EIGENVALUE RIGIDITY FOR ZARISKI-DENSE SUBGROUPS

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I would like to give a progress report on our work focused on a new form of rigidity that we call *eigenvalue rigidity*. Parts of this work are joint with various collaborators including G. Prasad, V. Chernousov and I. Rapinchuk.

As we all know, by rigidity in the classical sense we mean statements of the following nature:

Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be semi-simple Lie groups (or, more generally, the groups of points of semi-simple groups over an infinite field), and let $\Gamma_1 \subset \mathcal{G}_1$ and $\Gamma_2 \subset \mathcal{G}_2$ be "large" subgroups (lattice, commensurators of lattices, S-arithmetic subgroups). Then under appropriate assumptions a homomorphism/isomorphism $\phi\colon \Gamma_1 \to \Gamma_2$ (virtually) extends to a morphism/isomorphism $\tilde{\phi}\colon \mathcal{G}_1 \to \mathcal{G}_2$:

$$\begin{array}{ccc} \mathcal{G}_1 & \stackrel{\tilde{\phi}}{\dashrightarrow} & \mathcal{G}_2 \\ \bigcup & & \bigcup \\ \Gamma_1 & \stackrel{\phi}{\to} & \Gamma_2 \end{array} .$$

Such statements are very useful and powerful. One of the consequences is that the entire geometry of a compact hyperbolic manifold of dimension $\geqslant 3$ (including its volume, the Laplace spectrum, the lengths of closed geodesics, etc.) is determined by the structure of its fundamental group. I would like to point out another consequence of algebraic nature which is more relevant to our discussion.

Let $\Gamma = \mathrm{SL}_n(\mathbb{Z})$, where $n \geqslant 3$, and suppose we are given an absolutely almost simple simply connected algebraic group G over a number field K with ring of integers $\mathcal{O}$. If Γ is (virtually) isomorphic to $G(\mathcal{O})$ as an abstract group, then $K = \mathbb{Q}$ (and hence $\mathcal{O} = \mathbb{Z}$), and $G \simeq \mathrm{SL}_n$ as algebraic groups over $\mathbb{Q}$. Thus, the structure of a higher rank arithmetic group uniquely determines the *field of definition* and the *ambient group* as an algebraic group over this field.

This structural approach to rigidity obviously fails if we try to extend the results to arbitrary Zariski-dense subgroups because these may very well be free groups (in fact, by a famous theorem due to Tits, there are always Zariski-dense subgroups that are free groups on two generators). However, our results demonstrate that one can recover the field of definition and strongly suggest that one should also be able to almost recover the ambient algebraic group for any finitely generated Zariski-dense subgroup if instead of structural information one uses information about the eigenvalues of elements. This is an essential part of the phenomenon that we call *eigenvalue rigidity*.

So, basically what we want to say is that if the elements of two Zariski-dense subgroups have the "same" eigenvalues, then these subgroups have the same field of definition and almost the same ambient algebraic group. But before we can discuss any results to this effect, I need to tell you how we match the eigenvalues of elements of Zariski-dense subgroups. For one thing, they may be represented by matrices of differenet sizes, hence will have different numbers of eigenvalues. We gave the following definition.

Definition (Prasad-A.R.) Let F be an infinite field.

(1) Let $\gamma_1 \in \mathrm{GL}_{n_1}(F)$ and $\gamma_2 \in \mathrm{GL}_{n_2}(F)$ be *semi-simple (diagonalizable) matrices*, and let

$$\lambda_1, \ldots, \lambda_{n_1} \quad \text{and} \quad \mu_1, \ldots, \mu_{n_2}$$

be their eigenvalues (in a fixed algebraic closure $\overline{F}$). We say that γ_1 and γ_2 are *weakly commensurable* if there exist $a_1, \ldots, a_{n_1}$, $b_1, \ldots, b_{n_2} \in \mathbb{Z}$ such that

$$\lambda_1^{a_1} \cdots \lambda_{n_1}^{a_{n_1}} \;=\; \mu_1^{b_1} \cdots \mu_{n_2}^{b_{n_2}} \;\neq\; 1.$$

(2) Let $G_1 \subset \mathrm{GL}_{n_1}$ and $G_2 \subset \mathrm{GL}_{n_2}$ be reductive algebraic groups defined over F.
Two *Zariski-dense subgroups* $\Gamma_1 \subset G_1(F)$ and $\Gamma_2 \subset G_2(F)$ are called *weakly commensurable* if every semi-simple element $\gamma_1 \in \Gamma_1$ of infinite order is weakly commensurable to some semi-simple element $\gamma_2 \in \Gamma_2$ of infinite order, and vice versa.

If you see this definition for the first time, you may find it at least somewhat strange. First, what is the reason for lumping the eigenvalues together? Second, even more important, what are the consequences of this relation? We gave this definition while working on length-commensurable and isospectral locally symmetric spaces. Here is the set-up.

Let G be an absolutely almost simple algebraic groups over $\mathbb{R}$. We consider $\mathcal{G} = G(\mathbb{R})$ as a Lie group, and let $\mathcal{K}$ denote a maximal compact subgroup of $\mathcal{G}$ and $\mathfrak{X} = \mathcal{K}\backslash\mathcal{G}$ the associated symmetric spaces. Furthermore, given a discrete torsion-free subgroup $\Gamma \subset \mathcal{G}$, we let $\mathfrak{X}_\Gamma = \mathfrak{X}/\Gamma$ be the corresponding locally symmetric space. If Γ is *arithmetic*, then we say that $\mathfrak{X}_\Gamma$ is *arithmetically defined*.

For a Riemannian manifold M, we let $L(M)$ denote its (weak) length spectrum (the set of lengths of all closed geodesics), and - if M is compact - $E(M)$ its Laplace spectrum (the set of eigenvalues of the Beltrami-Laplace operator with multiplicities). Two Riemannian manifolds M_1 and M_2 are called

(1) *isospectral* if $E(M_1) = E(M_2)$ (assuming that M_1 and M_2 are compact);
(2) *iso-length-spectral* if $L(M_1) = L(M_2)$;
(3) *length-commensurable* if $\mathbb{Q} \cdot L(M_1) = \mathbb{Q} \cdot L(M_2)$.

On the other hand, M_1 and M_2 are *commensurable* if they have a common finite-sheeted cover:

$$\begin{array}{ccc} & M & \\ \swarrow \pi_1 & & \pi_2 \searrow \\ M_1 & & M_2 \end{array}$$

(where π_1 and π_2 are local isometries). In spectral geometry, one wants to understand when two isospectral/iso-length-spectral manifolds are necessarily isometric or at least commensurable.

In our work with Prasad, we addressed this problem for (arithmetically defined) locally symmetric spaces of simple algebraic $\mathbb{R}$-groups. So, let G_1 and G_2 be two absolutely almost simple algebraic $\mathbb{R}$-groups, and $\mathcal{G}_i = G_i(\mathbb{R})$ for $i = 1, 2$. Furthermore, we let $\mathfrak{X}_i = \mathcal{K}_i\backslash\mathcal{G}_i$ denote the associated symmetric spaces, and given discrete torsion-free subgroups $\Gamma_i \subset \mathcal{G}_i$ we let $\mathfrak{X}_{\Gamma_i}$ denote the corresponding locally symmetric spaces. We then have the following implications among the above properties:

- for $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ compact, $E(\mathfrak{X}_{\Gamma_1}) = E(\mathfrak{X}_{\Gamma_2})$ implies $L(\mathfrak{X}_{\Gamma_1}) = L(\mathfrak{X}_{\Gamma_2})$ (i.e., isospectral implies iso-length-spectral);
- trivially $L(\mathfrak{X}_{\Gamma_1}) = L(\mathfrak{X}_{\Gamma_2})$ implies $\mathbb{Q} \cdot L(\mathfrak{X}_{\Gamma_1}) = \mathbb{Q} \cdot L(\mathfrak{X}_{\Gamma_2})$ (i.e., iso-length spectrality implies length-commensurability);

- for Γ_1, Γ_2 lattices, $\mathbb{Q}\cdot L(\mathfrak{X}_{\Gamma_1}) = \mathbb{Q}\cdot L(\mathfrak{X}_{\Gamma_2})$ implies that Γ_1 and Γ_2 are weakly commensurable (i.e., length-commensurability implies weak commensurability of the fundamental groups).

Thus, weak commensurability should be viewed as an algebraic property that reflects isospectrality and more generally length-commensurability of locally symmetric spaces. On the other hand, $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are commensurable if and only if Γ_1 and Γ_2 are commensurable (with respect to an $\mathbb{R}$-isomorphism between the corresponding adjoint groups $\overline{G}_1$ and $\overline{G}_2$). So, the question becomes when weak commensurability of Γ_1 and Γ_2 implies their commensurability. At the first glance, the chances of proving a sufficiently general statement along these lines are not that great. Indeed, the following two matrices

$$A = \begin{pmatrix} 12 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1/24 \end{pmatrix} \quad , \quad B = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1/12 \end{pmatrix} \quad \in \quad \mathrm{SL}_3(\mathbb{C})$$

are weakly commensurable because

$$\lambda_1 = 12 = 4 \cdot 3 = \mu_1 \cdot \mu_2 \quad (\text{or} \quad \lambda_1 = \mu_3^{-1}).$$

However, no powers A^m and B^n $(m, n \neq 0)$ are conjugate, implying that the subgroups $\langle A \rangle$ and $\langle B \rangle$ are not commensurable even if one allows conjugation. What we discovered though is that the situation changes dramatically if instead of "small" (like cyclic subgroups) one considers "big" subgroups (e.g., Zariski-dense subgroups) of simple algebraic groups. In fact, the case of arithmetic subgroups can be worked out almost completely.

Theorem 0.1. (Prasad-R.) *Let G_1 and G_2 be absolutely almost simple algebraic groups over a field F of characteristic zero, and let $\Gamma_i \subset G_i(F)$ $(i = 1, 2)$ be Zariski-dense* arithmetic *subgroups.*

(1) *Assume that G_1 and G_2 are of the* same *Cartan-Killing type which is different from A_n, D_{2n+1} and E_6. If Γ_1 and Γ_2 are weakly commensurable, then they are commensurable.*

(2) *In all cases, (Zariski-dense) arithmetic subgroups $\Gamma_2 \subset G_2(F)$ weakly commensurable to a given arithmetic subgroup $\Gamma_1 \subset G_1(F)$ form finitely many commensurability classes.*

A few comments on this theorem are in order.

- The theorem remains valid in the context of S-arithmetic subgroups.
- The types excluded in (1) are precisely the types where (-1) is not in the Weyl group of the corresponding root system. In fact, they are honest exception. Namely, for each of those types, one can construct arbitrarily large, but finite, families of weakly commensurable but pairwise noncommensurable arithmetic subgroups.
- As we will see soon, the only situation where G_1 an G_2 can be of different types but still contain Zarsiki-dense weakly commensurable subgroups is when one of the groups is of type B_ℓ and the other of type C_ℓ. Weakly commensurable arithmetic subgroups in this case were completely classified in my work with Skip Garibaldi.

The theorem has a number of geometric applications. For example: Let M_1 and M_2 be compact *isospectral* hypebolic manifolds of dimension $d \not\equiv 1 (\mathrm{mod}\, 4)$. Assume that one of them is arithmetically defined. Then the other one is also arithmetically defined, and the manifolds are in fact commensurable.

The proof of the theorem is based on the following observation. Let G be an absolutely almost simple algebraic group over a field F of characteristic zero. Then the commensurability classes of Zariski-dense arithmetic subgroups $\Gamma \subset G(F)$ are classified by the pairs $(K, \mathcal{G})$ where K is a number field (the "field of definition" of Γ) and $\mathcal{G}$ is an F/K-form of the adjoint group $\overline{G}$ (i.e., $\mathcal{G}$ is a K-group such that $\mathcal{G} \times_K F$ is F-isomorphic to $\overline{G}$). More precisely, given Zariski-dense arithmetic subgroups $\Gamma_1 \subset G_1(F)$ and $\Gamma_2 \subset G_2(F)$ corresponding to the pairs $(K_1, \mathcal{G}_1)$ and $(K_2, \mathcal{G}_2)$, they are commensurable (up to an F-isomorphism between $\overline{G}_1$ and $\overline{G}_2$) if and only if $K_1 = K_2 =: K$ and $\mathcal{G}_1 \simeq \mathcal{G}_2$ over K. So, to prove that Zariski-dense weakly commensurable arithmetic subgroups Γ_1 and Γ_2 are commensurable we show that $K_1 = K_2$, and then using various local-global considerations (local classification, Hasse principle etc.) try to relate $\mathcal{G}_1$ and $\mathcal{G}_2$.

The important point here is that these invariants K and $\mathcal{G}$ can be defined and analyzed not only for arithmetic subgroups but for arbitrary Zariski-dense subgroups. In general, they do not determine the subgroup up to commensurability, but still carry important information. The problem is that the field of definition does not need to be a number field. For finitely generated Zariski-dense subgroups it can in principle be any finitely generated field. Of course, various arithmetic tools are simply not available in this generality. So, I would like to tell you about a new approach, based on good reduction, that appears to be quite useful and that already generated new results and also geometric applications.

Let me begin with two results for arbitrary Zariski-dense subgroups established in the work with Prasad. Let G_1 and G_2 be absolutely almost simple algebraic groups defined over a field F of characteristic zero, and let $\Gamma_i \subset G_i(F)$ $(i = 1, 2)$ be a finitely generated Zariski-dense subgroup.

Theorem 0.2. (Prasad-A.R.) *If Γ_1 and Γ_2 are weakly commensurable, then either G_1 and G_2 have the same Cartan-Killing type, or one of them is of type B_ℓ and the other of type C_ℓ for some $\ell \geqslant 3$.*

Now, given a Zariski-dense subgroup $\Gamma \subset G(F)$, where G is a semi-simple F-group, we let K_Γ denote the *trace field* of Γ, i.e. the subfield of F generated by the traces $\mathrm{tr}(\mathrm{Ad}\,\gamma)$ of all elements $\gamma \in \Gamma$ in the adjoint representation on the corresponding Lie algebra $\mathfrak{g} = L(G)$. By a result of Vinberg, the field $K = K_\Gamma$ is the minimal field of definition of $\mathrm{Ad}\,\Gamma$. This means that K is the minimal subfield of F such that all transformations in $\mathrm{Ad}\,\Gamma$ can be simultaneously represented by matrices over K in a suitable basis of $\mathfrak{g}$. If such a basis is chosen, then the Zariski closure of $\mathrm{Ad}\,\Gamma$ in $\mathrm{GL}(\mathfrak{g})$ is a semi-simple algebraic K-group $\mathcal{G}$. It is an F/K-form of the adjoint group $\overline{G}$, and we will call it the *algebraic hull* of $\mathrm{Ad}\,\Gamma$.

Theorem 0.3. (Prasad-A.R.) *If Γ_1 and Γ_2 are weakly commensurable, then $K_{\Gamma_1} = K_{\Gamma_2}$.*

For the sake of completeness, we mention one more result. Let K be the common trace field of two weakly commensurable Zariski-dense subgroups Γ_1 and Γ_2 as above, and let $\mathcal{G}_i$ be the algebraic hull of $\mathrm{Ad}\,\Gamma_i$ for $i = 1, 2$. We denote by L_i the minimal Galois extension of K over which $\mathcal{G}_i$ becomes an inner form of a split group.

Proposition 0.4. (Prasad-A.R.) *In the above notations, $L_1 = L_2$.*

Theorems 0.2 and 0.3 are the two basic results that opened an investigation into eigenvalue rigidity. Here is what they tell us. Let G_1 be an absolutely almost simple algebraic group over a field F of characteristic zero. Fix a finitely generated Zariski-dense subgroup $\Gamma_1 \subset G_1(F)$, and let K denote its trace field. If for some other absolutely almost simple algebraic F-group G_2 there exists a finitely generated Zariski-dense subgroup $\Gamma_2 \subset G_2(F)$ which is weakly commensurable to Γ_1, then either G_2 has the same type as G_1, or one of them has type B_ℓ, and the other type C_ℓ. So, it is enough to understand those Γ_2's weakly commensurable to the given Γ_1 that are contained in $G_2(F)$ for a fixed absolutely almost simple algebraic F-group G_2; in addition, without loss of generality we may assume G_2 to be *adjoint*. The trace field of such a Γ_2 is necessarily K, and let $\mathcal{G}(\Gamma_2)$ denote its algebraic hull, which is an F/K-form of G_2. The natural question arises of what one can say about the totality of $\mathcal{G}(\Gamma_2)$'s as Γ_2 runs through all possible finitely generated Zariski-dense subgroups $\Gamma_2 \subset G_2(F)$ that are weakly commensurable to Γ_1. The finiteness statement in Theorem 0.1 is proved by showing that when Γ_1 and Γ_2 are arithmetic, this collection of possible algebraic hulls of Γ_2's is finite. Our subsequent work strongly suggests that this finiteness should remain valid for all finitely generated Zariski-dense subgroups without any additional assumptions about the trace field K.

Finiteness Conjecture. In the above notations, there exists a *finite* collection $\mathcal{G}_1^{(2)}, \ldots, \mathcal{G}_r^{(2)}$ of F/K-forms of G_2 such that if $\Gamma_2 \subset G_2(F)$ is a finitely generated Zariski-dense subgroup that is weakly commensurable to Γ_1, then it is conjugate to a subgroup of one of the $\mathcal{G}_i^{(2)}(K)$'s $(\subset G_2(F))$.

This is a rather strong statement that an algebraic group is almost determined (i.e., determined up to finitely many possibilities) by the eigenvalues of elements of a Zariski-dense subgroup, however small this subgroup may be (e.g., it can very well be a free group on two generators). Here is what this conjecture means in some concrete situations.

Example 1. Let A be a central simple algebra over a finitely generated field K of characteristic zero, let $G = \mathrm{SL}_{1,A}$ be the algebraic K-group associated with norm 1 elements, and let $\Gamma \subset G(K)$ be a finitely generated Zariski-dense subgroup with the trace field K. Then there are only finitely many possibilities for a central simple K-algebra A' such that for $G' = \mathrm{SL}_{1,A'}$, the group $G'(K)$ may contain a finitely generated Zariski-dense subgroup Γ' which is weakly commensurable to Γ (and all these possible algebras have the same degree as A).

Example 2. Let q be a nondegenerate quadratic form in $n \geqslant 5$ variables over a finitely generated field K of characteristic zero, let $G = \mathrm{SO}_n(q)$, and let $\Gamma \subset G(K)$ be a finitely generated Zariski-dense subgroup with the trace field K. Then there exists finitely many similarity classes of n-dimensional quadratic forms q' over K such that for $G' = \mathrm{SO}_n(q')$, the group $G'(K)$ may contain a finitely generated Zariski-dense subgroup Γ' which is weakly commensurable to Γ.

The Finiteness Conjecture is known in the following cases:

(1) K is a number field although Γ_1 and Γ_2 do not need to be arithmetic (Prasad-A.R.);

(2) algebraic hull $\mathcal{G}(\Gamma_1)$ is of the form $\mathrm{SL}_{1,A}$ for some central simple K-algebra A, i.e. is an inner form of type A_n (Chernousov, A.R., I. Rapinchuk)

(3) for spinor groups of quadratic forms, some unitary groups, groups of type G_2 when K is a 2-dimensional global field, i.e. $K = k(C)$, the function field of a geometrically connected curve C over a number field k (Chernousov, A.R., I. Rapinchuk).

Items (1) and (2) together imply that the finiteness conjecture is true when Γ_1 is a lattice (arithmetic or not) in $\mathcal{G} = G(\mathbb{R})$ where G is an absolutely almost simple algebraic $\mathbb{R}$-group. I also would like to note the following consequence for Riemann surfaces (arithmetic or not). It is well-known that most Riemann surfaces are of the form $M = \mathbb{H}/\Gamma$ where $\mathbb{H}$ is the upper half-plane and $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ is a discrete torsion-free Zariski-dense subgroup. Then there is a natural way to associate to such Riemann surface a quaternion algebra $A(\Gamma)$ whose center is precisely the trace field of Γ - cf. Reid, MacLachlan (this algebra is an invariant of the commensurability class of Γ and in fact determines Γ if the latter is arithmetic).

Theorem 0.5. *Let $M_i = \mathbb{H}/\Gamma_i$ $(i \in I)$ be a family of length-commensurable Riemann surfaces where the subgroups Γ_i are Zariski-dense in PSL_2. Then the corresponding quaternion algebras $A(\Gamma_i)$ split into finitely many isomorphism classes over the common trace field K of all the Γ_i's.*

In the remaining part of the talk we will give an overview of the techniques involved in the proofs of the finiteness results over arbitrary finitely generated fields. It is based on the notion of *good reduction*. Let G be a (connected) reductive algebraic group over a field K, and let v be a discrete valuation of K. We let K_v denote the completion of K with respect to v, and $\mathcal{O}_v$ the valuation ring in K_v with the residue field k_v.

Definition. We say that G has *good reduction* at v if there exists a reductive group scheme $\mathcal{G}$ over $\mathcal{O}_v$ with generic fiber $\mathcal{G} \times_{\mathcal{O}_v} K_v$ isomorphic to $G \times_K K_v$. (Then the reduction $\underline{G}^{(v)} := \mathcal{G} \times_{\mathcal{O}_v} k_v$ is a (connected) reductive group k_v of the "same type" as G.)

Examples. (0) Every (simply connected) split K-group has good reduction at any v. For simple groups, it is provided by the Chevalley construction.

(1) $G = \mathrm{Spin}_n(q)$ has good reduction at v if q is equivalent to a quadratic form of the following shape

$$\lambda(a_1x_1^2 + \cdots + a_nx_n^2) \quad \text{with} \quad \lambda \in K_v^\times, \quad a_i \in \mathcal{O}_v^\times$$

(assuming that the residue characteristic is $\neq 2$).

(2) $G = \mathrm{SL}_{1,A}$, where A is a central simple K-algebra, has good reduction at v if $A \otimes_K K_v$ comes from an Azumaya algebra $\mathcal{A}$ over $\mathcal{O}_v$ (i.e., A is unramified at v).

(3) Let $L = \mathbb{Q}(\sqrt{p})$ where p is an odd prime, and let $T = \mathrm{R}^{(1)}_{L/\mathbb{Q}}(G_m)$ be the corresponding norm torus. Then T is represented by matrices of the form

$$\left\{ \begin{pmatrix} a & pb \\ b & a \end{pmatrix} \mid a^2 - pb^2 = 1 \right\}.$$

It follows that the reduction of T modulo p is $\{\pm 1\} \times G_a$, where G_a is the additive group. So, it is neither connected nor reductive, and therefore p is a prime of *bad* reduction in this case. One can similarly work out the case of $G = \mathrm{SL}_{1,D}$ where D is the quaternion algebra over $\mathbb{Q}$ corresponding to the pair $(-1, p)$ where p is a prime of the form $4k+3$.

Next, every finitely generated field K has an almost canonical set V of discrete valuations, called *divisorial*. More precisely, K can be viewed as the field of rational functions on a normal scheme X of finite type over $\mathbb{Z}$ (so-called *model* of K). Then the corresponding V consists of the discrete valuations of K associated to the prime divisors on X. A different choice of a model X results in a set of discrete valuations that differs from V in finitely many elements.

Example. Let $K = \mathbb{Q}(x)$. Then we can take $X = \mathrm{Spec}\, \mathbb{Z}[x]$. In this case $V = V_0 \cup V_1$ where V_0 consists of the valuations associated with the irreducible polynomials $f(x) \in \mathbb{Z}[x]$ having content 1, and V_1 of the natural (Gauss) extensions of the p-adic valuations for rational primes p.

We are now in a position to formulate the following.

Conjecture on groups with good reduction. Let G be a (connected) reductive algebraic group over a finitely generated field K, and let V be a divisorial set of discrete valuations. Then the set of (inner) K-forms G' of G that have good reduction at all $v \in V$ consists of finitely many isomorphism classes, provided that the characteristic of K is "good."

(For a semi-simple group G, the characteristic $p > 0$ is good if it does not divide the order of the Weyl group; for a nonsemi-simple reductive group (in particular, a torus) "good" characteristic means characteristic zero.)

One of the most famous results of G. Faltings (that earned him a Fields medal) is the theorem that the set of abelian varieties of a given dimension over a number field K having good reduction at all places of K lying outside a fixed finite set of places consists of finitely many isomorphism classes. This conjecture can be viewed as a hypothetical analog of this result for linear algebraic groups. Until recently, such questions were considered only in the situation where K is the quotient field of a Dedekind ring R and V consists of the discrete valuations associated with the maximal ideals of R. The higher-dimensional has been, and still remains, very much open.

The important point for us is that this conjecture would imply the truth of the Finiteness Conjecture due to the following result.

Theorem 0.6. (Chernousov, A.R., I. Rapinchuk) *Let G be an absolutely almost simple algebraic group over a finitely generated field K of characteristic zero, and let V be a divisorial set of places of K. Given a Zariski-dense subgroup $\Gamma \subset G(K)$ with the trace field K, there exists a finite subset $V(\Gamma) \subset V$ such that any absolutely almost simple algebraic group G' with the property that there exists a finitely generated Zariski-dense subgroup $\Gamma' \subset G'(K)$ that is weakly commensurable to Γ, has good reduction at all $v \in V \setminus V(\Gamma)$.*

Another implication of the conjecture along similar lines is the finiteness of the genus of an absolutely almost simple algebraic group over a finitely generated field of good characteristic. In a different direction, the conjecture immediately implies that for the adjoint group $\overline{G}$, the Tate-Shafarevich set

$$Ш(\overline{G}, V) := \mathrm{Ker}\left(H^1(K, \overline{G}) \longrightarrow \prod_{v \in V} H^1(K_v, \overline{G})\right)$$

is finite. However, we in fact expect the set $Ш(G, V)$ to be finite for any reductive group G and any divisorial set of places V (at least in characteristic zero).

We know the Conjecture on Good Reduction in the same cases as the Finiteness Conjecture (which were listed above) - in fact, this is the way we proved the Finiteness Conjecture in these cases. So, the general case remains wide open. Recently, however, we were able to prove the Conjecture on Good Reduction and the finiteness of the Tate-Shafarevich set for all tori over fields of characteristic zero.

Theorem 0.7. (A.R., I. Rapinchuk) *Let K be a finitely generated field of characteristic zero, V be a divisorial set of valuations of K. Then for every $d \geqslant 1$, there exists only finitely many isomorphism classes of d-dimensional K-tori having good reduction at all $v \in V$.*

Theorem 0.8. (A.R., I. Rapinchuk) *Let T be an algebraic torus over a finitely generated field K of characteristic zero, and let V be a divisorial set of places of K. Then*

$$Ш(T, V) := \mathrm{Ker}\left(H^1(K, T) \longrightarrow \prod_{v \in V} H^1(K_v, T)\right)$$

is finite.

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