Systole growth on arithmetic locally symmetric spaces

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BIRS Workshop on

Discrete Subgroups of Lie Groups

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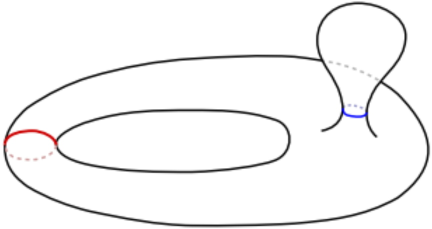
imagine **the impossible**

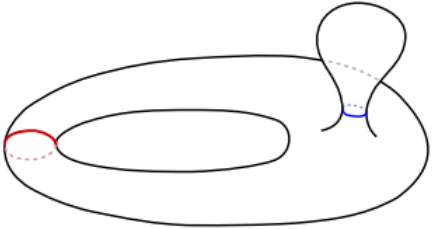
KIAS KOREA INSTITUTE FOR ADVANCED STUDY

A geometric invariant



A geometric invariant

Systole $=$ length of a shortest non-contractible closed geodesic in M.



Denoted by $\mathrm{sys}_1(M)$.

A topological result

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Theorem (Belolipetsky 2013)

Let M be a compact hyperbolic n-manifold with $\pi_1(S_g) \subset \pi_1(M)$.

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Let M be a compact hyperbolic n-manifold with $\pi_1(S_g) \subset \pi_1(M)$. Then, for any $\epsilon > 0$

$$g \geq e^{(\frac{1}{2}-\epsilon)\mathrm{sys}_1(M)}$$

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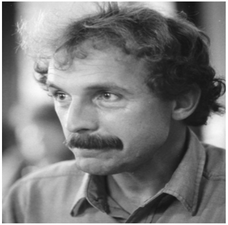
How to construct M with large systole?

The first event



P. Buser and **P. Sarnak** (1994):
Let $\Gamma \subset SL_2(\mathbb{R})$ be a cocompact arithmetic subgroup defined over $\mathbb{Q}$, and let $\Gamma(p)$ be a principal congruence subgroup.

For $S_p = \Gamma(p)\backslash\mathbb{H}^2$ we have

$$\mathrm{sys}_1(S_p) \geq \frac{4}{3}\log(\mathrm{area}(S_p)) - c,$$

where c is some constant independent of p.

Systole of congruence coverings

Mikhail Katz, **Mary Schaps** e **Uzi Vishne** (2007): Principal congruence subgroups $\Gamma(I)$ of any cocompact arithmetic group $\Gamma \subset \mathrm{SL}_2(\mathbb{C})$

$$\mathrm{sys}_1(M_I) \geq \frac{2}{3}\log(\mathrm{vol}(M_I)) - c_1.$$

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S. Makisumi (2013): $\frac{4}{3}$ is sharp in dimension 2.

Congruence coverings

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M_I is the congruence covering of M associated to I.

Systole of congruence coverings

Theorem (M. 2019)

Let M be a compact arithmetic hyperbolic n-orbifold as before, and M_I its congruence coverings. Then

$$\mathrm{sys}_1(M_I) \geq \frac{8}{n(n+1)} \log(\mathrm{vol}(M_I)) - c_2,$$

where c_2 is a constant.

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where c_2 is a constant.

Theorem (With C. Dória. 2019)

The constant $\frac{8}{n(n+1)}$ is sharp.

A topological result


Copyright: MFO

Theorem (Belolipetsky 2013)

Let M be a compact hyperbolic n-manifold with $\pi_1(S_g) \subset \pi_1(M)$. Then, for any $\epsilon > 0$

$$g \geq e^{(\frac{1}{2}-\epsilon)\mathrm{sys}_1(M)}$$

whenever $\mathrm{sys}_1(M)$ is large enough.

Proposition (Bel. 2013)

Let M be a compact arithmetic hyperbolic n-orbifold as before, and M_I its congruence coverings. If $\pi_1(S_{g_{min}}) \subset \pi_1(M_I)$, then

$$g_{min} \leq \mathrm{vol}(M_I)^{\frac{6}{n(n+1)}}.$$

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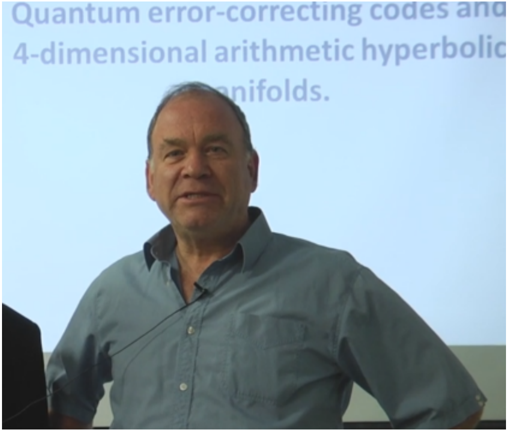
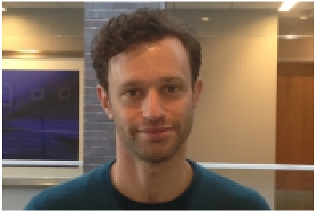
$$\mathrm{vol}(M_I)^{\frac{4}{n(n+1)}-\epsilon} \leq g_{min} \leq \mathrm{vol}(M_I)^{\frac{6}{n(n+1)}}.$$

Other applications



Limitation of parameters of error correcting codes constructed by **L. Guth** and **A. Lubotzky** in 2013.

Other applications



Construction of a special type of Einstein manifolds by **J. Fine** and **B. Premoselli** (2018).

Systole of congruence coverings (*Rank* > 1)



Sara Lapan, **Benjamin Linowitz** and **Jeffrey Meyer** (2018): Congruence subgroups $\Gamma(I)$ of non-cocompact arithmetic subgroups $\Gamma \subset \mathrm{SL}_n(\mathbb{R})$ such that

$$\mathrm{sys}_1(M_I) \geq \frac{2\sqrt{2}}{n(n^2-1)} \log(\mathrm{vol}(M_I)) - c_3.$$

$M_I = \Gamma(I)\backslash X$, $X = \mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n)$ and c_3 is a constant.

Systole of congruence coverings ($Rank = 1$)

$$\mathrm{sys}_1(M_I) \geq C \log(\mathrm{vol}(M_I)) - d,$$

d is a constant.

$$C = \begin{cases} \frac{2\sqrt{2}}{n(n+1)^2} & M \textit{ real } \text{hyperbolic} \\ \frac{1}{n(n+1)(n+2)} & M \textit{ complex } \text{hyperbolic} \\ \frac{1}{2\sqrt{2}(n+1)^2(2n+3)} & M \textit{ quaternionic } \text{hyperbolic} \end{cases}$$

Theorem (With V. Emery and I. Kim)

Let M be a compact quaternionic hyperbolic n-orbifold, and M_I its congruence coverings. Then

$$\mathrm{sys}_1(M_I) \geq \frac{4}{(n+1)(2n+3)} \log(\mathrm{vol}(M_I)) - c,$$

where c is a constant. Also, $\frac{4}{(n+1)(2n+3)}$ is sharp.

"A rare photo of Long and Reid".
Taken from Reid's homepage.

Theorem (Long and Reid, 2019)

There exists a sequence of congruence subgroups in $\mathrm{SL}_3(\mathbb{R})$ *all containing a genus* 3 *surface group.*

Thank you very much!