# Discrete subgroups of Lie groups

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## 1 Overview of the Field

Recent years have seen a great deal of progress in our understanding of "thin" subgroups, which are discrete matrix groups that have infinite covolume in their Zariski closure. (The subgroups of finite covolume are called "lattices" and, generally speaking, are much better understood.) Traditionally, thin subgroups are required to be contained in arithmetic lattices, which is natural in the context of number-theoretic and algorithmic problems but, from the geometric or dynamical viewpoint, is not necessary. Thin subgroups have deep connections with number theory (see e.g. [6, 7, 8, 18, 22]), geometry (e.g. [1, 14, 16, 38]), and dynamics (e.g. [3, 23, 27, 29]).

The well-known "Tits Alternative" [44] (based on the classical "ping-pong argument" of Felix Klein) constructs free subgroups of any matrix group that is not virtually solvable. (In most cases, it is easy to arrange that the resulting free group is thin.) Sharpening and refining this classical construction is a very active and fruitful area of research that has settled numerous old problems. For instance, Breuillard and Gelander [11] proved a quantitative form of the Tits Alternative, which shows that the generators of a free subgroup can be chosen to have small word length, with respect to any generating set of the ambient group. Kapovich, Leeb and Porti [26] provided a coarse-geometric proof of the existence of free subgroups that are Anosov. In a somewhat different vein, Margulis and Soifer [32] proved that $SL(n, \mathbb{Z}), n \geq 4$, contains free products of the form $\mathbb{Z}^2 \star F_k$ (where $F_k$ is the free group of rank $k$) for all $k \geq 1$. (Answering a question of Platonov and Prasad, this implies that $SL(n, \mathbb{Z})$ has maximal subgroups of infinite index that are not free groups.) Also, it has been shown that $SL(n, \mathbb{Z})$ contains Coxeter groups (and, hence, right-angled Artin groups) when $n$ is sufficiently large. Ping-pong type constructions have also emerged as an important technical tool in other contexts, such as for disproving the invariable generation property (by constructing a thin subgroup that intersects every conjugacy class [19]) and for proving the expander property for Cayley graphs of finite quotients of thin groups [22].

Conversely, there are also obstructions to the existence of thin subgroups. For example, no thin, Zariski-dense subgroup of $SL(n, \mathbb{Z})$ contains $SL(3, \mathbb{Z})$ [45]. Similarly, it has been shown in certain situations that thin, discrete, Zariski-dense subgroups cannot contain a lattice in a maximal unipotent subgroup of the ambient group [5, 35, 46].

The study of certain natural (and properly discontinuous) actions of thin groups is another important line of research. The famous Auslander Conjecture concerns actions of thin groups on affine spaces. (Namely, it is conjectured that if a group acts properly discontinuously and cocompactly on an affine space $\mathbb{R}^n$, then the group is virtually solvable.) Actions on more general geometric spaces (such as flag varieties) are also important. While it seems that nothing of interest can be said about such actions *in general*, a great deal of progress has been made in recent years analyzing actions of thin subgroups that satisfy further restrictions. The Anosov property has been of particular interest (see, for example, [23], [27], and [49]), as well as other forms of strengthening of discreteness, e.g. *regularity*, which has its origin in [3], see also [27].

This conference brought together a range of specialists whose expertise in order to educate each other in the broad spectrum of techniques and problems in the theory of discrete subgroups of Lie groups. A number of young researchers, including graduate students, actively participated in the conference as well.

# 2 Open Problems

On the first day of the workshop we conducted a problem session in the theory of discrete subgroups of Lie groups. Below are problems collected during and after the workshop.

## 2.1 Proper actions on affine spaces

Let $\Gamma$ be a finitely generated group, $\rho : \Gamma \to GL(n, \mathbb{R})$ be the given representation, $Z^1(\Gamma, \rho)$ be the space of $\rho$-cocycles with values in $\mathbb{R}^n$, i.e. maps

$$u : \Gamma \to \mathbb{R}^n, u(\alpha\beta) = u(\alpha) + \rho(\alpha)u(\beta), \alpha, \beta \in \Gamma.$$

Note that $Z^1(\Gamma, \rho)$ is a finite-dimensional real vector space. Each cocycle $u \in Z^1(\Gamma, \rho)$ determines an affine action $\rho_u$ of $\Gamma$ on $V = \mathbb{R}^n$:

$$\rho_u(\gamma) : \mathbf{x} \mapsto \rho(\gamma)\mathbf{x} + u(\gamma).$$

Let $C = C_\rho \subset Z^1(\Gamma, \rho)$ denote the subset consisting of cocycles such that the action $\rho_u$ of $\Gamma$ on $\mathbb{R}^n$ is properly discontinuous.

**Question 2.1** (N. Tholozan)**.** *Is $C$ open in $Z^1(\Gamma, \rho)$? Is it convex?*

Note that the answer is positive for $n = 3$, this follows from the results of [21].

**Question 2.2** (N. Tholozan)**.** *Does $C_\rho$ depend continuously on $\rho$?*

Here one has to be careful with the topology used on the set of subsets of $Z^1(\Gamma, \rho)$. For closed subsets one uses Shabauty topology. For instance, if the subsets $C$ are open, their complements are closed and, hence, one can interpret the question as of the continuity of the complement with respect to $\rho$.

For a finitely generated group $\Gamma$, let $P(\Gamma, n)$ denote the subset of $Hom(\Gamma, Aff(\mathbb{R}^n))$ consisting of representations defining proper actions of $\Gamma$ on $\mathbb{R}^n$. Let $PA(\Gamma, n) \subset P(\Gamma, n)$ denote the subset consisting of actions $\rho_u$ with $P$-Anosov linear part $\rho$ (for some parabolic subgroup $P < GL(n, \mathbb{R})$).

**Question 2.3** (G. Soifer)**.** *To which extent $P(\Gamma, n)$ is open?*

Note that in general $P(\Gamma, n)$ is not open even for $n = 3$, for instance, one can take a rank 2 free group $\Gamma$ and a representation $\rho_u \in P(\Gamma, 3)$ whose linear part $\rho$ contains unipotent elements. A small perturbation of $\rho_u$ will yield a representation (with linear part in $SO(2, 1)$) with nondiscrete linear part, hence, a non-proper affine action. One can also perturb a representation so that the linear part is deformed to a Zariski dense subgroup of $SL(3, \mathbb{R})$, again resulting in a non-proper action.

It is known that

$$PA(\Gamma, 3) \cap Hom(\Gamma, SO(2, 1) \ltimes \mathbb{R}^3)$$

is open in

$$Hom(\Gamma, SO(2, 1) \ltimes \mathbb{R}^3)$$

(this follows from the results of [21] and stability of Anosov representations).

**Question 2.4** (G. Soifer)**.** *To which extent $P(\Gamma, n)$ is open in general?*

**Conjecture 2.5** (The Auslander conjecture)**.** *If $\Gamma < \mathrm{Aff}(\mathbb{R}^n)$ is a subgroup which acts properly discontinuously and co-compactly on $\mathbb{R}^n$, then $\Gamma$ is virtually solvable.*

Abels, Margulis and Soifer proved the Auslander conjecture for the dimensions $n \le 6$ and observed that the following problem is important for the further progress towards the Auslander conjecture:

**Conjecture 2.6** (Abels, Margulis, Soifer)**.** *Let $\Gamma < O(4,3) \ltimes \mathbb{R}^7 < \mathrm{Aff}(\mathbb{R}^7)$ be a subgroup acting properly discontinuously and co-compactly on $\mathbb{R}^7$. Then the linear part of $\Gamma$ is not Zariski dense in $SO(4,3)$.*

**Question 2.7.** *Does there exist a properly discontinuous cocompact group of affine transformations isomorphic to the fundamental group of a closed hyperbolic manifold?*

Partial progress towards a negative answer to this problem was described at the talk by Suhyoung Choi at the workshop: He proved that such an action cannot exist under certain assumptions on its linear part (strengthening the $P$-Anosov condition).

**Question 2.8** (G. Mostow)**.** *Suppose that $M = \mathbb{R}^n/\Gamma$ is an affine manifold, such that the linear part of $\Gamma$ is in $SL_n(\mathbb{R})$. Thus, $\Gamma$ preserves the standard volume form on $\mathbb{R}^n$ and, hence, $M$ has a canonical volume form as well. Does $Vol(M) < \infty$ imply that $M$ is compact?*

This question is motivated by Mostow's theorem that lattices in solvable groups are cocompact.

**Question 2.9** (G. Soifer)**.** *Study subgroups $\Gamma < \mathrm{Aff}(\mathbb{C}^n)$ acting properly discontinuously on $\mathbb{C}^n$.*

More specifically:

**Question 2.10** (G. Soifer)**.** *Does there exist a free nonabelian subgroup $\Gamma < \mathrm{Aff}(\mathbb{C}^3)$ acting properly discontinuously on $\mathbb{C}^3$? For instance, consider the adjoint action of $SL_2(\mathbb{C})$ on $\mathbb{C}^3$ (identified with the Lie algebra of $SL_2(\mathbb{C})$). Consider a generic representation $\rho : F_2 = \langle a, b\rangle \to SL(2,\mathbb{C})$. Does there exist a cocycle $u \in Z^1(F_2, \mathbb{C}^3)$ and $m > 0$ such that the action on $\mathbb{C}^3$ of $\langle a^m, b^m\rangle$ given by $\rho_u$ is properly discontinuous?*

**Conjecture 2.11** (Markus Conjecture)**.** *Suppose that $M$ is a compact $n$-dimensional affine manifold whose linear holonomy is in $SL(n,\mathbb{R})$. Is $M$ complete?*

This conjecture is known when the linear holonomy of $M$ has "discompacity 1" (Y. Carriere, [13]), e.g. when $M$ is a flat Lorentzian manifold, and also for convex affine manifolds of dimension $\le 5$, [25].

**Conjecture 2.12** (M. Kapovich)**.** *Suppose that $M$ is a compact $n$-dimensional affine manifold whose linear holonomy is contained in a rank one subgroup of $SL(n,\mathbb{R})$. Is $M$ complete?*

## 2.2 Discrete subgroups of $SL(3,\mathbb{R})$

**Question 2.13** (M. Kapovich)**.**

1. *Does there exist a discrete a subgroup $\Gamma < SL(3,\mathbb{R})$ isomorphic to $\mathbb{Z}^2 \star \mathbb{Z}$ and containing only regular diagonalizable elements?*
2. *Does there exist a discrete subgroup $\Gamma < SL(3,\mathbb{Z})$ isomorphic to $\mathbb{Z}^2 \star \mathbb{Z}$?*

Note that there are known examples, [43], of discrete subgroups $\Gamma < SL(3,\mathbb{R})$ isomorphic to $\mathbb{Z}^2 \star \mathbb{Z}$ where $\mathbb{Z}^2$ is *super-singular*: It is generated by three singular diagonalizable matrices $A, B, C$ satisfying $ABC = 1$.

**Question 2.14** (K. Tsouvalas)**.** *Does there exist a discrete a subgroup $\Gamma < SL(3,\mathbb{R})$ isomorphic to $\pi_1(S) \star \mathbb{Z}$, where $S$ is a closed hyperbolic surface ?*

Note that it is impossible to find an Anosov subgroup $\Gamma < SL(3,\mathbb{R})$ isomorphic to $\pi_1(S) \star \mathbb{Z}$ with this property, since every Anosov subgroup of $SL(3,\mathbb{R})$ is either virtually free or a virtually surface group. Note, furthermore, that $SL(4,\mathbb{Z})$ contains subgroups isomorphic to $\mathbb{Z}^2 \star \mathbb{Z}$ and $\pi_1(S) \star \mathbb{Z}$.

## 2.3 Subgroups of $SL(n,\mathbb{Z})$, $n \geq 3$

While many "exotic" finitely generated groups embed in $SL(n,\mathbb{Z})$ for large $n$, very few subgroups of $SL(3,\mathbb{Z})$ are known: All currently known finitely generated thin subgroups of $SL(3,\mathbb{Z})$ are either virtually free or are virtually isomorphic to surface groups.

**Problem 2.15.** *Construct thin subgroups of $SL(3,\mathbb{Z})$ which are neither virtually free nor are virtually isomorphic to surface groups.*

For the next questions, we will need some group-theoretic definitions:

**Definition 2.1.** *A group $\Gamma$ is called* coherent *if every finitely generated subgroup of $\Gamma$ is finitely presented. A group $\Gamma$ is said to have the* Howson property *if the intersection of any two finitely generated subgroup of $\Gamma$ is again finitely generated.*

It is known that $SL(2,\mathbb{Z})$ is coherent (and, moreover, every discrete subgroup of $SL(2,\mathbb{C})$ is coherent), while $SL(4,\mathbb{Z})$ is non-coherent. Every discrete subgroup of $SL(2,\mathbb{C})$ which is not a lattice has the Howson property. However, there are (even arithmetic) lattices in $SL(2,\mathbb{C})$ which do not have the Howson property. The reason for this is the existence of finitely generated geometrically infinite subgroups of such lattices.

**Question 2.16** (J.-P. Serre)**.** *Is $SL(3,\mathbb{Z})$ coherent?*

The answer would be positive if every finitely generated thin subgroup of $SL(3,\mathbb{Z})$ were virtually isomorphic to either a free group or a surface group. While groups such as $\mathbb{Z}^2 \star \mathbb{Z}$ and $\pi_1(S) \star \mathbb{Z}$ (where $S$ is a surface) are coherent, the existence of embeddings of such groups in $SL(3,\mathbb{Z})$ might help us to find embeddings of more complicated subgroups and, hopefully, address the coherence problem.

**Problem 2.17** (J.-P. Serre)**.** *Is there a profinitely dense non-free subgroup in $SL(3,\mathbb{Z})$?*

**Question 2.18** (A. Detinko)**.** *Does $SL(3,\mathbb{Z})$ have the Howson property?*

**Question 2.19** (M. Kapovich)**.** *Suppose that $\Gamma_1, \Gamma_2$ are Anosov subgroups of $SL(3,\mathbb{Z})$. Is $\Gamma_1 \cap \Gamma_2$ finitely generated?*

**Problem 2.20** (T. Gelander, C. Meiri)**.** *An element $g \in SL(3,\mathbb{Z})$ is called* complex *if for every $m \geq 1$ the matrix $g^m$ has a non-real eigenvalue. Is it possible for a thin subgroup of $SL(3,\mathbb{Z})$ to contain a complex element?*

## 2.4 Algorithmic problems

**Question 2.21** (A. Detinko)**.** *Is freeness decidable for finitely generated subgroups of arithmetic groups (e.g. of $SL(n,\mathbb{Z}), n \geq 3$)?*

Note that freeness is undecidable for subsemigroups. Freeness is decidable for subgroups of $SL(2,\mathbb{Z})$. It is also decidable for some special classes of subgroups of arithmetic groups:

(a) Anosov subgroups.

(b) Subgroups which admit finitely-sided Dirichlet domains in associated symmetric spaces.

Freeness is likely to be, at least effectively, undecidable. The reason is the existence of *badly distorted* finitely generated free subgroups of $SL(n,\mathbb{Z})$ for large $n$: These are free subgroups whose distortion function is comparable to the $k$-th Ackerman function (for any $k$), see [15, 10] for the description of embeddings of such free groups in free-by-cyclic groups and [24, 48] for embeddings to $SL(n,\mathbb{Z})$.

**Question 2.22** (A. Detinko)**.** *Is arithmeticity decidable? More precisely, is there an algorithm that decides if a finitely generated Zariski dense subgroup $\Lambda$ of an* irreducible *arithmetic group $\Gamma$ (say, $SL(n,\mathbb{Z})$, $n \geq 3$) has finite index in $\Gamma$?*

Note that this problem is semidecidable: There is an algorithm which will terminate if $\Lambda < \Gamma$ has finite index. The problem is known to be decidable for subgroups of $SL(2,\mathbb{Z})$ and undecidable for subgroups of $SL(2,\mathbb{Z}) \times SL(2,\mathbb{Z})$.

**Question 2.23** (M. Kapovich)**.** *Is the membership problem in finitely generated subgroups of $SL(3,\mathbb{Z})$ decidable?*

Note that all *known* finitely generated subgroups of $SL(3,\mathbb{Z})$ have at most exponential distortion, hence, have decidable membership problem. In contrast, the membership problem is undecidable for finitely generated subgroups of $SL(4,\mathbb{Z})$. The reason is that that group contains $SL(2,\mathbb{Z}) \times SL(2,\mathbb{Z})$, which, in turn, contains a direct product of two free groups of large ranks. The latter admits finitely generated normal subgroups with undecidable membership problem (Mikhailova subgroups, [33]). However, in this case, the ambient lattice is reducible.

**Fact 2.24.** *There exist irreducible arithmetic groups $\Gamma$ such that for Zariski dense subgroups in $\Gamma$ the membership problem is undecidable.*

Very likely, the subgroups $\Gamma$ can be found in $SO(p,q)$ for suitable $p,q$. The existence of $\Gamma$ is an application of the Rips construction of small cancellation groups with non-recursively distorted normal subgroups [42], combined with with the Cubulation Theorem of Dani Wise [47] and the embedability of cubulated groups in RACGs (Right-Angled Coxeter groups), see [48], which, in turn, admit Zariski dense representations in $O(p,q) \cap GL(p+q,\mathbb{Z})$, see [4].

Note that the membership problem is decidable for subgroups with recursive distortion function, e.g. for quasiisometrically embedded subgroups, such as Anosov subgroups.

## 2.5 Maximal subgroups

Recall that a subgroup $\mathrm{M}$ of a group $\Gamma$ is said to be *maximal* if there is no *proper* subgroup $\Lambda < \Gamma$ containing $\mathrm{M}$.

According to [32], every Zariski dense subgroup $\Gamma$ in a semisimple Lie group $G$ (of positive dimension) admits *maximal subgroups of infinite index.* However, very little is known about maximal subgroups in this setting. The construction of maximal subgroups in [32] is a two-step process: First, construct an infinite rank free profinitely dense subgroup $\Lambda < \Gamma$ (this step is essentially constructive) and then, use Zorn's Lemma to get a maximal subgroup $\mathrm{M}$:

$$\Lambda < \mathrm{M} < \Gamma.$$

The second step is completely nonconstructive.

**Question 2.25** (G. Margulis, G. Soifer)**.** *Suppose that $\Gamma < G$ is as above.*

1. *Is it true that for every maximal subgroup $\mathrm{M} < \Gamma$ is not finitely generated?*
2. *Is it true that $\Gamma$ contains a free maximal subgroup?*

Note that M. Akka and T. Gelander proved that there exists a finitely generated profinitely dense subgroup $\Gamma$ of $SL_n(\mathbb{Z})$ such that the number of generators of $\Gamma$ does not depend on $n$.

**Question 2.26** (G. Soifer)**.** *Does there exist a profinitely dense subgroup of $SL_n(\mathbb{Z})$ generated by two elements?*

## 2.6 Other problems on thin subgroups

**Definition 2.2.** *For a finitely generated group $\Gamma$ with a finite generating subset $S$ the* Kazhdan constant *$\kappa(\Gamma, S)$ is defined as*

$$\kappa(\Gamma, S) = \inf_{\pi,v} \max_{g\in S} \|v - \pi_g v\|,$$

*where the infimum is taken over all unitary representations $(H_\pi, \pi)$ of $\Gamma$ without fixed unit vectors, and all unit vectors $v \in H_\pi$. Then $\Gamma$ is said to have Property T iff $\kappa(\Gamma, S) > 0$ for some/every finite generating subset $S$. A group $\Gamma$ is said to have* uniform Property T*, if $\inf_S \kappa(\Gamma, S) > 0$ where the infimum is taken over all finite generating subsets $S$.*

The following question goes back to [31]:

**Question 2.27** (A. Lubotzky)**.** *Does $SL(n,\mathbb{Z}), n \geq 3$, have the uniform Property T?*

Note that Lubotzky was asking the more general question whether Property T implies uniform Property T, which was answered in the negative independently by Gelander & Zuk [20], and Osin [36]. The problem is open for all $n$. In the case of many classes of higher rank uniform lattices, the answer is known to be negative.

**Question 2.28** (Bekka, de la Harpe, Valette)**.** *Are there thin subgroups of $SL(n,\mathbb{Z}), n \geq 3$, satisfying Property T?*

The difficulty is that the known examples of (infinite discrete) groups satisfying Property T tend to be: (a) super-rigid arithmetic groups, or (b) some combinatorially defined groups for which all real-linear representations are nondiscrete or nonfaithful.

One can attempt to combine super-rigid lattices, but such combinations tend to destroy Property T. Alternatively, one can attempt to use polygons of groups where vertex groups have Property T, with suitable spectral conditions on links of vertices, but such constructions tend to produce relatively compact subgroups of $SL(n,\mathbb{R})$.

Note that the property $\tau$ (with respect to the family of finite index normal subgroups which are kernels of homomorphisms to $SL(n,\mathbb{Z}/q\mathbb{Z})$) holds for thin subgroups of $SL(n,\mathbb{Z}), n \geq 3$, see [9].

For the next question, recall that standard proofs of the Tits Alternative yield Zariski dense free subgroups of the given semisimple Lie group $G$.

**Definition 2.3.** *A free subgroup $\Gamma < G$ is* hereditarily Zariski dense *(or* strongly dense*, see [12]) if every noncyclic subgroup of $\Gamma$ is Zariski dense in $G$.*

The following problem is raised in [12]:

**Question 2.29** (Breuillard, Green, Guralnick, Tao)**.** *Is it true that every Zariski dense subgroup of a real semisimple Lie group $G$ contains a hereditarily Zariski dense free subgroup? If so, is there a quantitative version of this result?*

It appears that the only case when the affirmative answer is known is when $G$ is 3-dimensional (in which case it is an immediate corollary of the Tits Alternative).

## 2.7 Characterization of higher rank lattices

**Definition 2.4** (Prasad–Raghunathan rank)**.** *Let $\Gamma$ be a group. Let $A_i$ denote the subset of $\Gamma$ that consists of those elements whose centralizer contains a free abelian group of rank at most $i$ as a subgroup of finite index. Thus, $A_0 \subset A_1 \subset \dots$. The* Prasad–Raghunathan rank*, $\operatorname{prank}(\Gamma)$, of $\Gamma$ is the minimal number $i$ such that $\Gamma = \gamma_1 A_i \cup \dots \cup \gamma_m A_i$ for some $\gamma_1, \dots, \gamma_m \in \Gamma$.*

For instance, if $\Gamma$ is a lattice in a semisimple Lie group of rank $n$, then $\operatorname{prank}(\Gamma) = n$. If $M$ is a compact Riemannian manifold of nonpositive curvature with $\Gamma = \pi_1(M)$, then $\operatorname{prank}(\Gamma)$ equals the geometric rank of $M$, i.e. the largest $n$ such that every geodesic in $M$ is contained in an immersed $n$-dimensional flat.

**Definition 2.5** (BGP, Bounded Generation Property)**.** *A group $\Gamma$ is said to have BGP if there exist elements $\gamma_1, ..., \gamma_k$ such that every $\gamma \in \Gamma$ can be written as a product*

$$\gamma = \gamma_1^{n_1} \gamma_2^{n_2} \cdots \gamma_k^{n_k}$$

*for some $n_1, ..., n_k \in \mathbb{Z}$. (Note that a power of each $\gamma_i$ appears only once.)*

**Question 2.30** (G. Prasad)**.** *Does there exist a discrete Zariski dense subgroup $\Gamma < G$ (with $G$ a simple real algebraic group) such that $\Gamma$ is not a lattice but $\operatorname{prank}(\Gamma) = rank_{\mathbb{R}}(G)$?*

**Question 2.31** (M. Kapovich)**.** *What algebraic properties distinguish higher rank (irreducible uniform) lattices?*

For instance, such groups $\Gamma$ have Prasad–Raghunathan rank, $\mathrm{prank}(\Gamma) \geq 2$. Are there discrete linear groups $\Gamma$ which are not virtually nontrivial direct products and are not lattices, satisfying $\mathrm{prank}(\Gamma) \geq 2$? In the case of groups $\Gamma$ of integer points of split semisimple algebraic groups over $\mathbb{Z}$, a defining feature are the *Serre relators*. However, Serre relators are for unipotent elements, which do not exist in uniform lattices. Uniform higher rank lattices satisfy *approximate* Serre relators. Do these determine whether a discrete linear group is a higher rank lattice?

Notice that there are some indirect signs that an algebraic characterization of lattices is possible:

1. Higher rank lattices are quasiisometrically rigid (Kleiner & Leeb [28], Eskin [17]).
2. Higher rank lattices are rigid in the sense of the 1st order logic (Avni, Lubotzky, Mieri [2]).
3. Appearance of Serre relators in profinite completions, (Prasad, Rapinchuk [39]).

The situation is not entirely clear even for nonuniform lattices. Many classes of higher rank nonuniform lattices satisfy the BGP. Nonlinear groups that satisfy the BGP were constructed by A. Muranov [34].

**Question 2.32** (M. Kapovich)**.** *Suppose that $\Gamma$ is an abstract (infinite) $\mathbb{R}$-linear group satisfying the BGP. Is it isomorphic to a lattice in a Lie group?*

**Problem 2.33** (M. Mj)**.** *Does $SL(3,\mathbb{Z})$ have the bounded generation property with respect to semisimple elements? I.e., is there a collection of $k$ semisimple elements $g_1, ..., g_k \in SL(3,\mathbb{Z})$ such that every element of $SL(3,\mathbb{Z})$ has the form*

$$g = g_1^{n_1} ... g_k^{n_k}?$$

Conjecturally, the answer is negative (for dynamical reasons related to ping-pong arguments) which should pave the way to prove that uniform lattices do not have bounded generation property.

## 2.8 Why are higher rank lattices super-rigid?

One way to say that an abstract group $\Gamma$ is *super-rigid* is to require that for every field $F$ and $n \in \mathbb{N}$, there are only finitely many conjugacy classes of representations $\Gamma \to GL(n, F)$. Of course, some groups do not admit any nontrivial linear representations, so it makes sense to restrict the discussion to finitely generated linear groups $\Gamma$.

Loosely speaking, such a group is (super) rigid if it satisfies some peculiar relators. There are many proofs of rigidity and super-rigidity of (higher rank irreducible) lattices, but none of these proofs (in the setting of uniform lattices) use relators satisfied by lattices, likely because such relators are simply unknown (see previous section). In contrast, there are known proofs of super-rigidity of some classes of higher rank non-uniform lattices (see [41] and references therein).

**Question 2.34** (M. Kapovich)**.** *What are group-theoretic reasons that make higher rank uniform lattices (super)-rigid? Are the approximate Serre relators responsible for this? Or high Prasad-Raghunathan rank?*

The only known result in this direction is that the BGP implies super-rigidity, see [37].

# 3 Presentation Highlights

### Suhyoung Choi (KAIST, Daejeon, Korea)
### "Closed affine manifolds with partially hyperbolic linear holonomy"

The overall goal to show that closed manifolds of negative curvature do not admit complete special affine structures whose linear parts are partially hyperbolic in the dynamical sense. Furthermore, they should not admit complete affine structures with semi-simple P-Anosov linear holonomy groups.

### Jeff Danciger (University of Texas, Austin)
### "Affine actions with Hitchin linear part"

Properly discontinuous actions of a surface group on $\mathbb{R}^d$ by affine transformations were shown to exist by Danciger–Gueritaud–Kassel. In the talk, representing a joint work with Tengren Zhang, it is shown, however, that if the linear part of an affine surface group action is in the Hitchin component, then the affine action is not properly discontinuous. The key case is that of linear part in $\mathrm{SO}(n, n-1)$, so that $\mathbb{R}^d = \mathbb{R}^{n,n-1}$ is the model for flat pseudo-Riemannian geometry of signature $(n, n-1)$. Here, the translational parts determine a deformation of the linear part into $\mathrm{SO}(n, n)$ Hitchin representations and the crucial step is to show that such representations are not Anosov in $\mathrm{SL}(2n, \mathbb{R})$ with respect to the stabilizer of an $n$-plane.

### Alla Detinko (University of Hull, Hull, UK)
### "Zariski density and computing with infinite linear groups"

This talk, presenting a joint work with Dane Flannery and Alexander Hulpke, describes recent developments in a novel domain of computational group theory: Computing with infinite linear groups. Special consideration is given to algorithms for Zariski dense subgroups. This includes a computer realization of the strong approximation theorem, and algorithms for arithmetic groups. These methods are then applied to the solution of problems further afield by computer experimentation.

### Cornelia Drutu (Oxford University, Oxford, UK)
### "Effective equidistribution of expanding horospheres"

The topic of the talk is a result of effective equidistribution of a special family of expanding horospheres (modulo $SL(d, \mathbb{Z})$) in the locally symmetric space of positive definite quadratic forms of determinant one on $\mathbb{R}^d$ modulo $SL(d, \mathbb{Z})$. The proof uses uniform lattice point asymptotics for compact sets of $d$-dimensional dilating ellipsoids.

### Anna Felikson (Durham University, Durham, UK)
### "Coxeter groups, quiver mutations and hyperbolic manifolds"

Mutations of quivers were introduced by Fomin and Zelevinsky in the beginning of 2000's in the context of cluster algebras. Since then, mutations appear (sometimes completely unexpectedly) in various domains of mathematics and physics. Using mutations of quivers, Barot and Marsh constructed a series of presentations of finite Coxeter groups as quotients of infinite Coxeter groups. The talk, presenting a joint work with Pavel Tumarkin, describes a generalization of this construction leading to a new invariant of bordered marked surfaces, and a geometric interpretation: It occurs that presentations constructed by Barot and Marsh give rise to a construction of geometric manifolds with large symmetry groups, in particular to some hyperbolic manifolds of small volume with proper actions of Coxeter groups.

### Ilya Gekhtman (University of Toronto, Toronto)
### "Gibbs measures vs. random walks in negative curvature"

The ideal boundary of a negatively curved manifold naturally carries two types of measures. On the one hand, we have conditionals for equilibrium (Gibbs) states associated to Hölder potentials; these include the Patterson-Sullivan measure and the Liouville measure. On the other hand, we have stationary measures coming from random walks on the fundamental group.

The talk, based on a joint work with Gerasimov–Potyagailo–Yang and, partly, on a joint work with Tiozzo, aims to compare and contrast these two classes. First, it is shown that both of these of these measures can be associated to geodesic flow invariant measures on the unit tangent bundle, with respect to which closed geodesics satisfy different equidistribution properties. Second, we show that the absolute continuity between a harmonic measure and a Gibbs measure is equivalent to a relation between entropy, (generalized) drift and critical exponent, generalizing previous formulas of Guivarc'h, Ledrappier, and Blachere–Haissinsky–Mathieu. This shows that if the manifold (or more generally, a $\mathrm{CAT}(-1)$ quotient) is geometrically finite but not convex cocompact, stationary measures are always singular with respect to Gibbs measures.

A major technical tool is a generalization of a deviation inequality due to Ancona saying the so called Green distance associated to the random walk is nearly additive along geodesics in the universal cover.

## Dmitry Kleinbock (Brandeis University, Waltham, MA)
## "Khintchine-type theorems via $L^2$ estimates for Siegel transform"

Recently there has been a surge of activity in quantifying the density of values of generic quadratic forms at integer points. The talk, presenting a joint work with Mishel Skenderi, describes some new and quite general results in this direction obtained via the so-called Siegel–Rogers method, which has recently been utilized by Athreya and Margulis, and later by Kelmer and Yu.

## Alex Kontorovich (Rutgers University, New Brunswick)
## "Sphere Packings and Arithmetic"

The talk describes recent progress in understanding Apollonian-like sphere packings and more general objects, with connections to arithmetic hyperbolic groups, both reflective and non-reflective.

## Arie Levit (Yale University, New Haven)
## "Quantitative weak uniform discreteness"

This talk, based on a joint work with Gelander and Margulis, describes a quantitative variant of the Kazhdan–Margulis theorem generalized to probability measure preserving actions of semisimple groups over local fields.

## Michael Lipnowski (McGill University, Montreal)
## "Building grids on $\Gamma\backslash X$ for $\Gamma < \mathrm{SL}_n(\mathbb{Z})$ admitting membership testing"

This talk, presenting a joint work with Aurel Page, describes an algorithm for building grids in metric spaces $(M, d_M)$. The algorithm uses repeated computations of (cutoff) distances $\min\{1, d_M\}$; computations are illustrated in the case of the locally symmetric space of lattices in $\mathbb{R}^n$. A pessimistic outlook of these results for locally symmetric spaces $G(\mathbb{Z})\backslash G(\mathbb{R})/K$, when $G(\mathbb{Z})$ satisfies the congruence subgroup property: The membership testing for finitely generated subgroups of $G(\mathbb{Z})$ is harder than certifying thinness. An optimistic outlook of these results for the same $G(\mathbb{Z})$ is that it gives a prospect for certifying thinness for finitely generated subgroups.

## Beibei Liu (Max Plank Institute for Mathematics, Bonn)
## "Hausdorff dimension and geometric finiteness in hyperbolic spaces"

Geometric finiteness is a nice property that discrete isometry group of a hyperbolic space can have. One way to define geometric finiteness is to require that the limit set of the group consists of conical limit points and parabolic fixed points. In the talk, it is shown that the limit sets of geometrically infinite Kleinian groups contain continuum of nonconical limit points. One can ask questions relating the measure-theoretic size of the limit set, conical limit set or non-conical limit set, in relation to the geometric finiteness. The talk reviews some recent results and conjectures about Kleinian groups with small Hausdorff dimension, and small critical exponents.

## Sara Maloni (University of Virginia, Charlottesville)
## "The geometry of quasi-Hitchin symplectic Anosov representations"

Quasi-Hitchin representations in $\mathrm{Sp}(4, \mathbb{C})$ are deformations of Fuchsian (and Hitchin) representations which remain Anosov. These representations acts on the space $\mathrm{Lag}(\mathbb{C}^4)$ of complex lagrangian grassmanian subspaces of $\mathbb{C}^4$. This theory generalizes the classical and important theory of quasi-Fuchsian representations and their action on the Riemann sphere $\mathbb{C}P^1 = \mathrm{Lag}(\mathbb{C}^2)$. The talk (based on a joint work with D.Alessandrini and A.Wienhard) reviews the classical theory as well as the geometry and topology of quasi-Hitchin representations, In particular, it is shows that the quotient of the domain of discontinuity for this action is a fiber bundle over the surface and fibers are described. The fibration map comes from an interesting parametrization of $\mathrm{Lag}(\mathbb{C}^4)$ as the space of regular ideal hyperbolic tetrahedra and their degenerations.

### Giuseppe Martone (University of Michigan, Ann Arbor)
### "Sequences of Hitchin representations of tree-type"

In this talk some non-trivial sufficient conditions are described for diverging sequences of Hitchin representations, whose limits in the Parreau boundary is given by an action on a tree. These non-trivial conditions are given in terms of Fock-Goncharov coordinates on moduli spaces of positive tuples of flags.

### Chen Meiri (Technion, Haifa, Israel)
### "First order rigidity of higher-rank arithmetic groups"

In many contexts, there is a dichotomy between lattices in Lie groups of rank one and lattices in Lie groups of higher-rank. In the talk, based on joint works with Nir Avni and Alex Lubotzky, some manifestations of this dichotomy are described in the context of the Model Theory.

### Plinio Murillo (KIAS, Seoul, Korea)
### "Systole growth on arithmetic locally symmetric spaces"

The systole of a Riemannian manifold is the shortest length of a non-contractible closed geodesic. The purpose of this talk is to survey recent results in systole growth along congruence covers of arithmetic manifolds, and how this information interact with the geometry and the topology of the covers.

### Joan Porti (Universitat Autònoma de Barcelona)
### "Twisted Alexander polynomials and hyperbolic volume for three-manifolds"

Given a hyperbolic 3-manifold with cusps, one considers the composition of a lift of its holonomy in $\mathrm{SL}(2,\mathbb{C})$ with the irreducible representation in $\mathrm{SL}(n,\mathbb{C})$, that yields a twisted Alexander polynomial $A_n(t)$, for each natural $n$. In the talk (based on a joint work with L.Bénard, J.Dubois and M.Heusener), it is proven that, for a complex number $z$ with norm one, $\log|A_n(z)|/n^2$ converges to the hyperbolic volume of the manifold divided by $4\pi$, as $n \to \infty$. This generalizes and uses a theorem of W. Mueller for closed manifolds on analytic torsion.

### Andrei Rapinchuk (University of Virginia, Charlottesville)
### "Eigenvalue rigidity for Zariski-dense subgroups"

This talk is a progress report on the work of Gopal Prasad and Andrei Rapinchuk focused on a new form of rigidity, called the *eigenvalue rigidity*. The latter is based on the notion of weak commensurability of Zariski-dense subgroups of semi-simple algebraic groups introduced in [38], which provides a convenient way of matching the eigenvalues of semi-simple elements of these subgroups. A detailed analysis of this notion for arithmetic groups allows to resolve some long-standing problems about isospectral compact locally symmetric spaces. Currently, there is growing evidence that some key results can be extended from arithmetic groups to arbitrary finitely generated Zariski-dense subgroups, yielding thereby certain rigidity statements, based on the eigenvalue information, in this generality (including the situations where the subgroups at hand are free groups). This work has led to new directions of research in the theory of algebraic groups, one of which is the analysis of forms of a given absolutely almost simple algebraic group that have good reduction at a given set of discrete valuations of the base field.

### Igor Rapinchuk (Michigan State University, East Lansing)
### "Abstract homomorphisms of algebraic groups and applications"

This talk presents several results on abstract homomorphisms between the groups of rational points of algebraic groups. The main focus will be on a conjecture of Borel and Tits formulated in their landmark 1973 paper. The presented results settle this conjecture in several cases; the proofs make use of the notion of an algebraic ring. Several applications are given to character varieties of finitely generated groups and representations of some non-arithmetic groups.

### Nicolas Tholozan (ENS, Paris)
### "Exotic compact quotients of pseudo-Riemannian symmetric spaces"

Let $M$ be a Gromov-Thurston manifold. This talk, presenting a joint work with Daniel Monclair and Jean-Marc Schlenker, describes a construction of proper and cocompact actions of the fundamental group of $M$ on a certain pseudo-Riemannian symmetric space. The talk also explain how the construction relates to the existence of a globally hyperbolic anti-de Sitter manifold with Cauchy hypersurface homeomorphic to $M$.

### Kostas Tsouvalas (University of Michigan, Ann Arbor)
### "Characterizing Benoist representations by limit maps"

Anosov representations of word hyperbolic groups form a rich class of discrete subgroups of semisimple Lie groups, generalizing classical convex cocompact groups of real rank one Lie groups. A large class of projective Anosov representations are Benoist representations. This talk, presenting a joint work with Richard Canary, gives a characterization of Benoist representations in terms of the existence of limit maps.

### Tengren Zhang (National University of Singapore)
### "Regularity of limit curves of Anosov representations"

Anosov representations are representations of a hyperbolic group $\Gamma$ to a non-compact semisimple Lie group that are "geometrically well-behaved." In the case when the target Lie group is $\mathrm{PGL}(d, \mathbb{R})$, these representations admit a limit set in the $d-1$ dimensional projective space that is homeomorphic to the boundary of $\Gamma$. In this talk, presenting a joint work with A. Zimmer, under some irreducibility conditions, necessary and sufficient conditions are given for when this limit set is a $C^{1,a}$ submanifold.

### Andrew Zimmer (Louisiana State University, Baton Rouge)
### "Convex co-compact actions of projective linear groups"

This talk, presenting a joint work with Mitul Islam, describes some results concerning convex cocompact subgroups of the projective linear group (as defined by Danciger–Guéritaud–Kassel). These are a special class of discrete subgroups which act convex cocompactly on a properly convex domain in real projective space. In the case when the subgroup is word hyperbolic, these are well studied objects: The inclusion representation is actually an Anosov representation. The non-hyperbolic case is less understood and is the main focus of the talk.

## 4 Outcome of the Meeting

Due to late cancellations, the number of participants was 31 instead of the planned 42, but there was a general consensus that the size of the group led to even closer interactions and more focused discussion than is common at such meetings. Some progress was made on problems raised at the meeting, but it is too early to predict the full impact on new research and collaborations from the meeting. Suffice it to say that in conversations at the meeting and since, participants have described the meeting as having been unusually successful.

# References


[1] H. Abels, G. Margulis, G. Soifer, The linear part of an affine group acting properly discontinuously and leaving a quadratic form invariant, *Geom. Dedicata* **153** (2011), pp. 1–46.

[2] N. Avni, A. Lubotzky, C. Meiri, First order rigidity of non-uniform higher rank arithmetic lattices, *Inventiones Math.* (2019)

[3] Y. Benoist, Propriétés asymptotiques des groupes linéaires, *Geom. Funct. Anal.* **7** (1997), no. 1, pp. 1–47.

[4] Y. Benoist, P. de la Harpe, Adhérence de Zariski des groupes de Coxeter, *Compositio Math.*, **140** (2004), pp. 1357–1366.

[5] Y. Benoist, H. Oh, Discrete subgroups of $SL_3(\mathbf{R})$ generated by triangular matrices, *Int. Math. Res. Not. IMRN* (2010), no. 4, pp. 619–632.

[6] J. Bourgain, E. Fuchs, A proof of the positive density conjecture for integer Apollonian circle packings, *J. Amer. Math. Soc.* **24** (2011), pp. 945–967.

[7] J. Bourgain, A. Gamburd, P. Sarnak, Affine linear sieve, expanders, and sum-product, *Invent. Math.* **179** (2010), no. 3, pp. 559–644.

[8] J. Bourgain, A. Kontorovich, On representations of integers in thin subgroups of $SL(2,\mathbf{Z})$, *GAFA* **20** (2010), no. 5, pp. 1144–1174.

[9] J. Bourgain, P. Varjú, Expansion in $SL_d(\mathbf{Z}/q\mathbf{Z})$, $q$ arbitrary, *Inventiones math.* **188** (2012) pp. 151–173.

[10] T. Brady, W. Dison, T. Riley, Hyperbolic hydra, *Groups Geom. Dyn.* **7** (2013), no. 4, pp. 961–976.

[11] E. Breuillard, T. Gelander, Uniform independence in linear groups, *Invent. Math.* **173** (2008), no. 2, pp. 225–263.

[12] E. Breuillard, B. Green, R. Guralnick, T. Tao, Strongly dense free subgroups of semisimple algebraic groups, *Israel Journal of Mathematics*, **192** (2012) pp. 347–379.

[13] Y. V. Carrière, Autour de la conjecture de L. Markus sur les variétés affines, *Invent. Math.* **95** (1989), pp. 615–628.

[14] J. Danciger, F. Guéritaud, and F. Kassel, Convex cocompactness in pseudo-Riemannian hyperbolic spaces, *Geometriae Dedicata*, **192** (2018) pp. 87–126.

[15] W. Dison, T. Riley, Hydra groups, *Comment. Math. Helv.* **88** (2013), pp. 507–540.

[16] D. Dumas, A. Sanders, Geometry of compact complex manifolds associated to generalized quasi-Fuchsian representations, arXiv:1704.01091.

[17] A. Eskin, Quasi-isometric rigidity of nonuniform lattices in higher rank symmetric spaces, *J. Amer. Math. Soc.* **11** (1998), no. 2, pp. 321–361.

[18] E. Fuchs, C. Meiri, P. Sarnak, Hyperbolic monodromy groups for the hypergeometric equation and Cartan involutions, *J. Eur. Math. Soc.* **16** (2014), no. 8, pp. 1617–1671.

[19] T. Gelander, C. Meiri, The Congruence Subgroup Property does not imply Invariable Generation, *Int. Math. Res. Not. IMRN* (2017), no. 15, pp. 4625–4638.

[20] T. Gelander, A. Zuk, Dependence of Kazhdan constants on generating subsets, *Israel J. Math.* **129** (2002) pp. 93–98.

[21] W. Goldman, F. Labourie, G. Margulis, Proper affine actions and geodesic flows of hyperbolic surfaces, *Annals of Math.*, **170** (2009) pp. 1051–1083.

[22] A. Golsefidy-Salehi, P. Varju, Expansion in perfect groups, *GAFA* **22** (2012), no. 6, pp. 1832–1891.

[23] O. Guichard, A. Wienhard, Anosov representations: Domains of discontinuity and applications, *Invent. Math.* **190** (2012), no. 2, pp. 357–438.

[24] M. Hagen, D. Wise, Cubulating hyperbolic free-by-cyclic groups: the general case, *GAFA*, **25** (2015) pp. 134–179.

[25] K. Jo, I. Kim, On the Markus conjecture in convex case, arXiv:1809.07918.

[26] M. Kapovich, B. Leeb, J. Porti, Morse actions of discrete groups on symmetric spaces, arXiv:1403.7671.

[27] M. Kapovich, B. Leeb, J. Porti, Some recent results on Anosov representations, *Transformation Groups*, **21** (2016), no. 4, pp. 1105–1121.

[28] B. Kleiner, B. Leeb, Rigidity of quasi-isometries for symmetric spaces and Euclidean buildings, *Inst. Hautes Études Sci. Publ. Math.* **86** (1997), pp. 115–197.

[29] F. Labourie, Anosov flows, surface groups and curves in projective space, *Invent. Math.* **165** (2006), no. 1, pp. 51–114.

[30] D. D. Long, A. Reid, Thin surface subgroups in cocompact lattices in $SL(3,\mathbf{R})$, *Illinois J. Math.* **60** (2016), no. 1, pp. 39–53.

[31] A. Lubotzky, *Discrete groups, expanding graphs and invariant measures*, Birkhauser, 1994.

[32] G. Margulis, G. Soifer, Nonfree maximal subgroups of infinite index of the group $SL_n(\mathbf{Z})$, *Uspekhi Mat. Nauk* **34** (1979), no. 4 (208), pp. 203–204.

[33] K. A. Mikhailova, The membership problem for free products of groups, *Doklady Akademii Nauk SSSR*, **127** (1959), pp. 746–748.

[34] A. Muranov, Diagrams with selection and method for constructing boundedly generated and boundedly simple groups, *Comm. Algebra* **33** (2005), no. 4, 1217–1258.

[35] H. Oh, Discrete subgroups generated by lattices in opposite horospherical subgroups, *J. Algebra* **203** (1998), no. 2, pp. 621–676.

[36] D. Osin, Kazhdan constants of hyperbolic groups, *Funct. Anal. Appl.* **36** (2002), pp. 290–297.

[37] V. Platonov, A. Rapinchuk, *Algebraic Groups and Number Theory,* Academic Press, 1993.

[38] G. Prasad, A. Rapinchuk, Weakly commensurable arithmetic groups and isospectral locally symmetric spaces, *Publ. Math. IHES* **109** (2009) 113–184.

[39] G. Prasad, A. Rapinchuk, Generic elements in Zariski-dense subgroups and isospectral locally symmetric spaces, pp. 211–252. In *Thin groups and superstrong approximation*, Math. Sci. Res. Inst. Publ., 61, Cambridge Univ. Press, Cambridge, 2014.

[40] G. Prasad, A. Rapinchuk, On the congruence kernel for simple algebraic groups, *Proceedings of the Steklov Institute of Mathematics*, **292** (2016) pp. 216–246.

[41] I. Rapinchuk, On abstract homomorphisms of Chevalley groups over the coordinate rings of affine curves, *Transformation Groups*, **24** (2019) pp 1241–1259

[42] E. Rips, Subgroups of small cancellation groups, *Bull. London Math. Soc.*, **14** (1982) pp. 45–47.

[43] G. Soifer, Embedding of $\mathbf{Z}^2 * \mathbf{Z}$ as a discrete subgroup into small linear groups, *Internat. J. Algebra Comput.* **22** (2012), no. 7, pp. 17–24.

[44] J. Tits, Free subgroups in linear groups, *J. Algebra* **20** (1972), pp. 250–270.

[45] T. N. Venkataramana, Zariski dense subgroups of arithmetic groups, *J. Algebra* **108** (1987), no. 2, pp. 325–339.

[46] T. N. Venkataramana, On systems of generators of arithmetic subgroups of higher rank groups, *Pacific J. Math.* **166** (1994), no. 1, pp. 193–212.

[47] D. Wise, Cubulating small cancellation groups, *GAFA*, **14** (2004) pp. 150–214.

[48] D. Wise, *From Riches to Raags: 3-Manifolds, Right-Angled Artin Groups, and Cubical Geometry.* CBMS Regional Conference Series in Mathematics, **117** (2012) 141 pp.

[49] A. Zimmer, Projective Anosov representations, convex cocompact actions, and rigidity, arXiv:1704.08582.