,2}(\mathcal{L})$	$D_{5,5}(\mathbb{K})$	$E_{6,1}(\mathbb{K})$	$E_{7,7}(\mathbb{K})$	$E_{8,8}(\mathbb{K})$
	$\{k+3\}$	$A_{2,1}(*) \times A_{2,1}(*)$	$A_{5,3}(\mathcal{L})$	$E_{6,2}(\mathbb{K})$			
$k+4$	$\{k+3, k+4\}$	$A_{4,2}(\mathcal{L})$	$D_{5,5}(\mathbb{K})$	$E_{6,1}(\mathbb{K})$	$E_{7,7}(\mathbb{K})$	$E_{8,8}(\mathbb{K})$	
	$\{k+3, k+5\}$	$A_{5,2}(\mathcal{L})$	$D_{6,6}(\mathbb{K})$	$E_{7,1}(\mathbb{K})$			
$k+6$	$\{k+5, k+6\}$	$E_{6,1}(\mathbb{K})$	$E_{7,7}(\mathbb{K})$	$E_{8,8}(\mathbb{K})$			

Table: The k-lacunary parapolar spaces with symplectic rank $d \geq k+3$.

- S: set of dimensions of maximal singular subspaces
- white cells: Strong & diam 2, grey cells: diam 3,
- Going one cell to the left: point-residual.
- name in white: non-strong, point-residual automatically yields strong ones.

Minimum symplectic rank $k+2$

symps	S	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
non-thick	$\{1,-\}$ $(k=0)$ $\{k+1, n-1\}$ $(k\geq 1)$	$A_1 \times LS$	$A_{n,2}(\mathcal{L})$ $n\geq 4$	$D_{5,5}(\mathbb{K})$	$E_{6,1}(\mathbb{K})$	$E_{7,7}(\mathbb{K})$	$E_{8,8}(\mathbb{K})$
				$D_{6,6}(\mathbb{K})$	$E_{7,1}(\mathbb{K})$		
				$D_{7,7}(\mathbb{K})$	$E_{8,1}(\mathbb{K})^h$		
				$D_{n,n}(\mathbb{K})^h$ $n\geq 9$	$E_{n,1}(*)^h$ $n\geq 8$		
	$\emptyset$	GQ					
	$\{k+1\}$	$A_1 \times A_1 \times A_1$	$D_{4,2}(\mathbb{K})$				
	$\{k+1, n+k-2\}$	$A_1 \times D_{n-1,1}(\mathbb{K})$ $n\geq 4$	$D_{n,2}(\mathbb{K})$ $n\geq 5$				
mixed	$\{k+1, n+k-2\}$	$A_1 \times B_{n-1,1}(*)$ $n\geq 3$	$B_{n,2}(*)$ $n\geq 4$				
thick	$\{k+1\}$	$B_{3,3}(*)$	$F_{4,1}(*)$				

Table: The k-lacunary parapolar spaces with minimum symplectic rank $k+2$.

- $(.^h)$: locally connected homomorphic image.
- (*): not necessarily uniquely determined algebraic structure
- $\emptyset$: no projective subspaces, $-$: not all projective.

Outline of the proof

- Induction on lacunarity using point-residuals.
- $k = 0$: Strong (assumption if min symp rank is 2) and for rank at least 3 use classification for $k = -1$ showing that all point-residuals have diameter 2. Also diameter is either 2 or 3. Then reduce to well-known theorems by "Co-Co-S".
- $k = 1$: First show that minimum symplectic rank is 3. Then show residuals have same Coxeter type, then "Co-Co-S".
- $k \geq 2$: no symps of rank 2 anymore. Induction and Shult haircut
- In fact, we don't need the above theorems, but the length of the paper would increase dramatically. Moreover one (probably) can obtain (versions of) these theorems as corollaries.

The main theorem

Let $\Omega = (X, \mathcal{L})$ be a locally connected lacunary parapolar space with symplectic rank at least 3. Then Ω is one of the following Lie incidence geometries ($\mathbb{K}$: any commutative field, $\mathbb{L}$: arbitrary skew field):

(A) $\mathsf{A}_{5,3}(\mathbb{L})$ *or the line Grassmannian of a not necessarily finite-dimensional projective space of dimension at least 4;*

(B) *The line Grassmannian of a thick polar space of rank at least 4;*

(D) $\mathsf{D}_{n,2}(\mathbb{K})$, *$n \geq 4$, or a homomorphic image of* $\mathsf{D}_{n,n}(\mathbb{K})$, *$n \geq 5$ (isomorphic image if $n \leq 9$);*

(E) $\mathsf{E}_{6,1}(\mathbb{K})$, $\mathsf{E}_{6,2}(\mathbb{K})$, $\mathsf{E}_{7,1}(\mathbb{K})$, $\mathsf{E}_{7,7}(\mathbb{K})$, $\mathsf{E}_{8,8}(\mathbb{K})$, *a homomorphic image of* $\mathsf{E}_{8,1}(\mathbb{K})$, *a homomorphic image of* $\mathsf{E}_{n,1}(*)$, *with $n \geq 9$ and* $\mathsf{E}_n(*)$ *any building of type* E_n*;*

(F) *Any metasymplectic space.*

Parameters of Some exceptional geometries

Ω	$\mathrm{Diam}\,\Omega$	d	S	strong	long root
$\mathsf{E}_{6,1}(\mathbb{K})$	2	5	$\{4,5\}$	✓	
$\mathsf{E}_{6,2}(\mathbb{K})$	3	4	$\{4\}$		✓
$\mathsf{E}_{7,7}(\mathbb{K})$	3	6	$\{5,6\}$	✓	
$\mathsf{E}_{7,1}(\mathbb{K})$	3	5	$\{4,6\}$		✓
$\mathsf{E}_{8,8}(\mathbb{K})$	3	7	$\{6,7\}$		✓
$\mathsf{E}_{8,1}(\mathbb{K})$	5	5	$\{4,7\}$		
$\mathsf{F}_{4,1}(*)$	3	3	$\{2\}$		✓

Local connectedness

Let $\Omega = (X, \mathcal{L})$ be a parapolar space and *p* one of its points. We call Ω *locally connected at p* if each two lines through *p* are contained in a finite sequence of singular planes consecutively intersecting in lines through *p*.

Lemma

Let $\Omega = (X, \mathcal{L})$ be a parapolar space with symplectic rank at least 3. Then Ω is locally connected if and only if Ω_p is connected for all $p \in X$.

Overview of the locally disconnected case

- We show a general theorem how locally disconnected parapolar spaces are built from locally connected ones, and then apply this to lacunary parapolar spaces with symplectic rank a least 3 to arrive at a universal construction for locally disconnected parapolar spaces, using as building blocks the locally connected ones classified in the previous sections.
- The case $k = 0$ does not show up since automatically locally connected
- The case $k = -1$ is special, since (-1)-lacunarity of the building blocks is not preserved under the construction of the global, locally disconnected, parapolar space, unlike the cases $k \geq 1$.

Local connectedness and the residu

- $\Omega = (X, \mathcal{L})$: arbitrary parapolar space of symplectic rank at least 3.
- For $p \in X$: $\mathfrak{C}_p$ is the set of connected components of Ω_p.
- Ω is locally connected if and only if Ω_p is connected.
- We now provide a construction which introduces a copy of a point p for each connected component of Ω_p.

Construction

Let $\Omega = (X, \mathcal{L})$ be an arbitrary parapolar space with symplectic rank at least 3. The *unbuttoning of* Ω is defined as the following point-line geometry $\widetilde{\Omega} = (\widetilde{X}, \widetilde{\mathcal{L}})$:

- $\widetilde{X} = \{(p, \Upsilon) : p \in X \text{ and } \Upsilon \in \mathfrak{C}_p\}$ (connected components of Ω_p);
- for each line $L \in \mathcal{L}$, we define $\widetilde{L} = \{(p, \Upsilon) \in \widetilde{X} : p \in L \in \Upsilon\}$,
- $\widetilde{\mathcal{L}} = \{\widetilde{L} : L \in \mathcal{L}\}$.

So two points (p_1, Υ_1) and (p_2, Υ_2), with $\Upsilon_i \in \Omega_{p_i}$ for $i = 1, 2$, are collinear in $\widetilde{\Omega}$ if and only if $p_1 \perp p_2$ and the line $p_1 p_2$ is an element of both Υ_1 and Υ_2.

Unbuttoning gives locally connected spaces

Proposition

Let $\Omega = (X, \mathcal{L})$ be a not necessarily locally connected parapolar space. Then its unbuttoning $\widetilde{\Omega}$ is the disjoint union of locally connected (para)polar spaces.

Characterisation of the locally disconnected case

There is a converse to unbuttoning where you "glue" a family of disjoint *k*-lacunary spaces together called *k*-buttoning.

Theorem

Let $\Omega = (X, \mathcal{L})$ be a k-lacunary parapolar space with symplectic rank at least 3 and $k \geq -1$. Then either Ω is locally connected (and hence is one of the parapolar spaces of rank at least 3 we mentioned), or Ω is a k-buttoned parapolar space.

The converse procedure

We are now interested in a reverse procedure.
Which parapolar spaces can we obtain by collecting connected locally (para)polar spaces and identifying certain points? (We may restrict to symplectic rank at least 3).
The next slide basically says that, in Ω, you cannot walk from a point p to itself in less than five steps using two different components of Ω_p to start and come back in.

A restriction

Let $\Omega = (X, \mathcal{L})$ have symplectic rank at least 3 and let $\widetilde{\Omega}$ be its unbuttoning. Let $p \in X$ be such that Ω_p is disconnected and let $\Upsilon_1^{(p)} \neq \Upsilon_2^{(p)}$.
Let $q, r, s \in X \setminus \{p\}$ be arbitrary (not necessarily distinct) and let $\Upsilon_1^{(q)}, \Upsilon_2^{(q)}, \Upsilon_1^{(r)}, \Upsilon_2^{(r)}, \Upsilon_1^{(s)}, \Upsilon_2^{(s)}$ be not necessarily distinct. Then

$$\begin{aligned} \ell := \delta((p, \Upsilon_2^{(p)}), (q, \Upsilon_1^{(q)})) + \delta((q, \Upsilon_2^{(q)}), (r, \Upsilon_1^{(r)})) \\ + \delta((r, \Upsilon_2^{(r)}), (s, \Upsilon_1^{(s)})) + \delta((s, \Upsilon_2^{(s)}), (p, \Upsilon_1^{(p)})) \geq 5. \end{aligned}$$

k-Buttoning (Case $k=-1$ is more subtle)

Let $\mathcal{F} = \{\Omega_i = (X_i, \mathcal{L}_i) : i \in I\}$ be a family of (disjoint) k-lacunary locally connected (para)polar spaces of symplectic rank at least 3, $0 \neq k \geq -1$. Let $\mathcal{R}$ be an equivalence relation on $\widetilde{X} = \bigcup_{i \in I} X_i$, satisfying:

(C1) Let $\widetilde{p}, \widetilde{q}, \widetilde{r}, \widetilde{s}$ be four (not necessarily distinct, but $\widetilde{p} \notin \{\widetilde{q}, \widetilde{r}, \widetilde{s}\}$) equivalence classes with respect to $\mathcal{R}$, and let $p_1, p_2 \in \widetilde{p}$, with $p_1 \neq p_2$. If $q_1, q_2 \in \widetilde{q}$, $r_1, r_2 \in \widetilde{r}$ and $s_1, s_2 \in \widetilde{s}$, then

$$\delta(p_2, q_1) + \delta(q_2, r_1) + \delta(r_2, s_1) + \delta(s_2, p_1) \geq 5.$$

(C2) The graph with vertex set $\mathcal{F}$, Ω_i where Ω_j, $i, j \in I$, are adjacent if they contain points in the same equivalence class, is connected.

Set $X = \widetilde{X}/\mathcal{R}$. For each line L contained in some member of $\mathcal{F}$, we put $\widetilde{L} := \{\widetilde{p} \mid p \in L\}$ and define $\mathcal{L}$ as $\{\widetilde{L} \mid L \in \mathcal{L}_i \text{ for some } i \in I\}$. Then we denote the geometry $\Omega = (X, \mathcal{L})$ by $\Omega(\mathcal{F}, \mathcal{R})$. If $\mathcal{R}$ is non-trivial, then we call Ω a *k-buttoned geometry*. ■